Cross-representation state approximation with a protected finite set

`MathlibAnnex.CStarAlgebra.CAR.exists_stageTests_crossRepresentation_path_approx`

theorem

Finite entrywise tests allow new state approximation while an entire unitary path nearly fixes earlier data.

Statement

Let A be the completed CAR algebra: the norm completion of the increasing union of $A_n = M_{2^n}(\mathbb{C})$ under the unital embeddings $a \mapsto a \otimes I_2$ (with the fixed coordinate reindexing). Write $\iota_n : A_n \to A$ for the canonical isometric inclusion and $e_{ij}^{(n)} = \iota_n(E_{ij})$ for its matrix units. A representation is a unital complex star homomorphism into the bounded operators on a complex Hilbert space. Irreducibility means nonzero action and no proper nonzero closed reducing subspace. We use the inner product conjugate linear in its first argument and write $\varphi_{\pi,v}(a) = \langle v, \pi(a)v \rangle$ for the vector functional. Fix an irreducible $\pi : A \to B(H)$ with $H \neq 0$. For every finite $F \subset A$ and $\varepsilon > 0$, there exist n and $\delta > 0$ with the following property. Let $\sigma : A \to B(K)$ be any representation and let $\xi \in H, \eta \in K$ be unit vectors whose vector states differ by less than δ on every $e_{ij}^{(n)}$. For every later finite set $F' \subset A$ and $\varepsilon' > 0$, there are $u \in U(A)$ and a norm-continuous path p from 1 to u such that $|\varphi_{\pi,\pi(u)\xi}(a) - \varphi_{\sigma,\eta}(a)| < \varepsilon'$ for every $a \in F'$. Throughout the path, both conjugation errors on F are less than ε.

Assumptions

The spaces H and K are complete complex Hilbert spaces. Only π is required to be irreducible. The order of choices is essential: F, ε come first, then n, δ, then the second representation and the unit vectors satisfying the tests, and finally the new approximation requests F', ε'.

Conclusion

Writing the all-time protection explicitly, $\|p(t)ap(t)^* - a\| < \varepsilon$ and $\|p(t)^*ap(t) - a\| < \varepsilon$ for every $t \in [0,1]$ and $a \in F$. At the endpoint the transported vector state approximates the target on the independently specified later set F'.

Proof route

Choose the tests using the exact local transport theorem in the first representation. Approximate the later set by a larger stage. Finite-stage purification realizes the second vector state exactly on that larger stage inside H, so the original tests still hold. Exact local transport reaches this purified vector while protecting F. Contractivity of unit vector states controls the approximation error on the later set.

Proof steps

1. Use the local exact-transport theorem for π, F, ε to fix n and δ. These choices do not depend on K, σ, or the later approximation request.

2. Approximate every element of F' in a common stage m within $\varepsilon'/3$, and pass to $N = \max(n, m)$. Purification gives a unit vector $\zeta \in H$ whose vector state agrees with $\varphi_{\sigma,\eta}$ on stage N. Compatibility of the inclusions gives agreement on the earlier stage n as well.

3. The assumed entrywise tests for ξ, η are therefore the tests for ξ, ζ in H. Exact local transport supplies u and p with $\pi(u)\xi = \zeta$ and the required all-time protection.

4. For $a \in F'$ and a stage approximant b, the two vector states agree at b. Each has norm at most 1, so their difference at a is bounded by $2\|a - b\| < \varepsilon'$.

Main citations

- The stated existence or structural result · Exact source
- Exact local transport preserving a finite set along the path · Exact source
- Exact purification on a larger finite stage · Exact source
- Representation and nonzero irreducibility conventions · Exact source
- Equivalence of the two irreducibility interfaces on a nonzero Hilbert space · Exact source

Lean source signature (exact)

```
theorem exists_stageTests_crossRepresentation_path_approx
    {H : Type*}
    [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
    [Nontrivial H]
    (rho : Representation Limit H) (hrho : StarAlgHom.IsIrreducible rho)
    (F : Finset Limit) {epsilon : ℝ} (hepsilon : 0 < epsilon) :
    ∃ n, ∃ delta > 0,
      ∀ {K : Type*}
        [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
        (sigma : Representation Limit K) (xi : H) (eta : K),
      ‖xi‖ = 1 → ‖eta‖ = 1 →
      (∀ i j : Fin (2 ^ n),
        ‖Representation.vectorFunctional rho xi (limitMatrixUnit n i j) -
          Representation.vectorFunctional sigma eta (limitMatrixUnit n i j)‖ < delta) →
      ∀ (F' : Finset Limit) (epsilon' : ℝ), 0 < epsilon' →
        ∃ u : unitary Limit,
          ∃ p : Path 1 u,
          (∀ t, ∀ a ∈ F,
            ‖(p t : Limit) * a * star (p t : Limit) - a‖ < epsilon ∧
            ‖star (p t : Limit) * a * (p t : Limit) - a‖ < epsilon) ∧
          ∀ a ∈ F',
            ‖Representation.vectorFunctional rho (rho (u : Limit) xi) a -
              Representation.vectorFunctional sigma eta a‖ < epsilon'
```

> Here `rho` is the fixed representation π, `sigma` is the later representation, and `F` is protected throughout `p`. The primed variables `F'` and `epsilon'` specify the later endpoint approximation. `Representation.vectorFunctional rho (rho u xi)` is the state after transport; the theorem compares it with the state of `eta` under `sigma`.

Lean realization notes

The endpoint approximation does not assert equality of the two states on all of A, unitary equivalence of the representations, or state approximation along every intermediate point of the path. The second representation need not be irreducible.

▼ Exact Card identity

Language: en
CARD_CONTENT_COMPLETE
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Card revision: 1 · SHA-256: 7711b3c3062f35d4f40941c530616797fcdd6ff68d6df057976f2c34a611097c
Exposition revision: 1 · SHA-256: 5c025351e5f794d88c7ae8ae8475b719da90a5ba389646a4db41b2537c5b2974
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