The closure-space homeomorphism induced by an orthogonality equivalence

`NormedSpace.geometricStructureHomeomorphOfBirkhoffJames`

def

Statement

Let E and F be normed spaces over real-or-complex scalar fields $\mathbb{K}$ and $\mathbb{L}$, and let $e : E \to F$ be a bijection preserving and reflecting their Birkhoff–James orthogonality relations. Define the induced closure-space homeomorphism

$$\Phi_e : S_{\mathbb{K}}(E) \longrightarrow S_{\mathbb{L}}(F), \qquad \Phi_e(I) = e(I),$$

with inverse $J \mapsto e^{-1}(J)$. Here $e(I) = \{e(x) : x \in I\}$ is a direct image and $e^{-1}(J) = \{x : e(x) \in J\}$ is a preimage. Well-definedness and closure preservation are supplied by the separately cited transport theorems.

Definition

Let E and F be normed spaces over real-or-complex scalar fields $\mathbb{K}$ and $\mathbb{L}$, respectively; the fields may differ. Define

$$x \perp_{\mathbb{K}} y \quad \Longleftrightarrow \quad \|x\| \le \|x + ay\| \text{ for every } a \in \mathbb{K}.$$

The same formula with $\mathbb{L}$ defines orthogonality on F. Let $e : E \to F$ be a bijection such that, for every $x, y \in E$,

$$e(x) \perp_{\mathbb{L}} e(y) \quad \Longleftrightarrow \quad x \perp_{\mathbb{K}} y.$$

Here is the geometric point set on which the induced map acts. Write E^* for the continuous $\mathbb{K}$-linear dual and $B_E = \{x \in E : \|x\| \le 1\}$. For a nonempty maximal proper real convex face M of B_E, put

$$\Sigma_{\mathbb{K}}(M) = \{f \in E^* : \|f\| \le 1,\ f|_M = 1\},$$
$$I_{\mathbb{K}}(M) = \bigcup_{f \in \Sigma_{\mathbb{K}}(M)} \ker f.$$

Here $f|_M = 1$ means $f(x) = 1$ for every $x \in M$.

A face is a convex subset containing both endpoints of each segment in the ball whose interior meets it; maximality is among proper convex faces. The set $S_{\mathbb{K}}(E)$ consists of the resulting subsets $I_{\mathbb{K}}(M)$, with equal subsets regarded as the same point. Its closure is

$$c_E(\mathscr{A}) = \left\{ I \in S_{\mathbb{K}}(E) : \bigcap_{J \in \mathscr{A}} J \subseteq I \right\}, \qquad \mathscr{A} \subseteq S_{\mathbb{K}}(E).$$

The empty intersection is E. Define $S_{\mathbb{L}}(F)$ and c_F by the same formulas for F. Define the point map by

$$\Phi_e : S_{\mathbb{K}}(E) \longrightarrow S_{\mathbb{L}}(F), \qquad \Phi_e(I) = e(I).$$

Its inverse sends $J \subseteq F$ to $e^{-1}(J)$, the preimage under e.

The geometric-point transport theorem makes this point map well-defined; the same theorem for the inverse bijection supplies the inverse. The incidence identity and hull–kernel closure-transport theorem supply its homeomorphism structure.

Assumptions

The spaces are normed, with no completeness or nontriviality assumption. Their real-or-complex scalar fields may differ. The given bijection preserves and reflects the displayed ordered orthogonality relation for every pair of vectors; additivity, homogeneity and norm continuity are not assumed.

Conclusion

The output is the specified point bijection Φ_e, together with closure continuity of it and its inverse. Its underlying map and inverse are exactly those displayed above. A closure-space homeomorphism is a bijection continuous in both directions for the given closure operators; the operators need not be topological closures. The cited transport theorem supplies the identity

$$\Phi_e\big(c_E(\mathscr{A})\big) = c_F\big(\Phi_e(\mathscr{A})\big) \qquad (\mathscr{A} \subseteq S_{\mathbb{K}}(E)).$$

Main citations

- Geometric points remain geometric under the given bijection: `NormedSpace.image_mem_geometricStructureFamily`.
- The point equivalence and its incidence identity: `NormedSpace.geometricStructureEquivOfBirkhoffJames` and `NormedSpace.mem_geometricStructureEquivOfBirkhoffJames`.
- Incidence-preserving equivalences commute with hull–kernel closure: `ClosureSpace.HullKernel.image_closure`. The following `homeomorphOfMemIff` definition packages this theorem.
- The geometric point family and closure; maximal proper real convex faces.
- Closure continuity and closure-space homeomorphisms and the homeomorphism structure.

Lean source signature (exact)

The construction uses the scalar, space and bijection hypotheses reproduced separately below.

```
def geometricStructureHomeomorphOfBirkhoffJames :
    ClosureSpace.Homeomorph (geometricStructureClosure 𝕜 E) (geometricStructureClosure 𝕝 F) :=
  ClosureSpace.HullKernel.homeomorphOfMemIff
    (univ_notMem_geometricStructureFamily 𝕜 E) (univ_notMem_geometricStructureFamily 𝕝 F)
    e (geometricStructureEquivOfBirkhoffJames 𝕜 𝕝 e h)
    (mem_geometricStructureEquivOfBirkhoffJames 𝕜 𝕝 e h)
```

In the source	Mathematical meaning

Scalar fields and normed spaces
`variable (𝕜 : Type v) (𝕝 : Type z) [RCLike 𝕜] [RCLike 𝕝]`
`variable {E : Type u} {F : Type w}`
`variable [NormedAddCommGroup E] [NormedSpace 𝕜 E]`
`variable [NormedAddCommGroup F] [NormedSpace 𝕝 F]`
	The fields $\mathbb{K}, \mathbb{L}$ may each be real or complex. The spaces E, F carry their respective norms and scalar actions; neither completeness nor nontriviality is required.

Bijection and orthogonality
`variable (e : E ≃ F)`
`variable (h : ∀ x y : E, BirkhoffJamesOrthogonal 𝕝 (e x) (e y) ↔`
`BirkhoffJamesOrthogonal 𝕜 x y)`
`include h`
	The given set bijection $e : E \to F$ preserves and reflects Birkhoff–James orthogonality for every ordered pair. The included hypothesis is the same h used in this construction; linearity of e is not assumed.

Two closure spaces
`def geometricStructureHomeomorphOfBirkhoffJames :`
`ClosureSpace.Homeomorph`
`(geometricStructureClosure 𝕜 E)`
`(geometricStructureClosure 𝕝 F) :=`
	The result is a homeomorphism from $(S_{\mathbb{K}}(E), c_E)$ to $(S_{\mathbb{L}}(F), c_F)$. These two complete closure arguments specify its domain and codomain.

Point map and incidence input
`ClosureSpace.HullKernel.homeomorphOfMemIff`
`(univ_notMem_geometricStructureFamily 𝕜 E)`
`(univ_notMem_geometricStructureFamily 𝕝 F)`
`e (geometricStructureEquivOfBirkhoffJames 𝕜 𝕝 e h)`
`(mem_geometricStructureEquivOfBirkhoffJames 𝕜 𝕝 e h)`
	The constructor receives the properness of both geometric families, the ambient bijection e, the point bijection Φ_e, and the incidence identity $e(x) \in \Phi_e(I) \iff x \in I$. These are the same maps as in the Statement.

Forward and inverse point maps
`def geometricStructureEquivOfBirkhoffJames :`
`GeometricStructureSpace 𝕜 E ≃`
`GeometricStructureSpace 𝕝 F where`
`toFun I := ⟨e '' I.val,`
`image_mem_geometricStructureFamily 𝕜 𝕝 e h`
`I.property⟩`
`invFun J := ⟨e.symm '' J.val,`
`image_mem_geometricStructureFamily 𝕝 𝕜 e.symm`
`(birkhoffJamesOrthogonal_symm_equiv 𝕜 𝕝 e h)`
`J.property⟩`
	The underlying point map is $\Phi_e(I) = e(I)$. Its inverse sends J to $e^{-1}(J)$; the image under the inverse bijection in the code is this same preimage. The cited geometric-point transport result supplies membership in the target family.

In the source	Mathematical meaning

Homeomorphism fields
`structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where`
`  continuous_toFun : c.Continuous d toFun`
`  continuous_invFun : d.Continuous c invFun`
	The bundle contains one bijection and continuity of that map and its inverse. Applied here, the two conditions are
	$$\Phi_e(c_E(\mathscr{A})) \subseteq c_F(\Phi_e(\mathscr{A})),$$
	$$\Phi_e^{-1}(c_F(\mathscr{B})) \subseteq c_E(\Phi_e^{-1}(\mathscr{B}))$$
	for every $\mathscr{A} \subseteq S_{\mathbb{K}}(E)$ and $\mathscr{B} \subseteq S_{\mathbb{L}}(F)$.

Closure continuity
`def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=`
`  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`
	Continuity of a map $r : (Y, c_Y) \to (Z, c_Z)$ means
	$$r(c_Y(E)) \subseteq c_Z(r(E)) \qquad (E \subseteq Y).$$
	In the preceding row this is used first for $r = \Phi_e$, then for its same inverse.

Exact surrounding source context (separate excerpts)

The surrounding declarations below are from the ambient source declarations; their mathematical meaning is included in the guide above.

```
variable (𝕜 : Type v) (𝕝 : Type z) [RCLike 𝕜] [RCLike 𝕝]
variable {E : Type u} {F : Type w}
variable [NormedAddCommGroup E] [NormedSpace 𝕜 E]
variable [NormedAddCommGroup F] [NormedSpace 𝕝 F]
variable (e : E ≃ F)
variable (h : ∀ x y : E, BirkhoffJamesOrthogonal 𝕝 (e x) (e y) ↔
  BirkhoffJamesOrthogonal 𝕜 x y)
include h
```

Related exact source for the reading guide

Forward and inverse point maps · Fixed source

```
def geometricStructureEquivOfBirkhoffJames :
    GeometricStructureSpace 𝕜 E ≃ GeometricStructureSpace 𝕝 F where
  toFun I := ⟨e '' I.val, image_mem_geometricStructureFamily 𝕜 𝕝 e h I.property⟩
  invFun J := ⟨e.symm '' J.val, image_mem_geometricStructureFamily 𝕝 𝕜 e.symm
    (birkhoffJamesOrthogonal_symm_equiv 𝕜 𝕝 e h) J.property⟩
```

Closure continuity · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

Homeomorphism fields · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```