Normalization and the trace identity determine the CAR trace

`MathlibAnnex.CStarAlgebra.CAR.eq_trace_of_apply_one_of_mul_comm`

theorem

Proves uniqueness without assuming positivity of the competing continuous functional.

Statement

Let A be the completion of $M_{2^n}(\mathbb{C})$ under $a \mapsto a \otimes I_2$, with normalized trace τ. If a continuous complex-linear functional $f : A \to \mathbb{C}$ satisfies $f(1) = 1$ and $f(ab) = f(ba)$ for every $a, b \in A$, then $f = \tau$.

Assumptions

Continuity, complex linearity, normalization and the trace identity are the hypotheses on f. Positivity, faithfulness and an a priori norm bound of one are not additional hypotheses. Let $\iota_n : M_{2^n}(\mathbb{C}) \to A$ be the canonical stage embedding. Write $e_{ij} = \iota_n(E_{ij})$ for the matrix units of a stage of size $d = 2^n$.

Conclusion

There is at most one normalized continuous trace on A, and the continuous functional τ defined in the CAR trace construction supplies it. In particular any tracial state equals τ, but the assertion applies to the larger class of functionals specified above.

Proof route

The trace identity and matrix-unit multiplication force all diagonal values $f(e_{ii})$ to agree and all off-diagonal values to vanish. Since $\sum_i e_{ii} = 1$, each diagonal value equals $1/d$. Hence f equals the normalized matrix trace on every stage. The union of stage images is dense in A, so continuity gives equality everywhere.

Proof steps

1. Apply the trace identity to $e_{i0}e_{0i} = e_{ii}$ and $e_{0i}e_{i0} = e_{00}$ to obtain equal diagonal values.
2. For $i \neq j$, the products $e_{ii}e_{ij} = e_{ij}$ and $e_{ij}e_{ii} = 0$ give $f(e_{ij}) = 0$. Normalization now gives $df(e_{00}) = 1$.
3. Expand any stage matrix in matrix units. Linear equality on all stages extends to the completion because f and τ are continuous and the stage union is dense.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.trace · Exact source
- MathlibAnnex.CStarAlgebra.CAR.limitMatrixUnit_mul · Exact source
- MathlibAnnex.CStarAlgebra.CAR.sum_limitMatrixUnit_diag · Exact source
- MathlibAnnex.CStarAlgebra.CAR.ofStage_eq_sum_smul_limitMatrixUnit · Exact source

- MathlibAnnex.CStarAlgebra.CAR.apply_limitMatrixUnit_eq_trace_of_apply_one_of_mul_comm · Exact source
- MathlibAnnex.CStarAlgebra.CAR.apply_ofStage_eq_trace_of_apply_one_of_mul_comm · Exact source
- MathlibAnnex.CStarAlgebra.CAR.dense_stageRange · Exact source
- MathlibAnnex.CStarAlgebra.CAR.trace_one · Exact source
- MathlibAnnex.CStarAlgebra.CAR.trace_mul_comm · Exact source

Lean source signature (exact)

```
theorem eq_trace_of_apply_one_of_mul_comm
    (f : Limit →L[ℂ] ℂ) (hf1 : f 1 = 1)
    (hf : ∀ a b, f (a * b) = f (b * a)) : f = trace
```

> Here `Limit` is A and `trace` is τ. The type of `f` already requires a continuous complex-linear functional. The hypotheses `hf1` and `hf` are respectively $f(1) = 1$ and $f(ab) = f(ba)$. Thus the source conclusion `f = trace` is the uniqueness assertion in the Statement, without a positivity hypothesis.

Lean realization notes

The argument depends on continuity at the last density step. It makes no uniqueness assertion for arbitrary discontinuous algebraic functionals, or for states without the trace identity.

▼ **Exact Card identity**

Language: en
CARD_CONTENT_COMPLETE
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