

The unique trace extension is tracial on the whole target

MathlibAnnex.CStarAlgebra.CAR.traceExtension_mul_comm

theorem

Unitary invariance, strong shell sums, and a closed centralizer extend the trace identity beyond the source.

Statement

Let A be the completed CAR algebra and τ its normalized trace. Fix a representative shell family and its chosen unitary links L_j in the selected atomic representation π . Write $T = C^*(\pi(A), \{L_j\})$ for the resulting concrete target and $\iota : A \rightarrow T$, $\iota(a) = \pi(a)$, for its faithful unital source map. This fixes one target and one source map throughout. Choose the state extension $\phi : T \rightarrow \mathbb{C}$ of τ supplied by state extension along ι . Its GNS representation is $(H_\phi, \rho_\phi, \xi_\phi)$, with

$$\|\xi_\phi\| = 1, \quad \phi(t) = \langle \xi_\phi, \rho_\phi(t) \xi_\phi \rangle.$$

The inner product is linear in its second argument. The T -orbit of ξ_ϕ is dense by the GNS construction. The chosen state is tracial: $\phi(ab) = \phi(ba)$ for every $a, b \in T$.

Assumptions

The representative shell family is fixed. The state ϕ is the chosen extension of the CAR trace, not an arbitrary state. Its uniqueness among all extensions has already been established by the separately cited extension-uniqueness theorem.

Conclusion

Every element of T lies in the centralizer of ϕ . Thus the trace identity holds for arbitrary pairs of target elements, not just pairs coming from A .

Proof route

Use extension uniqueness to prove source-unitary invariance. Linear spanning gives the source centralizer; vector-functional limits give the shell generators. The norm-closed star centralizer then contains the generated algebra.

Proof steps

1. For a unitary $u \in A$, put $v = \iota(u)$ and $\zeta = \rho_\phi(v) \xi_\phi$. Unitarity gives $\|\zeta\| = 1$, so $\psi_u(t) = \langle \zeta, \rho_\phi(t) \zeta \rangle$ is a state. Its source restriction is

$$\psi_u(\iota(b)) = \phi(v^* \iota(b) v) = \tau(u^* b u) = \tau(b).$$

The last equality uses the source trace and $uu^* = 1$. Extension uniqueness applies and gives $\psi_u = \phi$, hence $\phi(v^* t v) = \phi(t)$ for all $t \in T$. This is `traceExtension_star_unitary_mul_mul`.

2. Substitute $t = va$ into that invariance equality. Since $v^*v = 1$,

$$\phi(av) = \phi(v^*(va)v) = \phi(va).$$

The four-unitary spanning theorem for a unital complex C*-algebra writes each $b \in A$ as $b = \sum_{k=1}^4 c_k u_k$. Linearity therefore gives

$$\phi(\iota(b)a) = \sum_k c_k \phi(\iota(u_k)a) = \sum_k c_k \phi(a\iota(u_k)) = \phi(a\iota(b)).$$

The exact application of `CStarAlgebra.exists_sum_four_unitary` is in `traceExtension_shellFamilySourceHom_mul`.

3. For a class j , let $g_j = L_j$ regarded as an element of T , let $w_{j,n}$ be its source shell partial isometries, and set $x_N = \iota(\sum_{n < N} w_{j,n})$. The actual-representation shell-sum theorem applies to ρ_ϕ, ξ_ϕ : the vector functional restricts to τ , and the target orbit is dense. It gives $\rho_\phi(x_N) \rightarrow \rho_\phi(g_j)$ strongly.

4. Fix $a \in T$. The source centralizer equality holds for every x_N . Strong convergence at both $\rho_\phi(a)\xi_\phi$ and ξ_ϕ , followed by continuity of $\rho_\phi(a)$ and the inner product, gives

$$\begin{aligned}\phi(x_N a) &= \langle \xi_\phi, \rho_\phi(x_N) \rho_\phi(a) \xi_\phi \rangle \longrightarrow \phi(g_j a), \\ \phi(a x_N) &= \langle \xi_\phi, \rho_\phi(a) \rho_\phi(x_N) \xi_\phi \rangle \longrightarrow \phi(a g_j).\end{aligned}$$

Uniqueness of the scalar limit proves $\phi(g_j a) = \phi(a g_j)$. This is the precise input/output of `vectorFunctional_mul_eq_mul_of_stronglyConverges` in `traceExtension_shellFamilyGenerator_mul`.

5. Define $C_\phi = \{t \in T : \forall a \in T, \phi(ta) = \phi(at)\}$. State positivity supplies $\phi(t^*) = \overline{\phi(t)}$, so C_ϕ is a unital star subalgebra; continuity makes it norm closed. The preceding steps put $\iota(A)$ and all g_j in C_ϕ . The concrete-generation theorem `eq_top_of_source_mem_of_generator_mem` now yields $C_\phi = T$. Substituting any $a \in T$ into its defining property gives $\phi(ab) = \phi(ba)$.

Main citations

- The stated existence or structural result · Exact source
- The fixed concrete target and source map · Exact source
- Faithfulness of the source embedding · Exact source
- Uniqueness before traciality · Exact source
- The target GNS vector functional · Exact source
- Conjugation invariance from extension uniqueness · Exact source
- Four-unitary span gives every source element · Exact source
- Strong shell sums put generators in the centralizer · Exact source
- Shell convergence in the actual cyclic representation · Exact source
- The closed star centralizer · Exact source
- Closedness of the centralizer · Exact source
- Passing the vector-functional identity to a strong limit · Exact source
- Norm generation of the fixed target · Exact source

Lean source signature (exact)

```
theorem traceExtension_mul_comm (family : RepresentativeShellFamily)
  (a b : ShellFamilyTarget family) :
  traceExtension family (a * b) = traceExtension family (b * a)
```

`traceExtension family` is ϕ on the fixed target. The displayed signature states the final identity for arbitrary a b . The linked proof's `stateCentralizer` is C_ϕ . Its source and generator membership suppliers carry the unitary argument and strong-limit argument described above; they are separately linked proof results, not extra signature assumptions. Stars on operators in the explanation mean Hilbert adjoints; the source uses its exact operator notation.

Lean realization notes

Strong convergence is used inside the actual GNS representation ρ_ϕ . No norm convergence of shell partial sums in T , and no strong continuity of an arbitrary representation, is asserted.

▼ Exact Card identity

Language: en

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