Isolated character away from infinity

`MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.exists_isolated_character_ne_infinity`

Finds an isolated character distinct from the scalar character in a countable character space.

Statement

Let A be a complex C*-algebra, not assumed unital. Let D be a closed commutative unital star subalgebra of Unitization ℂ A with countable character space, and let 0 ≠ d ∈ D have zero scalar coordinate. There exists a character χ distinct from the restricted infinity character for which {χ} is open in the character space. This theorem itself does not assume that any representation represents a unique irreducible-representation class.

Assumptions

Let A be a complex C*-algebra, not assumed unital. Let D be a closed commutative unital star subalgebra of Unitization ℂ A with countable character space, and let 0 ≠ d ∈ D have zero scalar coordinate.

Conclusion

There exists a character χ distinct from the restricted infinity character for which {χ} is open in the character space. This theorem itself does not assume that any representation represents a unique irreducible-representation class.

Proof idea

Gelfand-transform injectivity finds a character not annihilating d; the zero scalar coordinate excludes the infinity character. The complement of that closed singleton is a nonempty open Baire subspace. The countable T1 Baire lemma yields an isolated point there, and openness transfers back to the full character space.

Proof steps

1. Exact Lean statement: ∀ {A : Type u} [inst : NonUnitalCStarAlgebra A] [inst_1 : PartialOrder A] [StarOrderedRing A] (D : StarSubalgebra ℂ (Unitization ℂ A)) [inst_3 : IsClosed ↑D] [inst_4 : IsMulCommutative ↥D] [Countable ↑(WeakDual.characterSpace ℂ ↥D)] (d : ↥D), d ≠ 0 → (↑d).toProd.1 = 0 → ∃ chi, chi ≠ MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.infinityCharacterOn D ∧ IsOpen {chi}

2. ∃ χ, χ ≠ infinityCharacterOn D ∧ IsOpen {χ}.

3. Gelfand-transform injectivity finds a character not annihilating d; the zero scalar coordinate excludes the infinity character. The complement of that closed singleton is a nonempty open Baire subspace. The countable T1 Baire lemma yields an isolated point there, and openness transfers back to the full character space.

Main citations

MathlibAnnex.Topology.exists_isOpen_singleton

Exact formal dependency; inspect the linked Card and exact source.

Lean source signature (exact)

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theorem exists_isolated_character_ne_infinity
(D : StarSubalgebra ℂ (Unitization ℂ A))
[IsClosed (D : Set (Unitization ℂ A))]
[IsMulCommutative D]
[Countable (WeakDual.characterSpace ℂ D)]
(d : D) (hd : d ≠ 0)
(hdfst : (d : Unitization ℂ A).fst = 0) :
∃ chi : WeakDual.characterSpace ℂ D,
chi ≠ infinityCharacterOn (A := A) D ∧
IsOpen ({chi} : Set (WeakDual.characterSpace ℂ D))
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Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: f13185dafb7e72e6ddc0775a43c0210faf488dbee88282e3fdd08359cc7d6ad9

Card SHA-256: 86c77d8996979dd6453b5b81a6f1314b1a2a146adc1a437fd80d77e3a94c0ee0

Approved exposition SHA-256: 9c7bafd852afbd5cb9640610f01ff1a397b58672466285c1bda0c4292c1f6757

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.