A simple C*-algebra with a unique irreducible representation class

`MathlibAnnex.CStarAlgebra.CAR.atomicCounterexampleEndpoint`

theorem

Establishes the properties that make the fixed algebra T a counterexample to Naimark's problem.

Statement

Let A be the completed CAR algebra. Let $\pi : A \to B(\mathcal{H})$ be the direct sum of one chosen GNS representation from each pure-state GNS equivalence class, and let $(L_i)_{i\in I}$ be the unitaries obtained from the fixed CAR homogeneity and shell construction. Put $T = C^*(\pi(A) \cup \{L_i : i \in I\}) \subseteq B(\mathcal{H})$, and write $\iota(a) = \pi(a)$ for the embedding into T. Let $\rho_0 : T \to B(\mathcal{H})$ be the inclusion representation. Then T is a nonzero, norm-closed, infinite-dimensional unital C*-algebra, ι is injective and unital, and ρ_0 is faithful and nonzero irreducible. Every nonzero irreducible $*$-representation $\rho : T \to B(K)$ on a complex Hilbert space K is unitarily equivalent to ρ_0, even if $\rho(1) = 1$ is not assumed: there is a unitary $U : \mathcal{H} \to K$ with

$$U\rho_0(t) = \rho(t)U \qquad (t \in T).$$

The algebra T is simple: every norm-closed two-sided ideal is $\{0\}$ or T. Moreover, for no complex Hilbert space K is there an injective $*$-representation $e : T \to B(K)$ with $e(T) = \mathcal{K}(K)$.

Assumptions

The automorphisms and shell unitaries are obtained from CAR homogeneity and fixed once. No simplicity or uniqueness-of-representation assumption is supplied to the theorem. The Hilbert space K in the representation and compact-operator assertions is arbitrary; separability and preservation of the identity by the given representation are not assumed.

Conclusion

Thus T has exactly one unitary-equivalence class of nonzero irreducible $*$-representations, but is not isomorphic to the full algebra of compact operators on any Hilbert space. The nonzero, norm-closed algebra T, its injective unital embedding ι and its faithful irreducible inclusion ρ_0 are the same throughout these assertions.

Proof route

Apply the cited theorem for a fixed shell family to the family supplied by CAR homogeneity. The construction and the representation-classification theorem give the embedding and unitary equivalences; their cited consequences give simplicity and the exclusion of an algebra of compact operators.

Proof steps

1. Let o denote the class of the distinguished pure state, let f_n be the decreasing root projections, and write $p_n = f_n - f_{n+1}$ for their shells. At o choose $\alpha_o = \mathrm{id}$ and $w_{o,n} = p_n$. In each other class, CAR homogeneity supplies one automorphism α_i, and exact shell matching supplies all $w_{i,n}$ for that same α_i. These choices are made once for the entire construction.

2. The shell construction supplies an injective unital embedding ι and an irreducible inclusion ρ_0. Since ι embeds the infinite-dimensional algebra A, the algebra T is infinite dimensional. Its norm closure is part of its construction, and its nonzeroness follows from injectivity of ι.

3. The representation-classification theorem for this fixed family gives unitary equivalence with ρ_0 for every unital nonzero irreducible representation. The cited unitality lemma gives $\rho(1) = I_K$ for every nonzero irreducible $*$-representation ρ of T, so the same conclusion applies without initially assuming that ρ preserves the identity.

4. The cited simplicity theorem applies because ρ_0 is faithful and all nonzero irreducible representations are unitarily equivalent to it. The cited compact-operator exclusion applies because T is unital and infinite dimensional. The Lean theorem collects these proved properties for the same algebra.

Main citations

- The stated existence or structural result · Exact source
- The abbreviation for the full structural assertion · Exact source
- The ten properties in the exact statement · Exact source
- The structural theorem for a fixed shell family · Exact source
- The once-chosen automorphisms and shell partial isometries · Exact source
- Unitality forced by nonzero irreducibility · Exact source
- Simplicity of the constructed algebra · Exact source
- Infinite dimensionality of the constructed algebra · Exact source

Lean source signature (exact)

```
theorem atomicCounterexampleEndpoint : AtomicCounterexampleEndpoint.{v}
```

> `AtomicCounterexampleEndpoint` abbreviates `ShellFamilyEndpoint homogeneityShellFamily`; the separate abbreviation and the full structure are linked. Its ten fields are the properties expanded above. In those fields, `atomicCounterexampleRepresentation` is ρ_0, `atomicCounterexampleSourceHom` is ι, and `captures_nonunital` is the unitary-equivalence assertion for an arbitrary nonzero irreducible representation, without an initial unit-preservation assumption. `not_compactOperatorModel` rules out an injective representation whose range is all of $\mathcal{K}(K)$. These names occur in the linked structure and proof, not as extra arguments in this short signature.

Lean realization notes

In the displayed inclusion representation, one also has $T \cap \mathcal{K}(\mathcal{H}) = \{0\}$. This follows from the stated properties: the intersection is a closed two-sided ideal of T; if it were nonzero, simplicity would give $T \subseteq \mathcal{K}(\mathcal{H})$. Then $I_{\mathcal{H}}$ would be compact, so $\mathcal{H}$, and hence $T \subseteq B(\mathcal{H})$, would be finite dimensional, a contradiction. This is a consequence of the theorem, not an extra field of its Lean statement.

▼ **Exact Card identity**

Language: en
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Card UID: lfh:lfh-declaration-card:sha256:fd0aa4fb4ef7616c41e13cea03169a5a1011d24ad2c287adbff4903787fd5470
Card revision: 2 · SHA-256: d10f0a91f19a1e8ff87ebdfcdb4948f6e061fdea3cf835a3652348f1e8084674
Exposition revision: 3 · SHA-256: f161a599b091c62218ee716a708cbc5306b024b90354aedfad6344dc8c94adb3
Source commit: 437e6e46228bbb8e91211ebded349d7a30020e73
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