Elementary C*-algebras from singleton spectrum and finite conjugation

`MathlibAnnex.Analysis.CStarAlgebra.exists_pureState_compactOperatorModel_iff_singleton_finiteConjugation`

theorem

Characterizes a pure-GNS compact model by singleton spectrum and continuous finite conjugation.

Statement

Let A be a nonzero complex C*-algebra with its compatible order. Write $\widehat{A}$ for its irreducible spectrum, the pure states modulo unitary equivalence of their GNS representations, and $S(A)$ for its geometric structure closure space. For $I \in S(A)$, let $C_A(I)$ be its finite-chain component, with the closure induced from $S(A)$. For a complex Hilbert space H, let $\mathcal{K}(H)$ denote the algebra of compact operators on H.

The following conditions are equivalent.

1. There is a pure state φ of A whose GNS representation $\pi_\varphi : A \to \mathcal{B}(H_\varphi)$ is faithful and satisfies

$$\pi_\varphi(A) = \mathcal{K}(H_\varphi). \tag{1}$$

2. The irreducible spectrum $\widehat{A}$ is a singleton, and there is a point $I \in S(A)$ whose component $C_A(I)$ satisfies the conjugation condition below.

For a closure space Y, write c_Y for its closure operator. For the geometric structure spaces $S(A)$ and $S(B)$, abbreviate these operators to c_A and c_B, respectively. Write $r(E)$ for the image of a subset E under a map r. The condition concerns every closure space X that is homeomorphic to $S(B)$ for some nonzero complex C*-algebra B. Here $S(B)$ is the same geometric structure construction used for $S(A)$. A homeomorphism $r : S(B) \to X$ is a bijection preserving closure on every subset:

$$r\big(c_B(E)\big) = c_X\big(r(E)\big) \qquad (E \subseteq S(B)).$$

The algebra B may vary; no unit, separability assumption or chosen representation of B is part of this condition.

For every such X, consider each of its finite-chain components D, with the induced closure $c_D(E) = D \cap c_X(E)$ for $E \subseteq D$, and all bijections

$$f : D \longrightarrow C_A(I), \qquad g : D \longrightarrow D.$$

A finite-chain component consists of points joined by a finite chain of linked pairs. Two points are linked if they are equal or their two-point set is not closed. These components are taken in the specified closure space, not as connected components of a topology.

A bijection $r : Y \to Z$ of closure spaces is a finite homeomorphism if it preserves closure on every finite subset:

$$r\big(c_Y(F)\big) = c_Z\big(r(F)\big) \qquad (F \subseteq Y \text{ finite}). \qquad (2)$$

A map $r : Y \to Z$ is continuous if

$$r\big(c_Y(E)\big) \subseteq c_Z\big(r(E)\big) \qquad (E \subseteq Y). \qquad (3)$$

The required condition is that whenever f and g are finite homeomorphisms and are also continuous in the sense of (3), transporting g to $C_A(I)$ by f gives a homeomorphism. More explicitly, both

$$\begin{matrix} h = f \circ g \circ f^{-1}, \\ h^{-1} = f \circ g^{-1} \circ f^{-1} \end{matrix} \qquad : C_A(I) \longrightarrow C_A(I) \qquad (4)$$

must be continuous on arbitrary subsets. This is required for every X, every component D and every pair f, g described above. Continuity of f^{-1} or g^{-1} on arbitrary subsets is not assumed. The inverses in (4) are those of the same bijections f and g.

Assumptions

A is a nonzero complex C*-algebra with its compatible order. It need not be unital or separable, and no finite-dimensionality assumption is imposed. The geometric condition requires both a singleton irreducible spectrum and the existence of $I \in S(A)$; neither condition is interpreted vacuously. The source guide records the fixed size bounds for the spaces in the conjugation condition.

Conclusion

The two geometric conditions characterize the existence of a pure state whose faithful GNS representation identifies A, as a C*-algebra, with $\mathcal{K}(H_\varphi)$. The Hilbert space and representation in this identification are those of the same state φ in (1).

Main citations

The compact-model characterization states this equivalence in every Hilbert dimension. Its two directions use the existence of a compact-operator model and the conjugation property of compact-operator models. The conjugation condition specifies the inputs and the two continuous maps in (4); the realization of closure spaces by C*-algebras specifies the spaces X. The conjugate and its inverse act by the formulas in (4).

Lean source signature (exact)

The preceding `variable` declarations supply `A` and its algebra and order structures; `[Nontrivial A]` appears explicitly in the theorem. The outer ↔ compares the GNS-model condition with a conjunction ∧: a singleton irreducible spectrum and the existence of a geometric point `I` with the component condition. The following is the contiguous prefix of the selected declaration, ending before the proof assignment.

```
theorem exists_pureState_compactOperatorModel_iff_singleton_finiteConjugation
    [Nontrivial A] :
    (∃ phi : NonUnitalPureState A, IsCompactOperatorModel phi.gnsRepresentation) ↔
      (∃ q : IrreducibleSpectrum A, ∀ r : IrreducibleSpectrum A, r = q) ∧
      ∃ I : NormedSpace.GeometricStructureSpace ℂ A,
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component
I)).HasContinuousFiniteConjugation
            (ClosureCat.IsCStarRealizable.{u, max u w})
```

In the source	Mathematical meaning

Ambient algebra
`universe u v w uA uL`
`variable {A : Type u} [NonUnitalCStarAlgebra A]`
`variable [PartialOrder A] [StarOrderedRing A]`
	The complex C*-algebra A with its compatible order; the size parameters are fixed for the selected theorem. No identity is required.

Nonzero algebra and one GNS model
`[Nontrivial A] :`
`(∃ phi : NonUnitalPureState A,`
`IsCompactOperatorModel phi.gnsRepresentation) ↔`
	The nonzero algebra A and one pure state φ whose same GNS representation $\pi_\varphi : A \to \mathcal{B}(H_\varphi)$ satisfies (1).

Compact-operator model
`def IsCompactOperatorModel [NonUnitalCStarAlgebra A]`
`(e : A →⋆ₙₐ[ℂ] (H →L[ℂ] H)) : Prop :=`
`Function.Injective e ∧`
`(∀ a : A, IsCompactOperator (e a)) ∧`
`∀ T : H →L[ℂ] H, IsCompactOperator T → ∃ a : A, e a = T`
	Applied to π_φ, this means faithfulness and the two inclusions giving
	$$\pi_\varphi(A) = \mathcal{K}(H_\varphi).$$
	Every image operator is compact, and every compact operator is an image of this same representation.

Singleton spectrum
`(∃ q : IrreducibleSpectrum A, ∀ r :`
`IrreducibleSpectrum A, r = q) ∧`
	The irreducible spectrum $\widehat{A}$ has exactly one point, including its existence. This is conjoined with the component condition that follows.

Point and applied component condition
`∃ I : NormedSpace.GeometricStructureSpace ℂ A,`
`((NormedSpace.geometricStructureClosure ℂ A).subspace`
`((NormedSpace.geometricStructureClosure ℂ A).component I)).HasContinuousFiniteConjugation`
`(ClosureCat.IsCStarRealizable.{u, max u w})`
`:= by`
	A point $I \in S(A)$ selects its finite-chain component $C_A(I)$. The ambient closure c_A is restricted to this component, and the resulting closure is the input of the conjugation condition, with precisely the stated class of test spaces.

In the source	Mathematical meaning
Geometric points `def GeometricStructureSpace : Type u := {I : Set E // I ∈ geometricStructureFamily 𝕜 E}`	A point of $S(A)$ is a subset of A belonging to the geometric structure family. Thus the quantified I is a geometric point, not an arbitrary subset or a normal-part point.
Ambient geometric closure `def geometricStructureClosure : ClosureSpace (GeometricStructureSpace 𝕜 E) := ClosureSpace.HullKernel.space (geometricStructureFamily 𝕜 E) (univ_notMem_geometricStructureFamily 𝕜 E)`	The hull–kernel closure c_A on the whole space $S(A)$, rather than the relative closure on a normal part.
Finite-chain component `def component (c : ClosureSpace α) (x : α) : Set α := {y | c.Related x y}`	The points related to I by finite chains form $C_A(I)$. This is not a claim about ordinary topological connectedness.
Induced component closure `def subspace (c : ClosureSpace α) (A : Set α) : ClosureSpace A := c.induced (Subtype.val : A → α)`	Restricting the ambient closure along inclusion gives the closure on the component. For $E \subseteq C_A(I)$, $$c_{C_A(I)}(E) = C_A(I) \cap c_A(E).$$
Class of test spaces `def IsCStarRealizable.{uA, uX} (X : ClosureCat.{uX}) : Prop := ∃ (A : Type uA) (instA : NonUnitalCStarAlgebra A), letI : NonUnitalCStarAlgebra A := instA Nontrivial A ∧ Nonempty (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A) X.str)`	A test space X is homeomorphic, for full closure, to $S(B)$ for some nonzero complex C*-algebra B. The bound algebra in this definition is the realizing B, not necessarily the original A.
All tests, components and first map `def HasContinuousFiniteConjugation (c : ClosureSpace α) (D : ClosureCat.{s} → Prop) : Prop := ∀ (L : ClosureCat.{s}), D L → ∀ t : L, ∀ f : FiniteHomeomorph (L.str.subspace (L.str.component t)) c, (L.str.subspace (L.str.component t)).Continuous c f.toEquiv →`	Apply the closure parameter to $c_{C_A(I)}$ and the test class to the preceding realizable spaces. For every such X, every chosen point selects a component $D \subseteq X$ with induced closure; the map $f : D \to C_A(I)$ is a finite homeomorphism and is fully continuous in the forward direction.
Second map and the two conclusions `∀ g : FiniteHomeomorph (L.str.subspace (L.str.component t)) (L.str.subspace (L.str.component t)), (L.str.subspace (L.str.component t)).Continuous (L.str.subspace (L.str.component t)) g.toEquiv → c.Continuous c (f.conjugate g).toEquiv ∧ c.Continuous c (f.conjugate g).toEquiv.symm`	For every finite homeomorphism $g : D \to D$ with forward full continuity, both the conjugate and its actual inverse are fully continuous on $C_A(I)$: $$h = f \circ g \circ f^{-1},$$ $$h^{-1} = f \circ g^{-1} \circ f^{-1}.$$ These are the same maps in (4); no full continuity of f^{-1} or g^{-1} is an input.

In the source	Mathematical meaning

Finite homeomorphism
`structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where`
`  finitelyContinuous_toFun : c.FinitelyContinuous d toFun`
`  finitelyContinuous_invFun : d.FinitelyContinuous c invFun`
	One bijection with finite continuity for it and its inverse. For f and g this gives the finite-set equality (2) in their respective induced closure spaces.

Finite continuity
`def FinitelyContinuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=`
`  ∀ s : Set α, s.Finite → f '' c.closure s ⊆ d.closure (f '' s)`
	Closure preservation by inclusion for every finite subset, including the empty subset. Applying this in both directions to the same bijection gives (2).

Full continuity
`def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=`
`  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`
	The inclusion (3) for every subset, without a finiteness restriction. In the conclusion the source and target closure are both $c_{C_A(I)}$.

The same conjugation
`def FiniteHomeomorph.conjugate (f : FiniteHomeomorph c d)`
`  (g : FiniteHomeomorph c c) : FiniteHomeomorph d d :=`
`  f.symm.trans (g.trans f)`
	The equivalence obtained by composing the same f^{-1}, then g, then f. Its forward map is $h = f \circ g \circ f^{-1}$ and its inverse is $h^{-1} = f \circ g^{-1} \circ f^{-1}$, not an independently chosen inverse.

Size convention: in `.{u, max u w}`, the theorem's algebra A and each realizing algebra B have level u; the underlying set of X has level $\max(u, w)$. The conjugation test ranges over all the stated spaces at these fixed levels, not over all universes at once.

Exact surrounding source context (separate excerpts)

Exact ambient source:

```
universe u v w uA uL
variable {A : Type u} [NonUnitalCStarAlgebra A]
variable [PartialOrder A] [StarOrderedRing A]
```

Exact ambient source installs the C*-algebra structure and closedness of representation images locally; this is separate from the selected declaration:

```
attribute [local instance] HomKernelQuotient.imageCStarAlgebra HomKernelQuotient.isClosed_image
```

Related exact source for the reading guide

Compact-operator model · Fixed source

```
def IsCompactOperatorModel [NonUnitalCStarAlgebra A]
    (e : A →⋆ₙₐ[ℂ] (H →L[ℂ] H)) : Prop :=
  Function.Injective e ∧
    (∀ a : A, IsCompactOperator (e a)) ∧
    ∀ T : H →L[ℂ] H, IsCompactOperator T → ∃ a : A, e a = T
```

Class of test spaces · Fixed source

```
def IsCStarRealizable.{uA, uX} (X : ClosureCat.{uX}) : Prop :=
  ∃ (A : Type uA) (instA : NonUnitalCStarAlgebra A),
    letI : NonUnitalCStarAlgebra A := instA
    Nontrivial A ∧ Nonempty
      (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A) X.str)
```

Point and applied component condition · Fixed source

```
      ∃ I : NormedSpace.GeometricStructureSpace ℂ A,
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component
I)).HasContinuousFiniteConjugation
            (ClosureCat.IsCStarRealizable.{u, max u w}) := by
```

Geometric points · Fixed source

```
def GeometricStructureSpace : Type u := {I : Set E // I ∈ geometricStructureFamily 𝕜 E}
```

Ambient geometric closure · Fixed source

```
def geometricStructureClosure : ClosureSpace (GeometricStructureSpace 𝕜 E) :=
  ClosureSpace.HullKernel.space (geometricStructureFamily 𝕜 E)
    (univ_notMem_geometricStructureFamily 𝕜 E)
```

Finite-chain component · Fixed source

```
def component (c : ClosureSpace α) (x : α) : Set α := {y | c.Related x y}
```

All tests, components and first map · Fixed source

```
def HasContinuousFiniteConjugation (c : ClosureSpace α) (D : ClosureCat.{s} → Prop) : Prop :=
  ∀ (L : ClosureCat.{s}), D L → ∀ t : L,
    ∀ f : FiniteHomeomorph (L.str.subspace (L.str.component t)) c,
      (L.str.subspace (L.str.component t)).Continuous c f.toEquiv →
      ∀ g : FiniteHomeomorph (L.str.subspace (L.str.component t))
          (L.str.subspace (L.str.component t)),
        (L.str.subspace (L.str.component t)).Continuous
          (L.str.subspace (L.str.component t)) g.toEquiv →
        c.Continuous c (f.conjugate g).toEquiv ∧
          c.Continuous c (f.conjugate g).toEquiv.symm
```

The same conjugation · Fixed source

```
def FiniteHomeomorph.conjugate (f : FiniteHomeomorph c d)
    (g : FiniteHomeomorph c c) : FiniteHomeomorph d d :=
  f.symm.trans (g.trans f)
```

Induced component closure · Fixed source

```
def subspace (c : ClosureSpace α) (A : Set α) : ClosureSpace A :=
  c.induced (Subtype.val : A → α)
```

Full continuity · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

Finite continuity · Fixed source

```
def FinitelyContinuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, s.Finite → f '' c.closure s ⊆ d.closure (f '' s)
```

Finite homeomorphism · Fixed source

```
structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  finitelyContinuous_toFun : c.FinitelyContinuous d toFun
  finitelyContinuous_invFun : d.FinitelyContinuous c invFun
```