The matching GNS fiber retains exactly its cyclic line

`MathlibAnnex.CStarAlgebra.CAR.selected_commonFixedProjection_eq_rankOne`

theorem

Identifies the residual projection of a transported CAR flag in its matching pure-state representation.

Statement

For the fixed shell family and index i described below, set $Q_n = \pi_i(f_{i,n})$ and $F_i = \bigcap_{n\geq 0} \operatorname{ran} Q_n$. The orthogonal projection P_i onto F_i satisfies $P_i x = \langle \xi_i, x\rangle \xi_i$ for every $x \in H_i$. In particular, $F_i = \mathbb{C}\xi_i$.

Assumptions

Let A be the completed CAR algebra, the completion of the matrix stages $M_{2^n}(\mathbb{C})$ under $a \mapsto a \otimes I_2$. Write f_n for the image of the first diagonal matrix unit. These are decreasing projections with $f_0 = 1$. Write ι_n for the canonical embedding of stage n. The root state ω is determined by $\omega(\iota_n(a)) = a_{00}$ on every stage.

Choose one pure state φ_j from each unitary-equivalence class $j \in I$ of pure-state Gelfand-Naimark-Segal (GNS) representations, retaining ω at the root index o. Write (H_j, π_j, ξ_j) for its complete complex GNS space, unital star representation and unit cyclic vector. Thus $\varphi_j(a) = \langle \xi_j, \pi_j(a)\xi_j\rangle$, and the vectors $\pi_j(a)\xi_j$ are dense in H_j. Inner products are linear in the second argument.

A fixed shell family supplies unital complex star automorphisms α_j and elements $w_{j,n} \in A$ with $\varphi_j \circ \alpha_j = \omega$, $w_{j,n}^* w_{j,n} = \alpha_j(f_n - f_{n+1})$, and $w_{j,n} w_{j,n}^* = f_n - f_{n+1}$. At the root, α_o is the identity and $w_{o,n} = f_n - f_{n+1}$. Fix $i \in I$ and put $f_{i,n} = \alpha_i(f_n)$. The family is supplied; its existence is not an extra conclusion here.

Conclusion

The one-dimensional common range is a proved property of the transported flag, not a field imposed on the shell family. Since Q_n are decreasing orthogonal projections, the cited decreasing-projection theorem also gives $Q_n x \to \langle \xi_i, x\rangle \xi_i$ in norm for each fixed $x \in H_i$.

Proof route

The state identity gives $\varphi_i(f_{i,n}) = 1$, so every Q_n fixes ξ_i. The transported CAR compression estimate gives $\|f_{i,n} b f_{i,n} - \varphi_i(b) f_{i,n}\| \to 0$ for each $b \in A$. Contractivity of π_i and fixedness of vectors in F_i yield $P_i \pi_i(b) P_i = \varphi_i(b) P_i$. Applying this to ξ_i determines P_i on its dense cyclic orbit and hence on all of H_i.

Proof steps

1. For a projection Q_n, the identity $\langle \xi_i, Q_n \xi_i\rangle = 1 = \|\xi_i\|^2$ implies $\|(I - Q_n)\xi_i\|^2 = 0$. Therefore $P_i \xi_i = \xi_i$.
2. On vectors fixed by every Q_n, the matrix coefficient of the represented compression error is independent of n and tends to zero. This proves the displayed compressed-operator identity. The source error converges in the norm of A.

3. Consequently $P_i\pi_i(b)\xi_i = \varphi_i(b)\xi_i = \langle \xi_i, \pi_i(b)\xi_i\rangle\xi_i$. Both sides are continuous linear maps of the orbit vector, and that orbit is dense. The unit norm of ξ_i identifies the resulting rank-one map as an orthogonal projection.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.transportedFlag · Exact source
- MathlibAnnex.CStarAlgebra.CAR.tendsto_representative_transported_compression · Exact source
- MathlibAnnex.CStarAlgebra.CAR.selectedVector_fixed_transportedFlag · Exact source
- MathlibAnnex.CStarAlgebra.PureState.selected_vectorFunctional · Exact source
- MathlibAnnex.Analysis.CStarAlgebra.commonFixedProjection_eq_rankOne_of_dense_orbit · Exact source
- MathlibAnnex.Analysis.CStarAlgebra.projection_eq_rankOne_of_dense_orbit · Exact source
- Submodule.tendsto_starProjection_iInf · Exact source
- The root state bundled with its purity proof · Exact source

Lean source signature (exact)

```
theorem selected_commonFixedProjection_eq_rankOne
    (family : RepresentativeShellFamily) (i : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit) :
    commonFixedProjection (fun n ↦
      MathlibAnnex.CStarAlgebra.PureState.selectedRepresentation completedRootPureState i
        (transportedFlag family i n)) =
      InnerProductSpace.rankOne ℂ
        (MathlibAnnex.CStarAlgebra.PureState.selectedVector completedRootPureState i)
        (MathlibAnnex.CStarAlgebra.PureState.selectedVector completedRootPureState i)
```

> Here `Limit` is A and `completedRootPureState` is the root state ω, bundled with its purity proof. `selectedRepresentation completedRootPureState i` is π_i, `selectedVector completedRootPureState i` is the unit vector ξ_i, and `transportedFlag family i n` is $f_{i,n}$. `commonFixedProjection` projects onto the vectors fixed by every Q_n; for projections this is F_i. The displayed `rankOne` sends x to $\langle \xi_i, x\rangle\xi_i$. The equality itself is not a filter-limit statement.

Lean realization notes

The exact declaration identifies the common fixed projection. The pointwise convergence just stated is a consequence using the separately cited decreasing-projection theorem. Neither statement asserts operator-norm convergence, and no strong limit is transported through a representation of a different algebra.

▼ **Exact Card identity**

Language: en
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