

The CAR algebra as a completion of finite matrix stages

`MathlibAnnex.CStarAlgebra.CAR.Limit`

abbrev

Names the completed CAR algebra in which finite matrix calculations can be extended by norm approximation.

Statement

For $n \geq 0$, let $A_n = M_{2^n}(\mathbb{C})$ with its operator norm, and include A_n in A_{n+1} by $a \mapsto a \otimes I_2$. The completed CAR algebra A is the norm completion of the resulting increasing algebraic union. The declaration names this completion; the cited constructions supply its algebra and involution.

Definition

Identify two finite-stage elements when their images agree in a common later stage. Addition, multiplication, scalar multiplication and adjoint can then be computed after passing to such a common stage. Isometry of the embeddings makes the norm of a represented element independent of its stage. Denote this normed union by A_{alg} . Define $A = \widehat{A_{\text{alg}}}$, its Hausdorff norm completion. Equivalently, elements of A are limits of Cauchy sequences from A_{alg} , with two sequences identified when their difference tends to zero. The Lean name `Limit` denotes this CAR algebra A ; `PreCAR` denotes the normed union A_{alg} . Here `abbrev` introduces a readily unfoldable name for `UniformSpace.Completion PreCAR`, not a second algebra or an additional theorem. Its full defining expression is displayed below.

Assumptions

The stages carry their usual complex matrix algebra operations and adjoint. The fixed embeddings are unital, preserve adjoints and are isometric. The index begins at $n = 0$, where $A_0 = \mathbb{C}$. There is no freely chosen representation or additional hypothesis on another algebra.

Conclusion

Each stage has a canonical unital star homomorphism $\iota_n : A_n \rightarrow A$. These maps preserve norms, hence are injective, satisfy $\iota_{n+1}(a \otimes I_2) = \iota_n(a)$, and have a dense union of ranges. With the operations extended from the union, A is a unital complex C*-algebra. These structural facts are furnished by the separately cited stage maps, density result and completion instances; they explain how the named completion is used.

Main citations

- Exact declaration and proof · Exact source
- `MathlibAnnex.CStarAlgebra.CAR.Stage` · Exact source
- `MathlibAnnex.CStarAlgebra.CAR.step` · Exact source
- `MathlibAnnex.CStarAlgebra.CAR.norm_step` · Exact source
- `MathlibAnnex.CStarAlgebra.CAR.PreCAR` · Exact source
- `MathlibAnnex.CStarAlgebra.CAR.preStarAlgEquiv` · Exact source
- `MathlibAnnex.CStarAlgebra.CAR.exists_common_stage` · Exact source
- `MathlibAnnex.CStarAlgebra.CAR.norm_stageHom` · Exact source

- [MathlibAnnex.CStarAlgebra.CAR.limitStarRing](#) · Exact source
- [MathlibAnnex.CStarAlgebra.CAR.limitStarModule](#) · Exact source
- [MathlibAnnex.CStarAlgebra.CAR.limitCStarRing](#) · Exact source
- [MathlibAnnex.CStarAlgebra.CAR.limitCStarAlgebra](#) · Exact source
- [MathlibAnnex.CStarAlgebra.CAR.ofStage](#) · Exact source
- [MathlibAnnex.CStarAlgebra.CAR.norm_ofStage](#) · Exact source
- [MathlibAnnex.CStarAlgebra.CAR.ofStage_step](#) · Exact source
- [MathlibAnnex.CStarAlgebra.CAR.dense_stageRange](#) · Exact source

Lean source signature (exact)

```
abbrev Limit := UniformSpace.Completion PreCAR
```

Here `isometry_stage` certifies the isometric connecting maps. `Metric.InductiveLimit` forms their metric union, and `UniformSpace.Completion` completes it. Thus `Limit` denotes the CAR algebra $\mathcal{A}$, not a generic limit operator.

Related definition — separate exact excerpt

```
abbrev PreCAR := Metric.InductiveLimit isometry_stage
```

Lean realization notes

The finite matrix stages form a dense subalgebra. The completion allows limits of norm-Cauchy sequences from that union. The construction uses the compatible operator norms of the stages; no new norm is chosen by taking a supremum over representations.

▼ Exact Card identity

Language: en

CARD_CONTENT_COMPLETE

Card UID: lfh:lfh-declaration-card:sha256:ce382ff1e89d804269092f5f9b219d81b789e39081e5f7a8e2dcd4763c85acfa

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Exposition revision: 1 · SHA-256: efe5359e61738d6c8116b12db794edcfb0abfafc587a9a02ea3a662872d2384d

Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73

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