Reconstructing a represented unitary from its shells and residual corner

`MathlibAnnex.CStarAlgebra.CAR.exists_targetShellReconstruction`

theorem

Separates a represented generator into a strong shell sum and the exact operator between its limiting fixed spaces.

Statement

In the setting below, fix $i \in I$. Put $P_n = \sigma(f_{i,n})$, $Q_n = \sigma(f_n)$ and $W_n = \sigma(w_{i,n})$, and let $F = \bigcap_n \operatorname{ran} P_n$ and $G = \bigcap_n \operatorname{ran} Q_n$. There are contractions S, S^* such that, for every $x \in K$,

$$\sum_{n<N} W_n x \longrightarrow Sx, \qquad \sum_{n<N} W_n^* x \longrightarrow S^* x.$$

If P, Q are the orthogonal projections onto F, G, then $S^*S = I - P$ and $SS^* = I - Q$. The remaining operator $R = V_i P$ satisfies

$$V_i = S + R, \qquad R^*R = P, \qquad RR^* = Q, \qquad R = QRP.$$

Moreover, $x \in F$ if and only if $V_i x \in G$, for every $x \in K$.

Assumptions

Let A be the completed CAR algebra, obtained by completing the matrix stages $M_{2^n}(\mathbb{C})$ with embeddings $a \mapsto a \otimes I_2$. Let f_n be the image of the first diagonal matrix unit. Thus $f_0 = 1$ and the f_n form a decreasing sequence of projections. The root state ω has value a_{00} on a matrix a in any stage.

Choose a pure state φ_j in each unitary-equivalence class $j \in I$ of pure-state GNS representations, with $\varphi_o = \omega$ at the root index o. Denote its complete complex GNS space, unital star representation and unit cyclic vector by (H_j, π_j, ξ_j). Set $\mathcal{H} = \bigoplus_{j\in I} H_j$ and $\pi = \bigoplus_{j\in I} \pi_j$. The index set need not be countable.

A supplied shell family consists of unital complex star automorphisms α_j with $\varphi_j \circ \alpha_j = \omega$ and elements $w_{j,n} \in A$. Put $f_{j,n} = \alpha_j(f_n)$. Their support identities are $w_{j,n}^* w_{j,n} = f_{j,n} - f_{j,n+1}$ and $w_{j,n} w_{j,n}^* = f_n - f_{n+1}$. At the root, α_o is the identity and $w_{o,n} = f_n - f_{n+1}$.

Let L_j be unitaries on $\mathcal{H}$ satisfying $L_j \pi(f_{j,n} - f_{j,n+1}) = \pi(w_{j,n})$ for every j, n. Let $T \subseteq B(\mathcal{H})$ be the norm-closed unital star algebra generated by $\pi(A)$ and all L_j. Write $\iota : A \to T$ for $\iota(a) = \pi(a)$ and $g_j \in T$ for the element represented by L_j. For a unital complex star representation $\rho : T \to B(K)$ on a complete complex Hilbert space, put $\sigma = \rho \circ \iota$ and $V_j = \rho(g_j)$. Each V_j is unitary because ρ preserves the unit, products and adjoints. The index i is arbitrary and fixed for this assertion. No irreducibility of ρ, root normalization $L_o = I$, or prescribed action on the selected cyclic vectors is assumed.

Conclusion

The shell sum is a partial isometry from $F^\perp$ onto $G^\perp$, and the residual corner is a partial isometry from F onto G. Together they recover the actual represented generator. The bounds are $\|S\| \le 1$ and $\|S^*\| \le 1$; all displayed limits are norm limits for each fixed vector.

Proof route

The finite shell identities pass through the star homomorphism and give $V_i(P_n - P_{n+1}) = W_n$. Orthogonality of the initial and final difference projections yields the strong sums of the W_n and their adjoints. Finite telescoping identifies the sum with the complementary part of V_i. Strong convergence of decreasing projections then identifies the remaining corner and its two support projections.

Proof steps

1. Applying ρ to the finite identity $g_i\iota(f_{i,n} - f_{i,n+1}) = \iota(w_{i,n})$ gives the represented shell relation. Also $W_n^*W_n = P_n - P_{n+1}$ and $W_nW_n^* = Q_n - Q_{n+1}$. The two projection sequences start at I and decrease.

2. The difference projections for different n are orthogonal. The orthogonal-shell theorem therefore makes both partial-sum sequences converge strongly to contractions. Taking inner products of the two finite sums identifies their limits as adjoints; the theorem also gives $S^*S = I - P$ and $SS^* = I - Q$.

3. For $s_N = \sum_{n<N} W_n$, telescoping gives $s_N = V_i(I - P_N)$. Its final support identity is $s_N s_N^* = I - Q_N$, hence $V_iP_NV_i^* = Q_N$. Since $P_N \to P$ and $Q_N \to Q$ strongly, these finite identities yield $S = V_i(I - P)$ and $V_iPV_i^* = Q$. Only multiplication by fixed bounded operators is used in these limit passages.

4. Set $R = V_iP$. Unitarity and the projection identities give $V_i = S + R$, $R^*R = P$, $RR^* = Q$ and $R = QRP$. The equality $QV_i = V_iP$ gives the fixed-space equivalence, in both directions, by injectivity of V_i.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.AtomicTarget · Exact source
- MathlibAnnex.CStarAlgebra.CAR.restrictedRepresentation · Exact source
- MathlibAnnex.CStarAlgebra.CAR.representedGeneratorUnitary · Exact source
- MathlibAnnex.CStarAlgebra.CAR.representedGenerator_comp_sourceShell · Exact source
- MathlibAnnex.Analysis.CStarAlgebra.exists_represented_unitaryCompletion_of_sourceShells · Exact source
- ContinuousLinearMap.unitaryCompletion_of_finite_relations · Exact source
- The root state bundled with its purity proof · Exact source

Lean source signature (exact)

```
theorem exists_targetShellReconstruction
    (family : RepresentativeShellFamily)
    (L : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit →
      MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState →L[ℂ]
        MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState)
    (hLunit : ∀ i, L i ∈ unitary
      (MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState →L[ℂ]
        MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState))
    (hLsource : ∀ i n, (L i).comp (selectedAtomicRepresentation
      (transportedFlag family i n - transportedFlag family i (n + 1))) =
        representedShellLink family i n)
    {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K]
    [CompleteSpace K] (rho : Representation (AtomicTarget L) K)
    (i : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit) :
    let sigma := restrictedRepresentation L rho
    let p := transportedFlag family i
    let q := rootFlag
    let w := (representativeShellData family i).link
    let U : ℕ → Submodule ℂ K := fun n ↦ (sigma (p n)).range
    let V : ℕ → Submodule ℂ K := fun n ↦ (sigma (q n)).range
    ∃ S T P Q R : K →L[ℂ] K,
      ContinuousLinearMap.StronglyConverges
        (ContinuousLinearMap.partialSum (fun n ↦ sigma (w n))) atTop S ∧
      ContinuousLinearMap.StronglyConverges
        (ContinuousLinearMap.partialSum (fun n ↦ (sigma (w n))†)) atTop T ∧
      ‖S‖ ≤ 1 ∧ ‖T‖ ≤ 1 ∧ T = S† ∧
      IsStarProjection P ∧ P.range = ⨅ n, U n ∧
      IsStarProjection Q ∧ Q.range = ⨅ n, V n ∧
      (S†).comp S = 1 - P ∧ S.comp (S†) = 1 - Q ∧
      (Unitary.linearIsometryEquiv
        (representedGeneratorUnitary L hLunit rho i) : K ≃ₗᵢ[ℂ] K) =
          S + R ∧
      R = ((Unitary.linearIsometryEquiv
        (representedGeneratorUnitary L hLunit rho i) : K ≃ₗᵢ[ℂ] K) :
          K →L[ℂ] K).comp P ∧
      (R†).comp R = P ∧ R.comp (R†) = Q ∧
      R = (Q.comp R).comp P ∧
      (∀ x, x ∈ ⨅ n, U n ↔
        (Unitary.linearIsometryEquiv
          (representedGeneratorUnitary L hLunit rho i) : K ≃ₗᵢ[ℂ] K) x ∈
            ⨅ n, V n)
```

> Here `Limit` is A, `completedRootPureState` is the root state ω bundled with its purity proof, and `SelectedAtomicHilbert completedRootPureState` is the atomic Hilbert space $\mathcal{H}$. `AtomicTarget L` is T, `restrictedRepresentation L rho` is σ, and `representedGeneratorUnitary L hLunit rho i` is V_i. In the source, `p`, `q`, `w` are the algebra-valued sequences $f_{i,n}$, f_n, $w_{i,n}$; `U n` and `V n` are their represented ranges. The source witness `T` is the adjoint limit S^*, not the algebra T. `P`, `Q`, `R` are the projection and corner operators above; `comp` denotes composition and `†` the Hilbert-space adjoint.

Lean realization notes

The strong sums are formed on K. No continuity of ρ for the strong-operator topology is used, and operator-norm convergence is not asserted. The spaces F, G are not claimed to be one-dimensional in an arbitrary representation. This theorem reconstructs a generator for a supplied family; it does not prove existence of that family.

▼ **Exact Card identity**

Language: en

CARD_CONTENT_COMPLETE

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Exposition revision: 1 · SHA-256: 64fb2500bbff7edb7b7f6e92966fb3d3ded511e67e335fff122b9ef2c3c7e12f

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