MATHLIBANNEX / CANONICAL DECLARATION CARD

Constructing the normalized trace on the completed CAR algebra

`MathlibAnnex.CStarAlgebra.CAR.trace`

def

Carries compatible bounded matrix traces through the algebraic limit, normed union and completion.

Statement

Let $A_n = M_{2^n}(\mathbb{C})$ with connecting maps $a \mapsto a \otimes I_2$, and let $\iota_n : A_n \to A$ embed the stages in the completed CAR algebra. The trace constructed here is the continuous complex-linear functional $\tau : A \to \mathbb{C}$ extending $\tau_n(a) = 2^{-n} \sum_i a_{ii}$. In particular, $\tau(\iota_n(a)) = \tau_n(a)$.

Definition

At a matrix stage of size $d = 2^n$, each diagonal entry satisfies $|a_{ii}| \leq \|a\|$. Consequently $|\tau_n(a)| \leq d^{-1} \sum_i |a_{ii}| \leq \|a\|$. Under $a \mapsto a \otimes I_2$, each diagonal entry is repeated twice and the normalizing denominator doubles, so the trace is unchanged. Iterating proves compatibility with every later-stage embedding.

The algebraic direct limit identifies two matrices when their images agree in a common later stage. Trace compatibility makes the value independent of the chosen representative. Transport this linear functional to the normed union. Every element there is represented at a finite stage; the stage embeddings preserve norm, so the same bound holds on the union.

For a Cauchy sequence x_k in that union, define the value at its limit by $\tau(\lim_k x_k) = \lim_k \tau(x_k)$. The bound makes the scalar sequence Cauchy and makes the limit independent of the representing sequence. This is the continuous extension used in the declaration.

Assumptions

The matrix norm is the C*-operator norm. The map $a \mapsto a \otimes I_2$ is a unital star homomorphism. Its entries with both added bit coordinates zero recover the entries of a, so it is injective; relabelling the coordinates does not affect injectivity. The C*-norm theorem for injective star homomorphisms then makes the stage map isometric. Thus the normed union has an unambiguous norm and its completion is A. These are proved features of the fixed construction, not new hypotheses about arbitrary directed systems.

Conclusion

The cited extension and stage-evaluation results identify one continuous functional on A. The contractivity, normalization, positivity, and trace-identity theorems show $|\tau(x)| \leq \|x\|$, $\tau(1) = 1$, $\tau(x^*x) \geq 0$, and $\tau(xy) = \tau(yx)$. Thus this extension is a normalized tracial state.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.Limit · Exact source
- MathlibAnnex.CStarAlgebra.CAR.Stage · Exact source
- MathlibAnnex.CStarAlgebra.CAR.step · Exact source

- MathlibAnnex.CStarAlgebra.CAR.step_injective · Exact source
- MathlibAnnex.CStarAlgebra.CAR.norm_step · Exact source
- MathlibAnnex.CStarAlgebra.CAR.embed · Exact source
- MathlibAnnex.CStarAlgebra.CAR.norm_stageHom · Exact source
- MathlibAnnex.CStarAlgebra.CAR.exists_common_stage · Exact source
- MathlibAnnex.CStarAlgebra.CAR.stageTrace · Exact source
- MathlibAnnex.CStarAlgebra.CAR.norm_stageTrace_le · Exact source
- MathlibAnnex.CStarAlgebra.CAR.stageTrace_step · Exact source
- MathlibAnnex.CStarAlgebra.CAR.stageTrace_embed · Exact source
- MathlibAnnex.CStarAlgebra.CAR.algTraceLinear · Exact source
- MathlibAnnex.CStarAlgebra.CAR.preTraceLinear · Exact source
- MathlibAnnex.CStarAlgebra.CAR.norm_preTraceLinear_le · Exact source
- MathlibAnnex.CStarAlgebra.CAR.preTrace · Exact source
- MathlibAnnex.CStarAlgebra.CAR.trace_ofStage · Exact source
- MathlibAnnex.CStarAlgebra.CAR.norm_trace_le · Exact source
- MathlibAnnex.CStarAlgebra.CAR.trace_one · Exact source
- MathlibAnnex.CStarAlgebra.CAR.trace_star_mul_self_nonneg · Exact source
- MathlibAnnex.CStarAlgebra.CAR.trace_mul_comm · Exact source
- MathlibAnnex.CStarAlgebra.CAR.amplify_injective · Exact source
- Uniqueness of the normalized continuous CAR trace · Exact source

Lean source signature (exact)

```
noncomputable def trace : Limit →L[ℂ] ℂ :=
  preTrace.extend (UniformSpace.Completion.toComplL : PreCAR →L[ℂ] Limit)
```

> Here `Limit` is A, `PreCAR` is the normed union A_{alg}, and `trace` is τ. The separately defined `preTrace` is the continuous trace on A_{alg}. The displayed `extend` applies the continuous-extension construction along the canonical dense inclusion `toComplL` of A_{alg} into A. It retains the finite-stage values described above.

Related definition — separate exact excerpt

```
noncomputable def preTrace : PreCAR →L[ℂ] ℂ :=
  preTraceLinear.mkContinuous 1 fun x ↦ by
    simpa only [one_mul] using norm_preTraceLinear_le x
```

Lean realization notes

The declaration defines the continuous extension. Positivity, normalization and traciality are separate proved results, not conclusions established merely by giving the definition. Uniqueness among all normalized continuous traces is proved in the CAR trace-uniqueness theorem. No statement about extending this trace to the larger shell-generated target is part of this construction.

▼ **Exact Card identity**

Language: en

CARD_CONTENT_COMPLETE

Card UID: lfh:lfh-declaration-card:sha256:ed2d0efd1c9929149eff4009da5c9011d720f4f178e084f1e099945c923d6987

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Exposition revision: 1 · SHA-256: f3fa7802702d378d0fece356ede75b11466fd9e70420b5fa42dd07d57ecbfc1d

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