A specified GSS homeomorphism preserves compact-operator equality

`MathlibAnnex.Analysis.CStarAlgebra.eq_compactOperatorStarSubalgebra_iff_of_geometricStructure_homeomorph`

theorem

Preserves equality with the compact operators under the specified full geometric homeomorphism.

Statement

Let H, K be complex Hilbert spaces, and let $A_1 \subseteq \mathcal{B}(H)$ and $A_2 \subseteq \mathcal{B}(K)$ be nonzero norm-closed complex $*$-subalgebras acting irreducibly on H and K, respectively. Write $S(D)$ for the geometric structure space of D, with hull–kernel closure c_D, and write $\mathcal{K}(L)$ for the algebra of compact operators on L.

For each maximal proper face of the closed unit ball of D, take the union of the kernels of all bounded complex-linear functionals of norm one that equal 1 on that face. The distinct subsets so obtained are the points of $S(D)$, by the geometric structure definition and the norm-one support characterization.

Suppose that $e : S(A_1) \to S(A_2)$ is a homeomorphism of closure spaces. Equivalently, e is a bijection preserving closure for all subsets:

$$e(c_{A_1}(E)) = c_{A_2}(e(E)) \qquad (E \subseteq S(A_1)). \qquad (1)$$

Then

$$A_1 = \mathcal{K}(H) \iff A_2 = \mathcal{K}(K). \qquad (2)$$

Assumptions

The Hilbert spaces are complete. Irreducibility supplies nonzero Hilbert spaces. No separability or finite-dimensionality is assumed. The homeomorphism uses the same e and its inverse. In terms of continuity, (1) requires

$$e(c_{A_1}(E)) \subseteq c_{A_2}(e(E)) \qquad (E \subseteq S(A_1)),$$

and

$$e^{-1}(c_{A_2}(F)) \subseteq c_{A_1}(e^{-1}(F)) \qquad (F \subseteq S(A_2)).$$

No hypothesis says that e directly maps $N(A_1)$ onto $N(A_2)$, the normal parts consisting of their coefficient kernels.

Conclusion

The conclusion is equality with the compact-operator algebra on each original Hilbert space, not merely containment or abstract $*$-isomorphism. The specified geometric homeomorphism is not asserted to be induced by a $*$-isomorphism. Finite continuity alone is not the hypothesis.

Main citations

Compact-equality transport is the equality branch for the specified full homeomorphism. It uses faithful normal-part transport, equality transport between normal parts, and equality reflection through a faithful irreducible image. The distinct compact-inclusion, proper-inclusion, and zero-intersection results are not substituted for equality.

Lean source signature (exact)

Exact source and declaration. Only the literal signature is displayed here; the linked source retains the proof.

```
theorem eq_compactOperatorStarSubalgebra_iff_of_geometricStructure_homeomorph :
    S = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H ↔
      T = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ K
```

In the source, `S` and `T` denote the algebras A_1 and A_2 used here, respectively.

In the source	Mathematical meaning

Hilbert spaces
`universe u v w`
`variable {H : Type u} [NormedAddCommGroup H]`
`[InnerProductSpace ℂ H] [CompleteSpace H]`
`variable {K : Type v} [NormedAddCommGroup K]`
`[InnerProductSpace ℂ K] [CompleteSpace K]`
	The complete complex Hilbert spaces H, K may have independent universe levels.

Closed operator algebras
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))`
`variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]`
	The source names S, T denote $A_1 \subseteq \mathcal{B}(H)$ and $A_2 \subseteq \mathcal{B}(K)$. Each is a norm-closed complex $*$-subalgebra.

Irreducible actions
`variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
	The inclusion representations of A_1 on H and A_2 on K are nonzero and irreducible. No finite-dimensionality or separability is assumed.

In the source	Mathematical meaning
Specified geometric homeomorphism `variable (e : ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ S) (NormedSpace.geometricStructureClosure ℂ T))`	The same given $e : S(A_1) \to S(A_2)$ is a homeomorphism of the ambient geometric closure spaces. It is not assumed to send one normal part directly onto the other.

Inherited algebra and order structures
`local instance : NonUnitalCStarAlgebra T := NonUnitalStarSubalgebra.nonUnitalCStarAlgebra T (h_closed := inferInstance)`
`local instance : PartialOrder T := _root_.CStarAlgebra.spectralOrder T`
`local instance : StarOrderedRing T := _root_.CStarAlgebra.spectralOrderedRing T`
	The closed operator algebra A_2 carries its inherited C*-algebra structure and compatible order. These local instances are structures on the same algebra; they add no hypothesis to the displayed theorem.

Effective theorem inputs `include hS hT e in`	The two irreducibility assumptions and the specified e are included in the theorem even though its displayed conclusion mentions only the algebras and compact operators.
Equality with compact operators `theorem eq_compactOperatorStarSubalgebra_iff_of_geometricStructure_homeomorph : S = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H ↔ T = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ K := by`	The conclusion is exactly $$A_1 = \mathcal{K}(H) \quad \Longleftrightarrow \quad A_2 = \mathcal{K}(K).$$ It concerns equality on the original Hilbert spaces, not merely compact containment or an abstract isomorphism.
Homeomorphism fields `structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where continuous_toFun : c.Continuous d toFun continuous_invFun : d.Continuous c invFun`	The underlying equivalence is the same e with inverse e^{-1}. The forward continuity field states, for every $E \subseteq S(A_1)$, $$e(c_{A_1}(E)) \subseteq c_{A_2}(e(E)).$$ The inverse continuity field states, for every $F \subseteq S(A_2)$, $$e^{-1}(c_{A_2}(F)) \subseteq c_{A_1}(e^{-1}(F)).$$ The inverse is not an independently selected map.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
universe u v w
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
```

Exact surrounding source:

```
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))
variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
S))
variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
T))
variable (e : ClosureSpace.Homeomorph
  (NormedSpace.geometricStructureClosure ℂ S)
  (NormedSpace.geometricStructureClosure ℂ T))

local instance : NonUnitalCStarAlgebra T :=
  NonUnitalStarSubalgebra.nonUnitalCStarAlgebra T
    (h_closed := inferInstance)
local instance : PartialOrder T := _root_.CStarAlgebra.spectralOrder T
local instance : StarOrderedRing T := _root_.CStarAlgebra.spectralOrderedRing T
```

Exact surrounding source:

```
include hS hT e in
```

Related exact source for the reading guide

Equality with compact operators · Fixed source

```
theorem eq_compactOperatorStarSubalgebra_iff_of_geometricStructure_homeomorph :
    S = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H ↔
      T = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ K := by
```

Homeomorphism fields · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```