Every irreducible representation is unitarily equivalent to the inclusion

`MathlibAnnex.CStarAlgebra.CAR.ambientInclusion_unitaryEquivalent`

theorem

Extends the unitary equivalence from the CAR source to all additional generators and the norm-closed target.

Statement

For the data and assumptions below, there is a complex-linear unitary $W : \mathcal{H} \to K$ such that $W(tx) = \rho(t)Wx$ for every $t \in T$ and $x \in \mathcal{H}$. Equivalently, the inclusion representation $\rho_0 : T \hookrightarrow B(\mathcal{H})$ is unitarily equivalent to ρ.

Assumptions

Let A be the completed CAR algebra, the completion of the matrix stages $M_{2^n}(\mathbb{C})$ under $a \mapsto a \otimes I_2$. Let f_n be the image of the first diagonal matrix unit, so $f_0 = 1$ and f_n decreases through projections. The root state ω takes the value a_{00} on each matrix stage.

Choose one pure state φ_j from each unitary-equivalence class $j \in I$ of pure-state Gelfand-Naimark-Segal (GNS) representations, with root index o and $\varphi_o = \omega$. Write (H_j, π_j, ξ_j) for its complete complex GNS space, unital star representation and unit cyclic vector. Set $\mathcal{H} = \bigoplus_{j \in I} H_j$, $\pi = \bigoplus_j \pi_j$, and let $e_j : H_j \to \mathcal{H}$ be the coordinate embedding. No countability of I is assumed. Inner products are linear in the second argument.

A supplied shell family consists of unital complex star automorphisms α_j and elements $w_{j,n} \in A$. Put $f_{j,n} = \alpha_j(f_n)$. The identities are $\varphi_j \circ \alpha_j = \omega$, $w_{j,n}^* w_{j,n} = f_{j,n} - f_{j,n+1}$ and $w_{j,n} w_{j,n}^* = f_n - f_{n+1}$. At the root, α_o is the identity and $w_{o,n} = f_n - f_{n+1}$. The family is an input, not an existence conclusion.

Let L_j be unitaries on $\mathcal{H}$ satisfying $L_j \pi(f_{j,n} - f_{j,n+1}) = \pi(w_{j,n})$ for all j, n, and assume $L_o = I_{\mathcal{H}}$. Let $T = C^*(\pi(A), \{L_j\}) \subseteq B(\mathcal{H})$ be the norm-closed unital star algebra they generate. Write $\iota(a) = \pi(a)$ as an element of T, and $g_j \in T$ for the element represented by L_j. Let $\rho : T \to B(K)$ be a unital complex star representation on a complete complex Hilbert space, put $\sigma = \rho \circ \iota$, and put $V_j = \rho(g_j)$. Assume ρ is nonzero and irreducible: its only closed reducing subspaces are $\{0\}$ and K. In addition, require $L_j(e_j \xi_j) = e_o \xi_o$ for every j. These are prescribed links between the selected cyclic vectors. The theorem applies to every complete complex target Hilbert space K and every nonzero irreducible ρ on it; no separability or fixed dimension is assumed.

Conclusion

A single unitary intertwines the whole target algebra, including its added generators. This identifies the irreducible representation with the concrete inclusion. Combining it with the construction of the faithful irreducible inclusion gives the fixed-family structural theorem. The family supplying the shell data remains a hypothesis.

Proof route

Use the same surjective isometry W supplied by surjective cyclic-sum assembly, now viewed as a unitary. For each generator, shell reconstruction gives a strong shell part and a residual corner on both Hilbert spaces. The source intertwiner compares the finite shell sums and hence their strong limits. The atomic common-range theorem identifies the source residual line, and the cyclic-vector transport identifies the remaining corner. Norm-closed generation extends the resulting generator identities to every element of the target.

Proof steps

1. Regard the same surjective isometry W as a unitary, without changing its underlying map. It satisfies $W\pi(a) = \sigma(a)W$ and $W(e_j\xi_j) = \eta_j$, where the compatible vectors obey $V_j\eta_j = \zeta$. The normalization $L_o = I_{\mathcal{H}}$ yields $\eta_o = \zeta$.
2. For each j, reconstruct $L_j = S_j^{\mathcal{H}} + R_j^{\mathcal{H}}$ on $\mathcal{H}$ and $V_j = S_j^K + R_j^K$ on K. The shell parts are the strong limits of the finite sums of $\pi(w_{j,n})$ and $\sigma(w_{j,n})$, respectively. The identity for each source element intertwines the two finite sums. Boundedness of W and uniqueness of vector limits give $WS_j^{\mathcal{H}} = S_j^K W$. This compares two constructed limits rather than applying ρ to a strong limit.
3. Let $P_j^{\mathcal{H}}$ and P_j^K be the projections onto the common ranges of the two represented j-flags. Unitary source intertwining identifies the fixed subspaces, so $WP_j^{\mathcal{H}} = P_j^K W$. The atomic common range is $\mathbb{C}e_j\xi_j$. If $P_j^{\mathcal{H}}x = ce_j\xi_j$, then $P_j^K Wx = c\eta_j$. Since $R_j^{\mathcal{H}} = L_j P_j^{\mathcal{H}}$ and $R_j^K = V_j P_j^K$, the prescribed identities $L_j(e_j\xi_j) = e_o\xi_o$ and $V_j\eta_j = \zeta$ give $WR_j^{\mathcal{H}}x = c\zeta = R_j^K Wx$.
4. Adding the shell and residual identities yields $WL_j = V_jW$ for every j. Together with source intertwining this holds on the algebraic star algebra generated by $\pi(A)$ and the L_j. Adjoint compatibility uses that W is unitary. Continuity in operator norm extends the equality to its closure T, proving $W\rho_0(t) = \rho(t)W$ for every t.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.selectedAtomicRepresentation · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_surjective_selectedAtomicCyclicIsometry · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_targetShellReconstruction · Exact source
- MathlibAnnex.CStarAlgebra.CAR.iInf_range_atomic_transportedFlag_eq_span · Exact source
- MathlibAnnex.CStarAlgebra.AtomicConstruction.intertwines_concreteTarget_of_generators · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_completedAtomicShellModel · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_completedAtomicTarget_uniqueIrreducibleModel · Exact source
- MathlibAnnex.CStarAlgebra.CAR.completedRootPureState · Exact source

Lean source signature (exact)

```
theorem ambientInclusion_unitaryEquivalent
    (family : RepresentativeShellFamily)
    (L : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit →
      MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState →L[ℂ]
        MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState)
    (hLunit : ∀ i, L i ∈ unitary
      (MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState →L[ℂ]
        MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState))
    (hLmap : ∀ i, L i
      (MathlibAnnex.CStarAlgebra.PureState.selectedEmbedding completedRootPureState i
        (MathlibAnnex.CStarAlgebra.PureState.selectedVector completedRootPureState i)) =
      MathlibAnnex.CStarAlgebra.PureState.selectedEmbedding completedRootPureState
        completedRootPureState.classOf
        (MathlibAnnex.CStarAlgebra.PureState.selectedVector completedRootPureState
          completedRootPureState.classOf))
    (hLsource : ∀ i n, (L i).comp (selectedAtomicRepresentation
      (transportedFlag family i n - transportedFlag family i (n + 1))) =
        representedShellLink family i n)
    (hLroot : L completedRootPureState.classOf = 1)
    {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K]
    [CompleteSpace K] (rho : Representation (AtomicTarget L) K)
    (hrho : rho.IsIrreducible) :
    (MathlibAnnex.CStarAlgebra.AtomicConstruction.ambientInclusion selectedAtomicRepresentation
L).UnitaryEquivalent
      rho
```

> Here `Limit` is A and `completedRootPureState` is the root state ω bundled with its purity proof. `SelectedAtomicHilbert ...` is $\mathcal{H}$, `selectedAtomicRepresentation` is π, and `AtomicTarget L` is T. `ambientInclusion selectedAtomicRepresentation L` is the inclusion ρ_0. At source index `i`, corresponding to j, `hLmap` is $L_j(e_j\xi_j) = e_o\xi_o$, `hLsource` is the shell equation, and `hLroot` is $L_o = I_{\mathcal{H}}$. `UnitaryEquivalent` requires a surjective complex-linear isometry intertwining every target element, not only the CAR source. `hrho` asserts nonzero irreducibility of ρ. The proof's local `E`, defined by `LinearIsometryEquiv.ofSurjective W hWsurj`, has the same underlying map as W; the mathematical exposition keeps the notation W.

Lean realization notes

The proof forms shell limits independently on $\mathcal{H}$ and K and compares them through a bounded unitary. It does not assume that an arbitrary representation preserves strong-operator limits, and it does not replace strong convergence by operator-norm convergence. The prescribed cyclic-vector action is used for the residual corner; source intertwining alone does not identify the added generators.

▼ **Exact Card identity**

Language: en

CARD_CONTENT_COMPLETE

Card UID: lfh:lfh-declaration-card:sha256:eb751b0f22dde8fb7e06ff32bb25d5db1cd55259e2b97594d65e2f3fc7657452

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Exposition revision: 1 · SHA-256: 9240c79b6251b9028caa42fe8be517775a4e56eb4a6f9cf135d5837c0b0b4e76

Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73

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