The ambient inclusion of the fixed shell-family target

`MathlibAnnex.CStarAlgebra.CAR.shellFamilyInclusion`

def

The same concrete target has a literal representation on the selected atomic Hilbert space.

Statement

Let A be the completed CAR algebra, the norm completion of the unital matrix system $M_{2^n}(\mathbb{C})$ with embeddings $a \mapsto a \otimes I_2$. Let ω be its distinguished pure product-vector state. Index pure-state GNS equivalence classes by I, choose one representative φ_j in each class with $\varphi_o = \omega$ at $o = [\omega]$, and denote its GNS data by (H_j, π_j, ξ_j). Write $\mathcal{H} = \bigoplus_{j \in I} H_j$ for the Hilbert direct sum, $e_j : H_j \to \mathcal{H}$ for coordinate inclusion, and $\pi(a)(x_j)_j = (\pi_j(a)x_j)_j$. Here the sum consists of square-summable families and has no countability restriction on I. Fix a representative shell family: for each class j it gives an automorphism α_j with $\varphi_j \circ \alpha_j = \omega$ and shell links $w_{j,n}$ satisfying $w_{j,n}^* w_{j,n} = \alpha_j(p_n)$ and $w_{j,n} w_{j,n}^* = p_n$. Here f_n is the image of the distinguished matrix projection $E_{00} \in M_{2^n}(\mathbb{C})$ in A, and $p_n = f_n - f_{n+1}$. At o, the family specifies $\alpha_o = \mathrm{id}$ and $w_{o,n} = p_n$. Choose once the family of unitary operators $L = (L_j)_j$ supplied for this shell family by the completed CAR atomic-shell realization theorem, and fix $T = C^*(\pi(A), \{L_j : j \in I\}) \subseteq B(\mathcal{H})$. The ambient inclusion is $\rho_0 : T \to B(\mathcal{H})$, $\rho_0(t) = t$. With the source map $\iota : A \to T$, $\iota(a) = \pi(a)$, it satisfies $\rho_0 \circ \iota = \pi$.

Definition

The generic ambient-inclusion construction takes a concrete generated star algebra to its inclusion in the existing operator algebra. Using the atomic CAR action and the fixed links gives exactly $t \mapsto t$. Since $\iota(a)$ is the same operator $\pi(a)$ regarded as an element of T, composing with this inclusion recovers $\pi(a)$ for every a; no new operator is constructed by the composition.

Assumptions

The representative shell family is the only input beyond the fixed CAR construction. Both T and its ambient Hilbert space $\mathcal{H}$ have already been determined by that family and its once-chosen links L. The target is a concrete norm-closed unital star subalgebra of $B(\mathcal{H})$.

Conclusion

The definition supplies the displayed unital representation ρ_0 of this same T on this same $\mathcal{H}$. Its injectivity is the injectivity of literal inclusion. The separately cited irreducibility theorem uses the construction of the chosen shell unitaries L to show that this representation has no proper nonzero closed reducing subspace.

Main citations

- Definition and its exact construction · Exact source
- The CAR atomic representation on the ambient Hilbert space · Exact source
- One fixed choice of the realized unitary links · Exact source
- The concrete target from that choice · Exact source
- Generic literal ambient inclusion · Exact source

- The source map into the same target · Exact source
- Underlying-operator identity giving the composition · Exact source
- Separate injectivity of this inclusion · Exact source
- Separate irreducibility from the CAR realization · Exact source

Lean source signature (exact)

```
noncomputable def shellFamilyInclusion (family : RepresentativeShellFamily) :
    Representation (ShellFamilyTarget family)
      (MathlibAnnex.CStarAlgebra.PureState.SelectedAtomicHilbert completedRootPureState) :=
  MathlibAnnex.CStarAlgebra.AtomicConstruction.ambientInclusion selectedAtomicRepresentation
    (shellFamilyLinks family)
```

> Here `family` fixes the shell data, `shellFamilyLinks family` is L, and `ShellFamilyTarget family` is T. `completedRootPureState` is the CAR root state ω bundled with its purity proof, so `SelectedAtomicHilbert completedRootPureState` is $\mathcal{H}$. The RHS `AtomicConstruction.ambientInclusion` is ρ_0. The separately defined `shellFamilySourceHom family` is ι; the underlying-operator formula gives $\rho_0 \circ \iota = \pi$. The cited irreducibility theorem is separate from this defining RHS.

Lean realization notes

The inclusion and source map have different domains: ρ_0 acts on T and ι on A. Irreducibility and universal capture are separate results, not fields of this definition. This construction does not identify arbitrary target representations or change the choice of L.

▼ **Exact Card identity**

Language: en
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Card revision: 1 · SHA-256: 608748718c9b2fa8d4a3f8b34ab899dc76f70c13f6b41b13231b5aa918686622
Exposition revision: 1 · SHA-256: dcb3559cfc538c702d2b99364db637cf2e502471606b7a64efecd50aae28b11e
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