Pure-state extension of a character

`MathlibAnnex.Analysis.CStarAlgebra.exists_pureState_extension`

Extends a character of a closed unital star subalgebra to a pure state of the ambient algebra.

Statement

Let A be a nonzero unital complex C*-algebra, D a closed unital star subalgebra of A, and χ a character of D. There is a continuous complex-linear functional φ on A in stateSpace A that is pure and restricts to χ: for every d in D, φ(d)=χ(d).

Assumptions

Let A be a nonzero unital complex C*-algebra, D a closed unital star subalgebra of A, and χ a character of D.

Conclusion

There is a continuous complex-linear functional φ on A in stateSpace A that is pure and restricts to χ: for every d in D, φ(d)=χ(d).

Proof idea

The source obtains an extreme point of the compact weak state-extension face. It maps that weak functional to the strong dual, preserves its restriction to D, and transports extremality to show purity. Closedness of D is used by the extension-face construction.

Proof steps

1. Exact Lean statement: ∀ {A : Type u} [inst : CStarAlgebra A] [inst_1 : PartialOrder A] [inst_2 : StarOrderedRing A] [Nontrivial A] (D : StarSubalgebra ℂ A) [inst_4 : IsClosed ↑D] (chi : ↑(WeakDual.characterSpace ℂ ↥D)), ∃ phi ∈ MathlibAnnex.Analysis.CStarAlgebra.stateSpace A, MathlibAnnex.Analysis.CStarAlgebra.IsPureState A phi ∧ ∀ (d : ↥D), phi ↑d = chi d
2. ∃ φ : A →L[ℂ] ℂ, φ ∈ stateSpace A ∧ IsPureState A φ ∧ ∀ d : D, φ d = χ d.
3. The source obtains an extreme point of the compact weak state-extension face. It maps that weak functional to the strong dual, preserves its restriction to D, and transports extremality to show purity. Closedness of D is used by the extension-face construction.

Main citations

MathlibAnnex.Analysis.CStarAlgebra.IsPureState

Exact formal dependency; inspect the linked Card and exact source.

Lean source signature (exact)

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theorem exists_pureState_extension (D : StarSubalgebra ℂ A) [IsClosed (D : Set A)]
(chi : WeakDual.characterSpace ℂ D) :
∃ phi : A →L[ℂ] ℂ,
phi ∈ stateSpace A ∧ IsPureState A phi ∧ ∀ d : D, phi d = chi d
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Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: f7ac96c3bcdff8225f7123d411c404580fab1d79b3975d601484f61837409226

Card SHA-256: b491c2ef82049b88e4d26e38eab76305c842ad0185cc3edeb40f129127bbe887

Approved exposition SHA-256: c07e120211e22943c73dc406c3dcd09263d2397d305a830ae7935ae0f479fa4d

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.