Exact norm density of the fixed atomic algebra

`MathlibAnnex.CStarAlgebra.CAR.hasDensityCharacter_atomicCounterexampleAlgebra`

theorem

Combines a cardinality bound from a faithful representation on a separable Hilbert space with a lower bound for every norm-dense subset.

Statement

Let A be the completed CAR algebra and $T = C^*(\pi(A) \cup \{L_i : i \in I\}) \subseteq B(\mathcal{H})$ the algebra generated by the selected direct sum π of pure-state GNS representations indexed by their unitary-equivalence classes $i \in I$ and the unitaries L_i from the fixed homogeneity and shell construction. Write $\iota(a) = \pi(a)$ for the embedding $A \to T$. Write $\mathfrak{c} = 2^{\aleph_0}$. The norm-density character of T is exactly $\mathfrak{c}$: there is a norm-dense subset $D_0 \subseteq T$ with $|D_0| = \mathfrak{c}$, and every norm-dense subset $D \subseteq T$ satisfies

$$\mathfrak{c} \leq |D|.$$

The density character is the least cardinality of a norm-dense subset.

Assumptions

The fixed algebra T is infinite dimensional, has exactly one unitary-equivalence class of nonzero irreducible representations, and has a faithful representation $\widetilde{\rho}$ on the separable space $H_\tau = L^2(A, \tau)$. These are established properties of this T, not additional hypotheses on an arbitrary algebra.

Conclusion

Thus $\mathrm{dens}_{\|\cdot\|}(T) = \mathfrak{c}$. Separately, the underlying set also has $|T| = \mathfrak{c}$. In particular T is not norm separable, although it is faithfully represented on H_τ.

Proof route

The faithful representation on H_τ gives an upper bound for $|T|$. The cited density theorem for an infinite-dimensional unital C*-algebra with a unique irreducible representation class gives a lower bound for every dense subset. Applying it also to T itself proves that this bound is attained.

Proof steps

1. Injectivity of $\widetilde{\rho} : T \to B(H_\tau)$ and separability of H_τ give

   $$|T| \leq |B(H_\tau)| \leq \mathfrak{c}.$$

   This is the cited cardinality bound for an algebra faithfully represented on a separable Hilbert space.

2. Let $\rho_0 : T \to B(\mathcal{H})$ be the inclusion representation and let $D \subseteq T$ be norm dense. The cited lower-bound theorem applies to the irreducible inclusion ρ_0, the fact that every nonzero irreducible unital representation is equivalent to it, and infinite dimensionality of T. It gives $\mathfrak{c} \leq |D|$. Its argument is by contradiction: $|D| < \mathfrak{c}$ would give a dense set of size less than $\mathfrak{c}$ in an irreducible representation space, and the finite-dimensionality theorem at that density would force T to be finite dimensional.

3. Taking $D = T$ gives $\mathfrak{c} \leq |T| \leq \mathfrak{c}$, hence $|T| = \mathfrak{c}$. The choice $D_0 = T$ then attains the lower bound, proving the exact density statement.

Main citations

- The stated existence or structural result · Exact source
- Cardinality bound from a faithful representation on a separable space · Exact source
- The lower bound for every dense subset of this algebra · Exact source
- The separate carrier-cardinality equality · Exact source
- The density bound for a single irreducible representation class · Exact source
- Faithful separable representations bound the carrier · Exact source

Lean source signature (exact)

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theorem hasDensityCharacter_atomicCounterexampleAlgebra :
    HasDensityCharacter AtomicCounterexampleAlgebra Cardinal.continuum
```

> `HasDensityCharacter AtomicCounterexampleAlgebra Cardinal.continuum` means both existence of a dense subset of size $\mathfrak{c}$ and minimality among all dense subsets. In the linked proof, the first supplier bounds `#AtomicCounterexampleAlgebra`; the second bounds `#s` for each dense set `s`. The `#` notation is cardinality, not dimension.

Lean realization notes

The dense-set lower bound is a statement about every norm-dense subset, not merely about the whole carrier. Neither cardinality is a Hamel dimension or a Hilbert-space dimension.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
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Card revision: 2 · SHA-256: bbaa38f0690d7b0ca134e7875ae37f1e48d6f2cd8dc0d5b1d6b4c3701fa214c4
Exposition revision: 3 · SHA-256: 971e28fa2f885c071331a719260720f3987fc339922a7783cd9b38fbd00a9a40
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