A fixed counterexample to Naimark's problem

`MathlibAnnex.CStarAlgebra.CAR.atomicCounterexampleEndpoint`

Proves the ordinary counterexample properties for the source-defined algebra, with no additional homogeneity premise.

Statement

The fixed CAR-based algebra A is a nontrivial unital simple infinite-dimensional C*-algebra. Its atomic inclusion is a faithful nonzero irreducible representation, every nonzero irreducible representation is unitarily equivalent to that inclusion, and A has no model as the algebra of all compact operators on a Hilbert space.

Assumptions

Let $A \subseteq B(H_{at})$ be the fixed unital C*-algebra obtained by adjoining the chosen shell-link unitaries to the atomic representation of the CAR algebra C, and then taking the norm-closed unital *-algebra they generate. The map $j: C \to A$ and the atomic inclusion π_{at} are the fixed maps from this construction. There are no additional theorem hypotheses. The comparison Hilbert spaces are arbitrary and need not be separable.

Conclusion

A is nontrivial and norm closed in $B(H_{at})$.

The CAR source map $j: C \to A$ is injective and satisfies $j(1) = 1$.

A is infinite-dimensional over $\mathbb{C}$.

The inclusion $\pi_{at}: A \to B(H_{at})$ is faithful, nonzero, and irreducible.

For every complex Hilbert space K and every nonzero irreducible *-representation $\sigma: A \to B(K)$, not initially required to preserve the unit, there is a surjective complex-linear isometry $U: H_{at} \to K$ with $U(\pi_{at}(a)x) = \sigma(a)(Ux)$ for all $a \in A$ and $x \in H_{at}$.

Every norm-closed two-sided ideal of A is either $\{0\}$ or A.

For every complex Hilbert space K, there is no injective complex-linear *-homomorphism $e: A \to B(K)$, not required to preserve the unit, whose image consists of compact operators and contains every compact operator on K.

Proof route

Apply the proved shell-family theorem to the one family selected by CAR homogeneity. This is a specialization of already established conclusions, not an assumption of those conclusions.

Proof steps

1. The chosen shell model gives closedness, the CAR embedding, faithfulness of the ambient inclusion, and infinite-dimensionality.

2. Its irreducibility and capture theorems supply a surjective isometric intertwiner for every nonzero irreducible comparison representation.

3. The corresponding ideal and compact-model theorems give simplicity and the exclusion of every full compact-operator model.

Lean realization notes

The exact proof is shellFamilyEndpoint homogeneityShellFamily. The comparison universe v is independent of the fixed carrier universe. Nonunital comparison maps are included. The result does not say that all irreducible representations are absent: π_{at} is one. Later declarations add separable and tracial conclusions.

Language: en · CARD_CONTENT_COMPLETE

Main citations

MathlibAnnex.CStarAlgebra.CAR.shellFamilyEndpoint

Lean source declaration (exact)

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/-- Closed ordinary Naimark main for the actual CAR construction. It has no
generic KOS, shell-data, rank-one, capture, simplicity, or compactness premise. -/
theorem atomicCounterexampleEndpoint : AtomicCounterexampleEndpoint.{v} :=
shellFamilyEndpoint homogeneityShellFamily
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Read exact MathlibAnnex v0.4.0 source

Exact Card identity

Stable Card ID: fd0aa4fb4ef7616c41e13cea03169a5a1011d24ad2c287adbff4903787fd5470

Card SHA-256: d10f0a91f19a1e8ff87ebdfcdb4948f6e061fdea3cf835a3652348f1e8084674

Approved exposition SHA-256: 850c30f86813ebccfa903a1c13f7f97291f350c66870764a2b5f1cc14f6fa99d

AI-assisted material. Selected, edited, and reviewed by Ryotaro Tanaka. The recorded review is source–exposition correspondence, not an independent proof audit.