The CAR source map into the fixed shell-family target

`MathlibAnnex.CStarAlgebra.CAR.shellFamilySourceHom`

def

Restricting the codomain of the selected atomic representation gives the source map into one fixed generated target.

Statement

Let A be the completed CAR algebra, the norm completion of the unital matrix system $M_{2^n}(\mathbb{C})$ with embeddings $a \mapsto a \otimes I_2$. Let ω be its distinguished pure product-vector state. Index pure-state GNS equivalence classes by I, choose one representative φ_j in each class with $\varphi_o = \omega$ at $o = [\omega]$, and denote its GNS data by (H_j, π_j, ξ_j). Write $\mathcal{H} = \bigoplus_{j \in I} H_j$ for the Hilbert direct sum, $e_j : H_j \to \mathcal{H}$ for coordinate inclusion, and $\pi(a)(x_j)_j = (\pi_j(a)x_j)_j$. Here the sum consists of square-summable families and has no countability restriction on I. Fix a representative shell family: for each class j it gives an automorphism α_j with $\varphi_j \circ \alpha_j = \omega$ and shell links $w_{j,n}$ satisfying $w_{j,n}^* w_{j,n} = \alpha_j(p_n)$ and $w_{j,n} w_{j,n}^* = p_n$. Here f_n is the image of the distinguished matrix projection $E_{00} \in M_{2^n}(\mathbb{C})$ in A, and $p_n = f_n - f_{n+1}$. At o, the family specifies $\alpha_o = \mathrm{id}$ and $w_{o,n} = p_n$. Choose once the family of unitary operators $L = (L_j)_j$ supplied for this shell family by the completed CAR atomic-shell realization theorem, and fix $T = C^*(\pi(A), \{L_j : j \in I\}) \subseteq B(\mathcal{H})$. The source map is the unital star homomorphism $\iota : A \to T$ defined by $\iota(a) = \pi(a)$, viewed as an element of T.

Definition

The generic map into a concrete generated target sends an algebra element to its represented operator, now with codomain that target. Apply this construction to the selected CAR atomic action π and the already chosen links L. The source and target units agree because T is a unital star subalgebra of $B(\mathcal{H})$. No new completion, quotient, or choice of representation is performed by this map.

Assumptions

The given representative shell family, its selected link family L, and the resulting target T are fixed throughout. The containment $\pi(A) \subseteq T$ follows from the definition of the generated algebra, so the codomain restriction is defined on every $a \in A$. No target representation or target Hilbert space is an additional input to this definition.

Conclusion

If $\rho_0 : T \to B(\mathcal{H})$ is literal inclusion, then $\rho_0 \circ \iota = \pi$. In particular $\iota(1) = 1_T$. The separate source-injectivity theorem shows that this particular ι is faithful; it uses the faithful direct-sum representation of CAR, rather than any simplicity assumption on T.

Main citations

- Definition and its exact construction · Exact source
- The CAR selected atomic action · Exact source
- The shell-family input and its root component · Exact source
- One fixed choice of the realized unitary links · Exact source

- The target formed from that same choice · Exact source
- The completed CAR realization supplying the link family · Exact source
- Generic codomain restriction to the generated algebra · Exact source
- Underlying-operator formula for that restriction · Exact source
- Separate injectivity of this source map · Exact source
- The inclusion of the same fixed target · Exact source

Lean source signature (exact)

```
noncomputable def shellFamilySourceHom (family : RepresentativeShellFamily) :
    Limit →⋆ₐ[ℂ] ShellFamilyTarget family :=
  MathlibAnnex.CStarAlgebra.AtomicConstruction.sourceHom selectedAtomicRepresentation (shellFamilyLinks
family)
```

> Here `family` is the specified representative shell family, `Limit` is A, and `selectedAtomicRepresentation` is π on $\mathcal{H}$. `shellFamilyLinks family` is the once-chosen L, and `ShellFamilyTarget family` is its fixed T. The RHS `AtomicConstruction.sourceHom` is ι, the codomain restriction of π. The identity with inclusion and the separate injectivity theorem are separately cited; the definition does not choose a second link family.

Lean realization notes

The choice of L is made once in the cited shell-family-links definition. The maps ι and ρ_0 use that same choice and the same T, not unrelated witnesses of existence statements. The defining RHS is a codomain restriction; injectivity is a separately proved property.

▼ Exact Card identity

Language: en

CARD_CONTENT_COMPLETE

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