Exact projection links for one approximately inner automorphism

`MathlibAnnex.CStarAlgebra.exists_shell_family_of_approximately_inner`

theorem

A point-norm approximation followed by close-projection conjugacy gives exact support identities for an arbitrary projection family.

Statement

Let A be a nonzero unital complex C*-algebra with its usual positive order, and let $\alpha : A \to A$ be a complex star automorphism. Suppose that for every finite $F \subset A$ and $\varepsilon > 0$ there is $g \in U(A)$ satisfying $\|\alpha(a) - gag^*\| < \varepsilon$ for all $a \in F$. For any set I and family of self-adjoint projections $(q_i)_{i \in I}$ in A, there are elements $(w_i)_{i \in I}$ such that $w_i^* w_i = \alpha(q_i)$ and $w_i w_i^* = q_i$ for every i.

Assumptions

The same automorphism α and its finite-set approximation property are fixed for the whole family. The index set is arbitrary. The projections need not be nonzero, orthogonal, or countably indexed, and no condition on their sum is imposed.

Conclusion

Each w_i is a partial-isometry link with initial support $\alpha(q_i)$ and final support q_i. The resulting links all use the one specified α, although the auxiliary approximating and correcting unitaries may depend on i.

Proof route

For one projection q, approximate $\alpha(q)$ within distance 1 by gqg^*. Close self-adjoint projections are exactly unitarily conjugate, so choose t with $t\alpha(q)t^* = gqg^*$. Then $w = g^*t\alpha(q)$ has the required supports. Apply this argument separately to each index and choose one link at each index.

Proof steps

1. Approximate on the singleton $\{q\}$ with tolerance 1. Both $e = \alpha(q)$ and gqg^* are self-adjoint projections and $\|e - gqg^*\| < 1$. The close-projection conjugacy theorem supplies $t \in U(A)$ with $tet^* = gqg^*$.
2. For $w = g^*te$, the unitary identities and $e = e^* = e^2$ give $w^*w = et^*gg^*te = e$. The other support is $ww^* = g^*tet^*g = q$. Thus the direction of the two support equations is fixed.
3. Use choice on the individual existence statements for q_i. This produces a function $i \mapsto w_i$ with both equations at every index, including the vacuous case of an empty index set.

Main citations

- The stated existence or structural result · Exact source
- Single-projection construction at tolerance one · Exact source
- Close-projection unitary conjugacy · Exact source

- Exact link formula · Exact source
- Both support identities for the link · Exact source

Lean source signature (exact)

```
theorem exists_shell_family_of_approximately_inner
    (A : Type u) {ι : Type v}
    [CStarAlgebra A] [PartialOrder A] [StarOrderedRing A] [Nontrivial A]
    (alpha : A ≃⋆ₐ[ℂ] A)
    (happrox : ∀ (F : Finset A) (epsilon : ℝ), 0 < epsilon →
      ∃ g : unitary A, ∀ a ∈ F,
        ‖alpha a - (g : A) * a * star (g : A)‖ < epsilon)
    (f : ι → A) (hf : ∀ i, IsStarProjection (f i)) :
    ∃ w : ι → A, ∀ i,
      star (w i) * w i = alpha (f i) ∧ w i * star (w i) = f i
```

> Here the source family `f i` is q_i in the prose, and `alpha` is the single fixed α. `hf` supplies self-adjoint idempotence. In the single-projection proof `matchedLink g t (alpha f)` is exactly $g^* t\alpha(q)$. The first support is `star w * w`; it must not be interchanged with the second.

Lean realization notes

The result gives algebraic support equations for the family. It asserts no convergence of a sum of links, no common correcting unitary, and no uniqueness of the chosen links. No pure-state or homogeneity hypothesis is needed once α and its approximate innerness are supplied.

▼ **Exact Card identity**

Language: en

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