A representation component is the full coefficient-kernel family

`MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.component_matrixCoefficientKernel_eq_family`

theorem

Describes each irreducible representation component by all of its nonzero vector coefficients.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, with no identity required, and let $\pi : A \to \mathcal{B}(H)$ be a nonzero irreducible $*$-representation on a complete complex Hilbert space. The inner product is conjugate-linear in its first argument. For nonzero $p, q \in H$, put

$$f^{\pi}_{p,q}(a) = \langle q, \pi(a)(p) \rangle, \qquad \mathcal{F}_{\pi} = \{\ker f^{\pi}_{p,q} : p, q \neq 0\}.$$

Write $S(A)$ for the geometric structure space of A. In this C*-algebra setting its points are precisely $\ker \rho$, where ρ is extreme, for real convexity, in the closed unit ball of the Banach dual; the extreme-kernel identification guarantees this description from the original face definition. For $\mathcal{T} \subseteq S(A)$, its closure is

$$c_A(\mathcal{T}) = \left\{ I \in S(A) : \bigcap_{J \in \mathcal{T}} J \subseteq I \right\}.$$

Two points are linked if they are equal or their pair is not closed. Let $C(I)$ be the equivalence class of I under finite chains of linked points. Then, for every nonzero $x, y \in H$,

$$C(\ker f^{\pi}_{x,y}) = \mathcal{F}_{\pi}.$$

Assumptions

The source uses the compatible C*-order on A. Irreducibility includes nonzero action and excludes proper nonzero closed reducing subspaces. No faithfulness, finite dimension or separability is required. Under this irreducibility hypothesis, $\ker f^{\pi}_{p,q}$ is a point of $S(A)$ for every $p, q \neq 0$. The component is defined by the closure relation above, without assuming that c_A is a topological closure.

Conclusion

Every point of the entire component, including one initially presented without reference to π, is a coefficient kernel of this fixed representation. Conversely, every pair of nonzero coefficient vectors gives a point of the component. Thus the choice of the initial pair (x, y) has no effect on this

component. This is an equality of subsets of $S(A)$; it does not identify an individual coefficient kernel with $\ker \pi$.

Main citations

- Exact whole-component theorem: `MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.component_matrixCoefficientKernel_eq_family`.
- Component–representation correspondence: the equivalence between finite-chain components and unitary equivalence.
- The coefficient-kernel family and its same-component inclusion: `matrixCoefficientKernelFamily` and `matrixCoefficientKernelFamily_subset_component`.
- Ryotaro Tanaka, arXiv:2504.05551v1, Lemma 5.10. The present declaration gives the component formula for an arbitrary irreducible representation; the pure-state specialization and the intersection calculation have separate explanations.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem component_matrixCoefficientKernel_eq_family
    (pi : NonUnitalCStarRepresentation A H) (hpi : pi.IsIrreducible)
    (x y : H) (hx : x ≠ 0) (hy : y ≠ 0) :
    (NormedSpace.geometricStructureClosure ℂ A).component
      (matrixCoefficientKernel pi hpi x y hx hy) = matrixCoefficientKernelFamily pi hpi
```

In the source	Mathematical meaning

Algebra and order
`variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]`
`variable [PartialOrder A] [StarOrderedRing A]`
	The nonzero complex C*-algebra A with its compatible order; no identity is required.

Representation space `variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`	The complex Hilbert space H is complete.
One irreducible representation `(pi : NonUnitalCStarRepresentation A H) (hpi : pi.IsIrreducible)`	The same fixed nonzero irreducible representation $\pi : A \to \mathcal{B}(H)$ supplies all coefficient kernels. It need not be faithful.
Starting vectors `(x y : H) (hx : x ≠ 0) (hy : y ≠ 0) :`	The starting pair is $x, y \in H \setminus \{0\}$, without a normalization assumption.

Component and full family
`(NormedSpace.geometricStructureClosure ℂ A).component`
`  (matrixCoefficientKernel pi hpi x y hx hy) = matrixCoefficientKernelFamily pi hpi := by`
	The component is taken in the ambient closure c_A, starting at $\ker f^{\pi}_{x,y}$. The complete equality is
	$$C(\ker f^{\pi}_{x,y}) = \mathcal{F}_{\pi}.$$
	It concerns all points of that component, not just kernels chosen in advance.

In the source	Mathematical meaning
The coefficient-kernel family `def matrixCoefficientKernelFamily (pi : NonUnitalCStarRepresentation A H) (hpi : pi.IsIrreducible) : Set (NormedSpace.GeometricStructureSpace ℂ A) := {I | ∃ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0), matrixCoefficientKernel pi hpi x y hx hy = I}`	A point $I \in S(A)$ belongs to $\mathcal{F}_\pi$ exactly when there are $p, q \in H \setminus \{0\}$ with $I = \ker f^\pi_{p,q}$. The vectors may depend on I; the representation is the same π throughout.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Exact surrounding source:

```
variable [PartialOrder A] [StarOrderedRing A]
```

Exact surrounding source:

```
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```

Related exact source for the reading guide

The coefficient-kernel family · Fixed source

```
def matrixCoefficientKernelFamily (pi : NonUnitalCStarRepresentation A H) (hpi :
pi.IsIrreducible) :
    Set (NormedSpace.GeometricStructureSpace ℂ A) :=
  {I | ∃ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0), matrixCoefficientKernel pi hpi x y hx hy = I}
```

Algebra and order · Fixed source

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
variable [PartialOrder A] [StarOrderedRing A]
```

Component and full family · Fixed source

```
    (NormedSpace.geometricStructureClosure ℂ A).component
      (matrixCoefficientKernel pi hpi x y hx hy) = matrixCoefficientKernelFamily pi hpi := by
```