

The fixed algebra on the CAR trace Hilbert space

MathlibAnnex.CStarAlgebra.CAR.separableCounterexampleRepresentation

def

Defines a representation of the same algebra T on the GNS Hilbert space of the CAR trace.

Statement

Let A be the completed CAR algebra and $T = C^*(\pi(A) \cup \{L_i : i \in I\}) \subseteq B(\mathcal{H})$ the algebra generated by the selected direct sum π of pure-state GNS representations indexed by their unitary-equivalence classes $i \in I$ and the unitaries L_i from the fixed homogeneity and shell construction. Write $\iota(a) = \pi(a)$ for the embedding $A \rightarrow T$. Let τ be the normalized CAR trace, $H_\tau = L^2(A, \tau)$ its GNS Hilbert space, and π_τ its GNS representation. Let ϕ be the chosen state on T with $\phi \circ \iota = \tau$, and let $(H_\phi, \rho_\phi, \xi_\phi)$ be its GNS representation with cyclic vector. The cited trace construction provides a chosen unitary $E : H_\tau \rightarrow H_\phi$ carrying the canonical trace vector to ξ_ϕ and intertwining the two representations of A . Define

$$\tilde{\rho}(t) = E^{-1} \rho_\phi(t) E \quad (t \in T).$$

Definition

The representation is obtained by conjugating ρ_ϕ with the same chosen unitary $E : H_\tau \rightarrow H_\phi$: $\tilde{\rho}(t) = E^{-1} \rho_\phi(t) E$. Its operators therefore act on H_τ , not on H_ϕ .

Assumptions

The algebra T and the state extension ϕ are fixed as in the cited construction. The same unitary E is used throughout. The Hilbert space H_τ is determined by A and τ , independently of the added unitaries L_i .

Conclusion

This is a unital $*$ -representation $\tilde{\rho} : T \rightarrow B(H_\tau)$. It extends the CAR trace representation in the sense that $\tilde{\rho}(\iota(a)) = \pi_\tau(a)$ for all $a \in A$.

Main citations

- Definition and its exact construction · Exact source
- The CAR trace Hilbert-space abbreviation · Exact source
- The transported representation and its exact RHS · Exact source
- The pointwise conjugation formula · Exact source
- Its source restriction · Exact source
- Separability of the trace Hilbert space · Exact source

Lean source signature (exact)

```
noncomputable def separableCounterexampleRepresentation :  
  Representation AtomicCounterexampleAlgebra SeparableCounterexampleHilbertSpace :=  
  traceModelRepresentation homogeneityShellFamily
```

`separableCounterexampleRepresentation` is the same $\tilde{\rho}$ obtained from `traceModelRepresentation` at the fixed family. `SeparableCounterexampleHilbertSpace` is H_T . The names `traceGNSUnitary` (E) and `tracialRepresentation` (ρ_ϕ) occur in the linked defining construction, explaining the conjugation although they are absent from this short signature.

Lean realization notes

The space H_T is separable by its dense CAR orbit, and injectivity is a separately cited theorem. Neither fact asserts norm separability of T or irreducibility of this representation.

▼ Exact Card identity

Language: en

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