A continuous finite comparison with a compact-operator GSS

`ClosureSpace.HasContinuousFiniteCompactComparison`

def

Requires a finite homeomorphism to a compact-operator geometric space whose forward map is fully continuous.

Statement

Let (X, c) be a closure space. A continuous finite compact comparison is the existence of a nonzero complex Hilbert space H and a finite homeomorphism from (X, c) to the geometric structure closure space $S(\mathcal{K}(H))$ whose forward map is also fully continuous.

Definition

Here $\mathcal{K}(H)$ denotes the algebra of compact operators on H. For a complex normed space D, $S(D)$ consists of the unions of kernels of norm-one bounded complex-linear functionals supporting each maximal proper face of its closed unit ball. Faces use real convexity. The support-functional norm theorem justifies the norm-one description for these nonempty faces. Its closure is $c_D(\mathcal{U}) = \{I \in S(D) : \bigcap_{J \in \mathcal{U}} J \subseteq I\}$.

For closure spaces $(X, c), (Y, d)$, full continuity of $e : X \to Y$ means $e(c(E)) \subseteq d(e(E))$ for every $E \subseteq X$. Finite continuity requires this only for finite E. The condition holds when there exists a bijection $e : X \to S(\mathcal{K}(H))$ such that e and e^{-1} are finitely continuous, and additionally

$$e(c(E)) \subseteq c_{\mathcal{K}(H)}(e(E)) \quad (E \subseteq X).$$

The norm and inner product belong to the same nonzero Hilbert space H. Full continuity of e^{-1} is not required by this condition.

Assumptions

The input is an arbitrary closure space. It need not already be presented as a GSS or an operator-algebra normal part. There is no representation of a C*-algebra in the input or conclusion. The model Hilbert space is nonzero; no separability or finite-dimensionality is required.

Conclusion

This is an existence property of a closure space, with forward full continuity added to finite continuity of a bijection and its inverse. No representation range condition is part of this definition.

Main citations

- Exact defining expression.
- The full-homeomorphism condition, which requires full continuity in both directions.

Lean source signature (exact)

Exact source and declaration. The complete defining right-hand side is included.

```
def HasContinuousFiniteCompactComparison.{uH, uX} {X : Type uX}
    (c : ClosureSpace X) : Prop :=
  ∃ (H : Type uH) (instN : NormedAddCommGroup H),
    letI := instN
    ∃ instI : InnerProductSpace ℂ H,
      letI := instI
      ∃ instC : CompleteSpace H,
        letI := instC
        ∃ instT : Nontrivial H,
          letI := instT
          ∃ e : FiniteHomeomorph c
              (NormedSpace.geometricStructureClosure ℂ
                (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)),
            c.Continuous (NormedSpace.geometricStructureClosure ℂ
              (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)) e.toEquiv
```

In the source	Mathematical meaning

Input closure
`def HasContinuousFiniteCompactComparison.{uH, uX} {X : Type uX}`
`(c : ClosureSpace X) : Prop :=`
	The input c is the closure on X. The following existential statement defines the requested model condition; it does not select a preferred model or map.

One normed model space
`∃ (H : Type uH) (instN : NormedAddCommGroup H),`
`letI := instN`
	There must exist one normed additive group H. The first `letI` installs that chosen normed structure on H for the inner-product type that follows.

Complex inner product on that model
`∃ instI : InnerProductSpace ℂ H,`
`letI := instI`
	The same H must have a compatible complex inner product. This `letI` installs it for the later Hilbert-space and operator-algebra types.

Completeness of that model
`∃ instC : CompleteSpace H,`
`letI := instC`
	The same normed space must be complete. The next `letI` installs this completeness, used in the compact-operator algebra on this Hilbert space.

A nonzero model
`∃ instT : Nontrivial H,`
`letI := instT`
	The same H must be nonzero. Its `letI` makes this nontriviality available in the remaining conditions; it does not introduce another space.

In the source	Mathematical meaning
Finite comparison with the compact-operator GSS `∃ e : FiniteHomeomorph c (NormedSpace.geometricStructureClosure ℂ (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)),`	There must exist one finite homeomorphism $$e : X \to S(\mathcal{K}(H)).$$ The target closure is $c_{\mathcal{K}(H)}$, the geometric closure of the compact-operator algebra on this same H, not a closure on H. For every finite $F \subseteq X$, $$e(c(F)) = c_{\mathcal{K}(H)}(e(F)).$$
Full forward continuity of the same map `c.Continuous (NormedSpace.geometricStructureClosure ℂ (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)) e.toEquiv`	`e.toEquiv` is the bijection underlying that same finite homeomorphism. The remaining condition is, for every $E \subseteq X$, $$e(c(E)) \subseteq c_{\mathcal{K}(H)}(e(E)).$$ It imposes no full-continuity condition on e^{-1}.
Finite continuity in both directions `structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where finitelyContinuous_toFun : c.FinitelyContinuous d toFun finitelyContinuous_invFun : d.FinitelyContinuous c invFun`	The inherited equivalence supplies e and e^{-1}. The two fields require finite continuity of these maps in opposite directions. Together they give the finite closure equality above; full continuity of the forward map is the extra condition in the preceding row.
Continuity for all subsets `def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop := ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`	For $r : (Y, d) \to (Z, k)$, the definition requires $$r(d(E)) \subseteq k(r(E)) \quad (E \subseteq Y).$$ There is no finiteness restriction in this condition.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
namespace ClosureSpace

open MathlibAnnex.Analysis.CStarAlgebra
attribute [local instance] ContinuousLinearMap.isClosed_compactOperatorStarSubalgebra
```

The model-space level `uH` and the input-space level `uX` are independent fixed parameters. All model structures and conditions above refer to the same quantified H.

Related exact source for the reading guide

Continuity for all subsets · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

Finite continuity in both directions · Fixed source

```
structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  finitelyContinuous_toFun : c.FinitelyContinuous d toFun
  finitelyContinuous_invFun : d.FinitelyContinuous c invFun
```