Faithfulness of the representation on the CAR trace Hilbert space

`MathlibAnnex.CStarAlgebra.CAR.separableCounterexampleRepresentation_injective`

theorem

Proves that the representation $\widetilde{\rho} : T \to B(H_\tau)$ is injective.

Statement

Let A be the completed CAR algebra and $T = C^*(\pi(A) \cup \{L_i : i \in I\}) \subseteq B(\mathcal{H})$ the algebra generated by the selected direct sum π of pure-state GNS representations indexed by their unitary-equivalence classes $i \in I$ and the unitaries L_i from the fixed homogeneity and shell construction. Write $\iota(a) = \pi(a)$ for the embedding $A \to T$. Let τ be the CAR trace and ϕ its chosen state extension to T. Write $\rho_\phi : T \to B(H_\phi)$ for the GNS representation of ϕ, and use the chosen unitary $E : H_\tau = L^2(A, \tau) \to H_\phi$ from the cited trace construction. For $\widetilde{\rho}(t) = E^{-1}\rho_\phi(t)E$,

$$\widetilde{\rho}(s) = \widetilde{\rho}(t) \implies s = t \qquad (s, t \in T).$$

Assumptions

The automorphisms and unitaries defining T are those fixed by CAR homogeneity. The cited injectivity theorem for the conjugated GNS representation applies to every such shell family, hence to this one.

Conclusion

Thus $\widetilde{\rho} : T \to B(H_\tau)$ is faithful. Together with the separately established separability of H_τ, it realizes this same T faithfully on a separable Hilbert space.

Proof route

Cancel the fixed unitary conjugation and use faithfulness of the GNS representation ρ_ϕ of the state extension.

Proof steps

1. If $\widetilde{\rho}(s) = \widetilde{\rho}(t)$, then

   $$\rho_\phi(s) = E\widetilde{\rho}(s)E^{-1} = E\widetilde{\rho}(t)E^{-1} = \rho_\phi(t).$$

   The cited injectivity theorem for ρ_ϕ gives $s = t$. The Lean proof specializes this same argument to the fixed family.

Main citations

- The stated existence or structural result · Exact source
- The exact representation being tested · Exact source
- Injectivity under the fixed unitary conjugation · Exact source
- Faithfulness of the GNS representation of the extended state · Exact source
- Separability of the fixed Hilbert space · Exact source

Lean source signature (exact)

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theorem separableCounterexampleRepresentation_injective :
    Function.Injective separableCounterexampleRepresentation
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> The exact theorem asserts `Function.Injective separableCounterexampleRepresentation`, namely injectivity of $\widetilde{\rho}$. Its proof calls `traceModelRepresentation_injective homogeneityShellFamily`; the unitary E and ρ_ϕ are in that linked construction, not additional assumptions in this signature.

Lean realization notes

This statement concerns injectivity of a representation, not faithfulness of a vector state, and does not assert irreducibility.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
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Card revision: 2 · SHA-256: 82dcb53f14784d8184619f6b558bae0a063457b36614f838624b46b2451cb4c8
Exposition revision: 3 · SHA-256: 3dc89042d82a9fcf010aae44694943627b4617c3e634a9718e986ae5401d93c6
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