The atomic Hilbert space has norm density continuum

`MathlibAnnex.CStarAlgebra.CAR.hasDensityCharacter_atomicCounterexampleHilbert`

Determines the size of a smallest norm-dense set in the irreducible Hilbert model.

Statement

The Hilbert space H_{at} of the atomic inclusion of the fixed CAR-based algebra A has norm density character exactly $\mathfrak{c} = 2^{\aleph_0}$.

Assumptions

Let $A \subseteq B(H_{at})$ be the fixed unital C*-algebra obtained by adjoining the chosen shell-link unitaries to the atomic representation of the CAR algebra C, and then taking the norm-closed unital *-algebra they generate. Let π_{at} be its distinguished atomic inclusion on H_{at}. This theorem concerns H_{at}, not the separable trace-GNS space H_τ.

Conclusion

There is a norm-dense subset $D \subseteq H_{at}$ of cardinality $\mathfrak{c}$, and every norm-dense $E \subseteq H_{at}$ satisfies $\mathfrak{c} \leq |E|$. Thus $\mathrm{dens}(H_{at}) = \mathfrak{c}$.

Proof route

The representation-space density obstruction supplies the lower bound. For the upper bound, take the cyclic orbit of a nonzero vector under A and use $|A| = \mathfrak{c}$.

Proof steps

1. Choose a nonzero vector in the nonzero irreducible atomic representation.
2. Its A-orbit is dense and has cardinality at most $|A| = \mathfrak{c}$.
3. The lower bound for every dense subset makes that witness exact and proves minimality.

Lean realization notes

This is a norm-density statement, not a carrier-cardinality or Hamel-rank computation. The usual infinite-dimensional Hilbert-space theorem then identifies the cardinality of an orthonormal basis with $\mathfrak{c}$; that consequence is not a dedicated Lean endpoint of this declaration. No Project level is fabricated for this scope-external declaration.

Language: en · CARD_CONTENT_COMPLETE

Main citations

MathlibAnnex.CStarAlgebra.CAR.continuum_le_cardinalMk_dense_atomicCounterexampleHilbert

MathlibAnnex.Analysis.CStarAlgebra.Representation.denseRange_orbitMap_of_isIrreducible

Lean source declaration (exact)

```
theorem hasDensityCharacter_atomicCounterexampleHilbert :
HasDensityCharacter
(MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState)
Cardinal.continuum := by
let H := MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState
let pi : Representation AtomicCounterexampleAlgebra H :=
shellFamilyInclusion homogeneityShellFamily
have hirr : Representation.IsIrreducible pi :=
isIrreducible_shellFamilyInclusion homogeneityShellFamily
letI : Nontrivial H := Representation.nontrivial_of_isNonzero pi hirr.1
obtain ⟨x, hx⟩ : ∃ x : H, x ≠ 0 := exists_ne 0
let s : Set H := Set.range (fun a : AtomicCounterexampleAlgebra => pi a x)
have hs : Dense s := Representation.denseRange_orbitMap_of_isIrreducible pi hirr hx
have hupper' : Cardinal.lift.{0} (#s) ≤
Cardinal.lift.{0} (#AtomicCounterexampleAlgebra) := Cardinal.mk_range_le_lift
have hupper : #s ≤ Cardinal.continuum := by
simpa [cardinalMk_atomicCounterexampleAlgebra] using hupper'
refine ⟨⟨s, hs, le_antisymm hupper
(continuum_le_cardinalMk_dense_atomicCounterexampleHilbert s hs)⟩,
continuum_le_cardinalMk_dense_atomicCounterexampleHilbert⟩
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Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: e3615f5ad0eee009f7d76eac1eef94455a6c979000af73eece1cc7b944d3c3d7

Card SHA-256: 79f1cbd6ad329c3e3023830d3ad3cca58491bc0bea89f63173261c36d58a54e7

Approved exposition SHA-256: 5dc44e2fea087f3d587ec430ebd43d7ec6096adafcec24e2ae2aeb3578eb30b1

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.