Linked points in a closure space

`ClosureSpace.Linked`

def

Linked points in a closure space

Statement

Two points of a closure space are linked if they are equal or their two-point set is not closed. It tests closure of a pair, without requiring an ambient topology.

Definition

Let X be a set with a closure operation $c : \mathcal{P}(X) \to \mathcal{P}(X)$: the map is extensive, monotone and idempotent, with $c(\varnothing) = \varnothing$. A subset $A \subseteq X$ is closed when $c(A) = A$.

For $x, y \in X$, write $x \leftrightarrow_c y$ for the relation

$$x \leftrightarrow_c y \quad \Longleftrightarrow \quad x = y \ \text{ or } \ c(\{x, y\}) \neq \{x, y\}.$$

If $x \neq y$, this says precisely that closing the pair adds a point outside it. The equality alternative ensures $x \leftrightarrow_c x$ even when the singleton $\{x\}$ is closed.

The relation is symmetric, since $\{x, y\} = \{y, x\}$. Reflexivity and symmetry are established by the cited elementary lemmas. Transitivity is supplied later by passing to the equivalence relation generated by linked pairs.

Assumptions

The only input is a closure space (X, c) and two points $x, y \in X$. Finite-union preservation and separation properties are not assumed. The formula therefore applies even when c is not a topological closure.

Conclusion

The output is the proposition that x and y are linked. It is the generating relation for finite-chain components: a chain may move between equal points or between points whose pair is not closed. The associated component construction is defined separately by the equivalence closure of this relation.

Main citations

- Exact relation, reflexivity and symmetry: `ClosureSpace.Linked`, `ClosureSpace.linked_refl` and `ClosureSpace.linked_symm`, lines 30–46.
- The meaning of closedness: `ClosureSpace.closure`, `ClosureSpace.IsClosed` and `ClosureSpace.isClosed_iff`, lines 37–61.
- Passing to finite-chain components: `ClosureSpace.componentSetoid`, `ClosureSpace.Related` and `ClosureSpace.component`, lines 49–68. This is the explanatory next construction.
- Ryotaro Tanaka, arXiv:2504.05551v1, Definition 3.20: the same generating relation and its finite-chain extension.

Lean source signature (exact)

The namespace is `ClosureSpace`; the surrounding carrier `α` denotes the set X.

```
def Linked (c : ClosureSpace α) (x y : α) : Prop := x = y ∨ ¬ c.IsClosed {x, y}
```

In the source	Mathematical meaning
Exact declaration `α, c : ClosureSpace α`	The underlying set X and the closure structure c: an extensive, monotone, idempotent operation on $\mathcal{P}(X)$ that preserves $\varnothing$. The operation itself is `c.closure`.
Exact declaration `x y : α`	The two given points $x, y \in X$; no distinctness is required.
Exact declaration `c.IsClosed {x, y}`	The equality $c(\{x, y\}) = \{x, y\}$.
Exact declaration `x = y ∨ ¬ c.IsClosed {x, y}`	The entire defining proposition $x = y$ or $c(\{x, y\}) \neq \{x, y\}$. The disjunction retains the equality case, and the negation applies to closedness of the pair.
Exact declaration `: Prop`	The output is the linked-pair relation $x \leftrightarrow_c y$, before taking any transitive closure or component.