Transported flags vanish on vectors generated by a trace vector

`MathlibAnnex.CStarAlgebra.CAR.tendsto_transportedFlag_orbit_zero`

theorem

Turns decay of projection traces into norm convergence on each source-orbit vector.

Statement

Let $\sigma : A \to B(K)$ be a unital star representation of the completed CAR algebra on a complete complex Hilbert space. Suppose $\xi \in K$ implements the normalized trace from the CAR trace construction: $\langle \xi, \sigma(a)\xi \rangle = \tau(a)$ for all $a \in A$. For the transported projections $f_{j,n}$ of a fixed shell family and every $b \in A$, one has $\sigma(f_{j,n})\sigma(b)\xi \to 0$ in norm as $n \to \infty$.

Assumptions

Let A be the completed CAR algebra with matrix stages $M_{2^n}(\mathbb{C})$ embedded by $a \mapsto a \otimes I_2$. Write f_n for the image of the first diagonal matrix unit, $p_n = f_n - f_{n+1}$, and ω for the root state characterized by $\omega(\iota_n(a)) = a_{00}$ on each stage, where ι_n is its canonical embedding.

Index chosen pure-state Gelfand-Naimark-Segal (GNS) representations by their unitary-equivalence classes $j \in I$, with root index o and representative states φ_j.

A fixed shell family consists of complex-linear unital star automorphisms α_j and elements $w_{j,n} \in A$ satisfying $\varphi_j(\alpha_j(a)) = \omega(a)$, $w_{j,n}^* w_{j,n} = \alpha_j(p_n)$ and $w_{j,n} w_{j,n}^* = p_n$.

At the root, α_o is the identity and $w_{o,n} = p_n$. Set $f_{j,n} = \alpha_j(f_n)$.

The space K lies in an arbitrary universe; it need not be separable. The vector-functional equality is assumed for every $a \in A$. Cyclicity of ξ in all of K is not assumed, nor is irreducibility of σ. Inner products are linear in their second argument. Here $B(K)$ is the algebra of bounded complex-linear operators on K.

Conclusion

Each fixed vector $\sigma(b)\xi$ is sent to zero in the limit. The estimate $\|\sigma(f_{j,n})\sigma(b)\xi\|^2 \le \|b\|^2 2^{-n}$ proves this directly.

Proof route

For a projection p, the vector-functional hypothesis gives $\|\sigma(p)\sigma(b)\xi\|^2 = \tau(b^* p b)$. Traciality rewrites this as $\tau(p b b^* p)$. Since $0 \le p b b^* p \le \|b\|^2 p$, positivity gives the bound $\|b\|^2 \tau(p)$. Apply it to $p = f_{j,n}$, whose trace is 2^{-n} by the transported-flag trace theorem, and let n tend to infinity.

Proof steps

1. Use $p^* = p = p^2$ and star preservation to express the squared norm as $\tau(b^* p b)$.
2. Use the trace identity and the positive order bound on $p b b^* p$ to obtain $\|\sigma(p)\sigma(b)\xi\|^2 \le \|b\|^2 \tau(p)$.

3. Insert the exact value $\tau(f_{j,n}) = 2^{-n}$. The nonnegative squared norms are squeezed to zero, which implies norm convergence of the vectors.

Main citations

- Exact declaration and proof · Exact source
- MathlibAnnex.CStarAlgebra.CAR.RepresentativeShellFamily · Exact source
- MathlibAnnex.CStarAlgebra.CAR.transportedFlag · Exact source
- MathlibAnnex.CStarAlgebra.CAR.isStarProjection_transportedFlag · Exact source
- MathlibAnnex.CStarAlgebra.CAR.trace_transportedFlag · Exact source
- MathlibAnnex.CStarAlgebra.CAR.tracePositive · Exact source
- MathlibAnnex.CStarAlgebra.CAR.trace_mul_comm · Exact source
- MathlibAnnex.Analysis.CStarAlgebra.norm_sq_projection_orbit_le · Exact source
- MathlibAnnex.Analysis.CStarAlgebra.tendsto_projection_orbit_zero · Exact source
- MathlibAnnex.CStarAlgebra.CAR.trace · Exact source

Lean source signature (exact)

```
theorem tendsto_transportedFlag_orbit_zero (family : RepresentativeShellFamily)
    (i : MathlibAnnex.CStarAlgebra.PureState.GNSClass Limit)
    (σ : Representation Limit H) (ξ : H)
    (hξ : ∀ a, Representation.vectorFunctional σ ξ a = trace a) (b : Limit) :
    Tendsto (fun n ↦ σ (transportedFlag family i n) (σ b ξ)) atTop (nhds 0)
```

> Here `Limit` is A, `trace` is τ, and `transportedFlag family i n` is $f_{j,n}$. The source index `i` is the index j used here. The source letter `H` denotes the comparison space called K above. `vectorFunctional σ ξ a` is $\langle \xi, \sigma(a)\xi \rangle$. The filter expression asserts that $\sigma(f_{j,n})\sigma(b)\xi$ tends in norm to zero as $n \to \infty$, for each fixed b; it makes no additional cyclicity assertion.

Lean realization notes

This declaration is a statement about orbit vectors and norm convergence of their images. It does not claim operator-norm convergence of the projections, or convergence on every vector of an arbitrary ambient representation without an additional argument.

▼ **Exact Card identity**

Language: en

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