Geometric Structure Spaces

From unit-ball geometry to C*-algebras

The geometric structure space connects faces and supporting functionals of the norm unit ball with irreducible representations and primitive ideals. The project follows those connections to normal parts, compact representations, decomposition and rigidity.

Read the mathematical overview · Research project · Reader Project PDF

Assumptions and limits

The geometric construction begins with real or complex normed spaces; the C*-algebra comparisons retain their complex, possibly nonunital hypotheses. The zero space has empty geometric structure space. Individual results retain their nonzero, closedness, irreducibility and continuity assumptions.

- For the elementary-model rigidity result, at least one algebra is *-isomorphic to the whole K(H) of a complex Hilbert space.
- For finite-dimensional rigidity, at least one algebra is finite-dimensional over the complex numbers.
- For bounded-operator rigidity, at least one algebra is *-isomorphic to the whole B(H) of a complex Hilbert space.

These conclusions equate the existence of a *-isomorphism with the existence of a GSS homeomorphism. They do not assert that a specified nonlinear homeomorphism is induced by that isomorphism. The exact manuscripts govern zero-algebra and zero-model cases. An abstract type-I factor is not substituted for the concrete B(H) model assumption.

GSS rigidity when either algebra has an elementary model · GSS rigidity when either algebra is finite-dimensional · GSS rigidity from a full bounded-operator model

Mathematical routes

These six groups follow the accepted reading guide. Choosing a group includes its prerequisite closure. The displayed levels always retain the full selected scope.

Closure spaces and norm geometry

Begin with hull–kernel closure on the kernel families obtained from faces. Closure need not be topological; finite continuity tests finite subsets, whereas continuity tests every subset.

Faces, pure states and spectra

Faces and extreme dual functionals lead to pure states and representation components. Unitary-equivalence classes of irreducible representations and their kernels remain different objects; the two spectral comparisons use their respective hull–kernel topologies.

Normal parts and semilinear structure

Pass from a representation component to its coefficient-kernel normal part with relative closure. Projective parametrization is a bijection. The same map, its inverse, the scalar automorphism and global orientation retain the hypotheses specified in each manuscript; full continuity requires extra conditions.

The central characterization

The faithful, fully normal representation condition and continuous finite conjugation meet in the elementary-algebra characterization. Retain the same GNS model, equality of the compact-operator image, singleton condition, and two-sided conjugate continuity. Manuscript 43 uses the exact source-bound exposition.

From components to classification

Use compact comparisons to read representation-theoretic conditions. Compact containment differs from equality; prescribed dimension differs from an upper bound. Transport through a specified homeomorphism retains that specified map.

Compact representations, decomposition and rigidity

Faithful compact representation leads to the canonical c_0-decomposition indexed by minimal nonzero closed ideals. Its compact-operator summands are distinct from compact operators on a Hilbert direct sum. The final rigidity results assert existence of an isomorphism, with the different model hypotheses below.

Boundary inputs

The mathematical background below follows reading A. Links identify fixed source entry points using these tools; they are not new graph edges or claims of new independent proofs.

- **Hahn–Banach** — Norming and extension functionals supply the connection from norm geometry to states. Fixed source entry
- **Krein–Milman and weak* compactness** — The weak* support family and extreme dual ball connect maximal faces to their supports. Fixed source entry
- **Pure states and GNS representations** — The pure-state construction and irreducibility comparison give representation classes, distinct from kernels. Fixed source entry
- **Simultaneous interpolation** — Simultaneous control of representation vectors enters the compact-representation results. Fixed source entry
- **Quotient by the kernel** — Representation images are compared through the quotient by the representation kernel. Fixed source entry

Technical boundary browser lists all 3,751 unique external names with consumers. The unchanged graph retains all 126,760 selected-consumer occurrences, 829 reachability witnesses and 205 reduced edges. Provider pins and full-graph identity are available there.

Source and qualification

Source edition. Source GSS v0.5.0 is fixed at `ac405356bb98afed954da9b9c795d23cc8d5d149`. The legacy published LFH v0.4.0 / R09 binding remains separate. Checkpoint-assisted qualification is inherited from run 38063627833, attempt 1. Independent clean single-runner qualification is NOT_RUN.

This edition contains 69 exact Card references and their source-bound expositions. The website edition, source releases and Card revisions are separate version axes. Whole-manuscript correspondence is not asserted.

Original paper, fixed v1 · Original v1 PDF · Saved graph and witnesses · edition record

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Geometric structure spaces and C*-algebras

How much of a C^*-algebra is determined by the geometry of its unit ball? The geometric structure space connects this geometry with irreducible representations and primitive ideals. Further results compare it with the geometric structure spaces of operator algebras and, under additional hypotheses, determine the algebra up to $*$-isomorphism.

From norm geometry to closure spaces

Let E be a nonzero real or complex normed space, with closed unit ball B_E. For each maximal proper face F of B_E, let I_F be the union of the kernels of all norm-one bounded linear functionals supporting F. Faces are taken in the underlying real vector space. The family $S(E)$ of these subsets is equipped with the hull–kernel closure

$$c(\mathcal{A}) = \left\{ I \in S(E) : \bigcap_{J \in \mathcal{A}} J \subseteq I \right\}, \qquad \mathcal{A} \subseteq S(E).$$

As usual, the intersection of the empty family is E. For the zero space, $S(E)$ is empty.

This closure need not come from a topology. Two distinct points are linked if the closure of the pair contains a third point, and components are defined by finite chains of linked points. A pairwise linked set, by contrast, requires every pair of distinct points to be linked. The geometric spectrum is the set of components. The primitive geometric spectrum records the intersections of the subsets belonging to each component.

For closure spaces X and Y, a map $f : X \to Y$ is finitely continuous if

$$f(c_X(F)) \subseteq c_Y(f(F))$$

for every finite subset F of X. Requiring this for every subset gives continuity. A finite homeomorphism is a bijection such that both it and its inverse are finitely continuous. There is no finiteness assumption on X or Y.

Faces, states and spectra

Let A be a nonzero C^*-algebra, not necessarily unital, and let A^* be its space of bounded linear functionals. The facial correspondence identifies $S(A)$ with the family of kernels of the extreme points of B_{A^*}. The pure-state translation theorem expresses these functionals as unitary translates of pure states, using the unitization $\widetilde{A} = \mathbb{C} \oplus A$.

A nonzero irreducible $*$-representation $\pi : A \to \mathcal{B}(H_\pi)$ determines a component consisting of its matrix-coefficient kernels. Their intersection is $\ker \pi$. The spectral comparison theorems identify the two geometric spectra with the irreducible spectrum, which distinguishes representations up to

unitary equivalence, and the primitive spectrum, which records their kernels. The spectral comparison maps are homeomorphisms for the corresponding hull–kernel topologies. The construction of the irreducible spectrum uses the GNS representations of pure states.

There is also a comparison with representation images. Given a nonzero $*$-representation $\pi : A \to \mathcal{B}(H)$ and a component C of $S(A)$ satisfying

$$\bigcap_{I \in C} I = \ker \pi,$$

it identifies the geometric structure space of $\pi(A)$ with the closure of C in $S(A)$. No faithfulness or irreducibility assumption on π is needed here. The spectral transport construction then uses the spectral comparisons to obtain maps between spectra from homeomorphisms of geometric structure spaces.

Normal parts and continuity

Let $A_1 \subseteq \mathcal{B}(H)$ be a complex $*$-subalgebra on a complex Hilbert space H. Its normal part consists of those points of $S(A_1)$ that are kernels of matrix coefficients

$$a \longmapsto \langle y, a(x) \rangle, \qquad x, y \in H \setminus \{0\}.$$

For norm-closed algebras whose inclusion representation is nonzero and irreducible, this family is a component. The projective parametrization records the lines through x and y; it is a bijection, not a claim that the normal part has the usual product topology on two projective spaces.

Semilinear changes of the two vectors give finite homeomorphisms. The two changes use the same involutive field automorphism, commuting with complex conjugation. Full continuity requires further conditions on their action on operators. The subsequent factorization and regularity theorems state the dimension and continuity hypotheses needed for those conclusions.

Component tests and compact models

One comparison condition asks for a faithful irreducible representation whose matrix-coefficient kernels exhaust $S(A)$. The project also studies continuity under conjugation by finite homeomorphisms. The latter condition specifies which maps are assumed continuous and requires continuity of both the conjugate and its inverse.

Comparisons with the geometric structure space of $\mathcal{K}(H)$ distinguish continuity in the forward direction from continuity in both directions. The finite-dimensional version uses a nonzero finite-dimensional complex Hilbert space. Prescribing its dimension and giving an upper bound for that dimension are different conditions.

The representation-theoretic results likewise distinguish images containing $\mathcal{K}(H)$, images equal to $\mathcal{K}(H)$, and finite-dimensional images. The Glimm–Sakai theorem is used when the compact-operator inclusion condition is expressed in terms of type I; it is not part of the definitions of the geometric comparisons.

Compact representations and rigidity

A faithful representation by compact operators, compact-operator images of all irreducible representations, and discreteness of the primitive spectrum are related by the characterization theorems. They are not interchangeable definitions.

For an algebra with a faithful representation by compact operators, the canonical decomposition gives a $*$-isomorphism

$$A \cong \bigoplus_{i \in I}^{c_0} \mathcal{K}(H_i).$$

Here I indexes the minimal nonzero closed ideals of A, and the Hilbert spaces H_i are obtained from the given representation. The c_0-sum is a sum of operator algebras, not the algebra of all compact operators on a Hilbert direct sum. The construction includes the zero algebra, for which the index set is empty.

The rigidity theorems establish the existence of a $*$-isomorphism under their respective compact-operator, elementary, finite-dimensional or $\mathcal{B}(H)$-model hypotheses. They do not assert that every given geometric homeomorphism is induced by that isomorphism. In the last case, an isomorphism with a concrete algebra $\mathcal{B}(H)$ is part of the model assumption; passage from an abstract type-I factor uses a separate classification theorem.

Mathematical background

The main tools are the Hahn–Banach and Krein–Milman theorems, weak* compactness, the GNS construction, the correspondence between pure states and irreducible representations, and simultaneous interpolation. Representation images are treated through the quotient by the kernel. The individual Cards give the definitions, statements and references needed to follow these connections.

Closure spaces before topology

`ClosureSpace`

structure

Closure spaces before topology

Statement

A closure space is a set with an extensive, monotone and idempotent operation on its subsets that sends the empty set to the empty set. Preservation of finite unions is a separate condition.

Definition

Let X be a set and let $\mathcal{P}(X)$ denote its power set, ordered by inclusion. A closure space on X consists of a map $c : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying, for all $A, B \subseteq X$,

$$\begin{aligned} A &\subseteq c(A), \\ A \subseteq B &\Longrightarrow c(A) \subseteq c(B), \\ c(c(A)) &= c(A), \\ c(\varnothing) &= \varnothing. \end{aligned}$$

A subset A is **closed** when $c(A) = A$. The first three axioms say that c is a closure operator on the ordered set $\mathcal{P}(X)$; the fourth supplies the empty-set condition.

The cited intersection theorem gives the useful equivalent description

$$c(A) = \bigcap \{ B \subseteq X : A \subseteq B \text{ and } c(B) = B \}.$$

Indeed, $c(A)$ is closed and contains A, and monotonicity puts it inside every closed superset of A. Thus $c(A)$ is the smallest closed set containing A.

Assumptions

The underlying set X is arbitrary and may be empty. To specify a closure space, one supplies the map c and the four displayed properties. No topology, metric or separation condition on X is required.

Conclusion

The resulting structure provides a closure operation and a notion of closed subset, with arbitrary intersections of closed sets again closed. A topological closure requires the additional identity

$$c(A \cup B) = c(A) \cup c(B).$$

This extra condition is called topologizability in the cited source. It is not one of the four axioms above.

Main citations

- Exact structure and closure laws: `ClosureSpace`, lines 28–30; its operation and laws, lines 37–61.
- Closed intersections and closure minimality: `ClosureSpace.closure_min`, `ClosureSpace.isClosed_sInter` and `ClosureSpace.closure_eq_sInter`, lines 78–99.
- The additional finite-union condition: `ClosureSpace.IsTopologizable`, lines 102–103.
- The underlying order-theoretic structure: Mathlib `ClosureOperator` in `Order/Closure.lean`, at the pinned mathlib commit.
- Ryotaro Tanaka, arXiv:2504.05551v1, Definition 3.1 and Remark 3.4: the four axioms and the intersection description.

Lean source signature (exact)

```
structure ClosureSpace (α : Type u) where
  toClosureOperator : ClosureOperator (Set α)
  map_empty' : toClosureOperator ∅ = ∅
```

In the source	Mathematical meaning
Exact declaration `α : Type u, Set α`	An arbitrary set X and its power set $\mathcal{P}(X)$; neither finiteness nor nonemptiness is required.
Exact declaration `toClosureOperator : ClosureOperator (Set α)`	The operation $c : \mathcal{P}(X) \to \mathcal{P}(X)$, with $A \subseteq c(A)$, $A \subseteq B \Rightarrow c(A) \subseteq c(B)$, and $c(c(A)) = c(A)$ for all subsets A, B. Its closed-set predicate agrees with $c(A) = A$.
Exact declaration `map_empty'`	The additional equality $c(\varnothing) = \varnothing$.
Exact declaration `ClosureSpace α`	The returned structure consists of this operation and all four laws. It does not include preservation of finite unions.

Exact declaration for the source guide

```
structure ClosureSpace (α : Type u) where
  toClosureOperator : ClosureOperator (Set α)
  map_empty' : toClosureOperator ∅ = ∅

namespace ClosureSpace

variable {α : Type u}

/-- The underlying set closure. -/
```

ClosureOperator · fixed Mathlib definition

Fixed source, lines 60–70 · Raw UTF-8 source

```
structure ClosureOperator [Preorder α] extends α →o α where
  /-- An element is less than or equal its closure -/
  le_closure' : ∀ x, x ≤ toFun x
  /-- Closures are idempotent -/
  idempotent' : ∀ x, toFun (toFun x) = toFun x
  /-- Predicate for an element to be closed.

  By default, this is defined as `c.IsClosed x := (c x = x)` (see `isClosed_iff`).
  We allow an override to fix definitional equalities. -/
  IsClosed (x : α) : Prop := toFun x = x
  isClosed_iff {x : α} : IsClosed x ↔ toFun x = x := by aesop
```

Closed hyperplane images force semilinear regularity

`LinearEquiv.regularity_of_isClosed_image_hyperplane`

theorem

Derives scalar regularity and bounded invertibility from closed images of closed hyperplanes.

Statement

Let X, Y be complex Banach spaces, with X infinite-dimensional. Let $\sigma : \mathbb{C} \to \mathbb{C}$ be a field automorphism and $T : X \to Y$ an algebraic σ-semilinear equivalence: T is bijective and

$$T(x + x') = Tx + Tx', \qquad T(cx) = \sigma(c)Tx. \tag{1}$$

Suppose that $T(M)$ is closed in Y for every closed complex hyperplane $M \subseteq X$. Then σ is the identity or complex conjugation. The same T and its inverse T^{-1} are continuous, and there exist real constants $C, D > 0$ such that

$$\|Tx\| \le C\|x\| \quad (x \in X), \qquad \|T^{-1}y\| \le D\|y\| \quad (y \in Y). \tag{2}$$

The scalar alternative and both continuity assertions hold together with this one pair of bounds.

Assumptions

Here a closed complex hyperplane is a closed complex-linear subspace of codimension one. The assumption requires its image under T to be closed.

Both X and Y are Banach spaces; only X is required to be infinite dimensional. No separability, Hilbert-space structure, or initial continuity or boundedness of σ, T, or T^{-1} is assumed.

Conclusion

The given semilinear equivalence is complex-linear or conjugate-linear and is boundedly invertible. The conclusion includes continuity of T and T^{-1} and one pair of positive constants satisfying both estimates (2). It does not require those constants to be optimal or equal.

Main citations

The closed-hyperplane regularity theorem gives the scalar alternative, both continuity assertions and both bounds. In the same file, scalar continuity, continuity of the equivalence and its inverse, and positive norm bounds are stated separately.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem regularity_of_isClosed_image_hyperplane
    (T : LinearEquiv σ.toRingHom (σ' := σ.symm.toRingHom) X Y) (hX : ¬ FiniteDimensional ℂ X)
    (hclosed : ∀ M : Submodule ℂ X, IsCoatom M → IsClosed (M : Set X) →
      IsClosed (T '' (M : Set X))) :
    (σ = RingEquiv.refl ℂ ∨ σ = starRingAut (R := ℂ)) ∧
      Continuous T ∧ Continuous T.symm ∧
      ∃ C D : ℝ, 0 < C ∧ 0 < D ∧
        (∀ x, ‖T x‖ ≤ C * ‖x‖) ∧ (∀ y, ‖T.symm y‖ ≤ D * ‖y‖)
```

In the source	Mathematical meaning
Exact declaration `X`, `Y`, their normed-space instances, and `[CompleteSpace X] [CompleteSpace Y]`	The complex Banach spaces X, Y.
Exact declaration `{σ : ℂ ≃+* ℂ}`	A field automorphism $\sigma : \mathbb{C} \to \mathbb{C}$, not assumed continuous.
Exact declaration `T : LinearEquiv σ.toRingHom (σ' := σ.symm.toRingHom) X Y`	The additive bijection $T : X \to Y$ with $T(cx) = \sigma(c)T(x)$; its inverse satisfies $T^{-1}(cy) = \sigma^{-1}(c)T^{-1}(y)$.
Exact declaration `hX : ¬ FiniteDimensional ℂ X`	$\dim_{\mathbb{C}} X = \infty$.
Exact declaration `M : Submodule ℂ X`	A complex-linear subspace $M \subseteq X$. This part alone does not require closedness.
Exact declaration `IsCoatom M`	$\dim_{\mathbb{C}}(X/M) = 1$: M is a complex hyperplane.
Exact declaration `IsClosed (M : Set X)`	M is closed in X. Together with the preceding condition, this specifies a closed complex hyperplane.
Exact declaration `hclosed`, ending in `IsClosed (T '' (M : Set X))`	$T(M)$ is closed in Y for every closed complex hyperplane $M \subseteq X$.
Exact declaration `σ = RingEquiv.refl ℂ ∨ σ = starRingAut (R := ℂ)`	$\sigma = \mathrm{id}_{\mathbb{C}}$ or $\sigma(c) = \overline{c}$ for every $c \in \mathbb{C}$.
Exact declaration `Continuous T ∧ Continuous T.symm`	Both $T : X \to Y$ and $T^{-1} : Y \to X$ are norm-continuous.
Exact declaration `∃ C D : ℝ, 0 < C ∧ 0 < D ∧ ...`	There are $C, D > 0$ such that $\|T(x)\| \le C\|x\|$ for every $x \in X$, and $\|T^{-1}(y)\| \le D\|y\|$ for every $y \in Y$, as in (2).

In the source	Mathematical meaning
The two local **Exact declaration** `RingHomInvPair` instances	$\sigma^{-1} \circ \sigma = \mathrm{id}_{\mathbb{C}}$ and $\sigma \circ \sigma^{-1} = \mathrm{id}_{\mathbb{C}}$; no assumption $\sigma^2 = \mathrm{id}_{\mathbb{C}}$ is made.

The literal signature above is the declaration prefix through line 298, ending before the proof assignment. The full declaration occupies lines 291–302; the comment at 304–305 belongs to the next theorem. The ambient variables and completeness assumptions are quoted separately below.

Exact surrounding source context (separate excerpts)

Source lines 35–39:

```
universe u v
variable {X : Type u} {Y : Type v}
variable [NormedAddCommGroup X] [NormedSpace ℂ X]
variable [NormedAddCommGroup Y] [NormedSpace ℂ Y]
variable {σ : ℂ ≃+* ℂ}
```

Source lines 41–44:

```
local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair σ.toRingHom σ.symm.toRingHom :=
  RingHomInvPair.of_ringEquiv σ
local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair σ.symm.toRingHom σ.toRingHom :=
  RingHomInvPair.of_ringEquiv_symm σ
```

Source lines 237–237:

```
variable [CompleteSpace X] [CompleteSpace Y]
```

Exact declaration for the source guide

```
theorem regularity_of_isClosed_image_hyperplane
    (T : LinearEquiv σ.toRingHom (σ' := σ.symm.toRingHom) X Y) (hX : ¬ FiniteDimensional ℂ X)
    (hclosed : ∀ M : Submodule ℂ X, IsCoatom M → IsClosed (M : Set X) →
      IsClosed (T '' (M : Set X))) :
    (σ = RingEquiv.refl ℂ ∨ σ = starRingAut (R := ℂ)) ∧
      Continuous T ∧ Continuous T.symm ∧
      ∃ C D : ℝ, 0 < C ∧ 0 < D ∧
        (∀ x, ‖T x‖ ≤ C * ‖x‖) ∧ (∀ y, ‖T.symm y‖ ≤ D * ‖y‖) := by
```

The dependent co-direct sum

`MathlibAnnex.Analysis.CStarAlgebra.C0DirectSum`

abbrev

Names the algebra of bounded families that vanish at infinity.

Statement

Let ι be any index set and let A_i be a complex C*-algebra for each $i \in \iota$, with no identity required. The dependent c_0-direct sum is the algebra of families $x = (x_i)$ in the bounded product satisfying the vanishing condition below.

Definition

The ambient bounded product consists of families $x_i \in A_i$ with $\sup_i \|x_i\| < \infty$, equipped with its supremum norm. Write

$$\bigoplus_{i \in \iota}^{c_0} A_i = \{x : \forall \varepsilon > 0,\ \{i : \varepsilon \le \|x_i\|\} \text{ is finite}\}$$

inside that bounded product. Operations are coordinatewise, and the norm is inherited. This subalgebra is norm closed, with the inherited C*-algebra structure.

Assumptions

Each fiber has its given complex C*-algebra structure. No identity, countability, separability, nonzero-fiber or nonempty-index assumption is imposed.

Conclusion

The abbreviation denotes this c_0 subalgebra, with the vanishing condition imposed for every positive threshold.

Main citations

- Exact complete definition.
- VanishesAtInfinity: the complete finiteness predicate.
- starSubalgebra: the carrier and algebra fields.
- isClosed_starSubalgebra: norm closedness.
- nonUnitalCStarAlgebra: the inherited structure.

Lean source signature (exact)

The following is the complete defining right-hand side, copied from the fixed source.

```
abbrev C0DirectSum {ι : Type uι} (A : ι → Type uA)
    [∀ i, NonUnitalCStarAlgebra (A i)] :=
  C0DirectSum.starSubalgebra A
```

In the source	Mathematical meaning
Exact declaration `{ι : Type uι} (A : ι → Type uA)`	An arbitrary index set ι and its family of algebras $(A_i)_{i\in\iota}$.
Exact declaration `[∀ i, NonUnitalCStarAlgebra (A i)]`	Each fiber carries the specified complex C*-structure.
Exact declaration `C0DirectSum.starSubalgebra A`	The subalgebra $\bigoplus_i^{c_0} A_i$ of bounded families: $\{i : \varepsilon \le \|x_i\|\}$ is finite for every $\varepsilon > 0$. Its operations are coordinatewise and its norm is the supremum norm.

Exact surrounding source context (separate excerpts)

Source lines 34–36:

```
namespace MathlibAnnex.Analysis.CStarAlgebra

universe uι uA uH
```

The irreducible spectrum as pure-GNS representation classes

`MathlibAnnex.Analysis.CStarAlgebra.IrreducibleSpectrum`

def

Defines the irreducible spectrum using unitary equivalence classes of pure-state GNS representations.

Statement

Let A be a complex C*-algebra with compatible order, not necessarily unital or nonzero. The irreducible spectrum here is the quotient of its pure states by unitary equivalence of their GNS representations of A.

Definition

A pure state is a positive norm-one bounded complex-linear functional $\varphi : A \to \mathbb{C}$, extreme among such functionals for real convexity. Let π_φ denote the GNS representation of its canonical extension to the forced unitization, restricted to the original algebra A. The pure-state GNS irreducibility theorem guarantees irreducibility of this action.

Write $\varphi \sim \psi$ when π_φ and π_ψ are unitarily equivalent: there is a surjective complex-linear isometry W between their Hilbert spaces such that $W\pi_\varphi(a) = \pi_\psi(a)W$ for every $a \in A$. The definition is

$$\widehat{A} = \{\varphi : \varphi \text{ is a pure state of } A\}/\sim.$$

The quotient uses this unitary-equivalence relation. Equality of representation kernels is not substituted for unitary equivalence. The definition assigns no topology; the later ordinary spectrum topology is induced through the representation-kernel map, not specified here as a pure-state quotient topology.

Assumptions

Only the complex C*-algebra and its compatible order are required. The set of pure states may be empty. No auxiliary Hilbert space or separability assumption is required.

Conclusion

The result is the set of unitary equivalence classes of pure-state GNS representations. The definition permits an empty quotient when there are no pure-state representatives.

Main citations

- Exact complete definition.
- NonUnitalPureState: the exact pure-functional subtype.
- gnsRepresentation: restriction to the original algebra.
- isIrreducible_gnsRepresentation: the irreducibility guarantee.
- pureStateGNSSetoid: the entire relation and equivalence fields.

Lean source signature (exact)

The following is the complete defining right-hand side, copied from the fixed source.

```
def IrreducibleSpectrum := Quotient (pureStateGNSSetoid A)
```

In the source	Mathematical meaning
Exact declaration `NonUnitalPureState A` inside the setoid	The pure states of A defined above: positive norm-one bounded complex-linear functionals that are extreme among such functionals.
Exact declaration `pureStateGNSSetoid A`	The relation $\varphi \sim \psi$ of unitary equivalence between π_φ and π_ψ.
Exact declaration `Quotient (pureStateGNSSetoid A)`	The quotient for precisely that relation. No kernel equivalence or topology is included in this one-line RHS.

Exact surrounding source context (separate excerpts)

Source lines 23–26:

```
namespace MathlibAnnex.Analysis.CStarAlgebra

universe u
variable (A : Type u) [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A]
```

A compact-operator model for a representation

`MathlibAnnex.Analysis.CStarAlgebra.IsCompactOperatorModel`

def

Packages faithfulness together with equality of the represented image and the compact operators.

Statement

Let A be a complex C^*-algebra, with no unit assumed, let H be a complex Hilbert space, and let $e : A \to \mathcal{B}(H)$ be a $*$-representation. The predicate below asserts that e identifies A exactly with the compact operators on H.

Definition

The three conditions are

$$e \text{ is injective}, \qquad e(A) \subseteq \mathcal{K}(H), \qquad \mathcal{K}(H) \subseteq e(A).$$

Equivalently, e is faithful and $e(A) = \mathcal{K}(H)$; hence A is $*$-isomorphic to $\mathcal{K}(H)$ through the specified representation.

Assumptions

No separability, irreducibility or singleton-spectrum hypothesis is part of this definition. A unit is not assumed.

Conclusion

The output is a property of the given representation e, not merely an assertion that some abstract $*$-isomorphism exists.

Main citations

- Exact definition and context

Lean source signature (exact)

The complete definition is copied from the pinned source.

```
/-- A nonunital star representation that is injective and has precisely the
compact operators as its range.  This is the unbundled form of a star
isomorphism with `K(H)`. -/
def IsCompactOperatorModel [NonUnitalCStarAlgebra A]
    (e : A →⋆ₙₐ[ℂ] (H →L[ℂ] H)) : Prop :=
  Function.Injective e ∧
    (∀ a : A, IsCompactOperator (e a)) ∧
    ∀ T : H →L[ℂ] H, IsCompactOperator T → ∃ a : A, e a = T
```

In the source	Mathematical meaning
Exact declaration `[NonUnitalCStarAlgebra A]`	The complex C^*-algebra A; a unit is not required.
Exact declaration `(e : A →⋆ₙₐ[ℂ] (H →L[ℂ] H))`	The given $*$-representation $e : A \to \mathcal{B}(H)$.
Exact declaration `Function.Injective e`	Equal represented operators imply equal algebra elements.
Exact declaration `(∀ a : A, IsCompactOperator (e a))`	Every element a of A is sent to a compact operator; this is $e(A) \subseteq \mathcal{K}(H)$.
Exact declaration `∀ T : H →L[ℂ] H, IsCompactOperator T → ∃ a : A, e a = T`	Choose any bounded operator T; if it is compact, choose an element a of the original algebra whose image is exactly T. This is the opposite inclusion.
Exact declaration `: Prop`	The conjunction of injectivity and the two inclusions $e(A) \subseteq \mathcal{K}(H)$ and $\mathcal{K}(H) \subseteq e(A)$.

Exact surrounding source context (separate excerpts)

```
universe u v

variable {A : Type u}
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```

The primitive spectrum as representation kernels

`MathlibAnnex.Analysis.CStarAlgebra.PrimitiveSpectrum`

def

Defines the primitive spectrum as the set of pure-state GNS representation kernels.

Statement

Let A be a complex C*-algebra with compatible order, not necessarily unital or nonzero. The primitive spectrum here is the set of subsets of A occurring as kernels of pure-state GNS representations.

Definition

A pure state is a bounded complex-linear functional $\varphi : A \to \mathbb{C}$ that is positive, has norm one, and is extreme among positive norm-one functionals, for real convexity. Its canonical state extension to the forced unitization has a GNS representation; π_φ denotes the restriction of that representation to A.

$$\mathrm{Prim}(A) = \{\ker \pi_\varphi : \varphi \text{ is a pure state of } A\}.$$

A point is the representation kernel itself, not a preferred pure state. The ideal identification regards it as a two-sided ideal of A. This definition assigns no topology; the spectrum topology is defined separately.

Assumptions

The source requires a nonunital complex C*-algebra, a partial order and compatible order. No nontriviality, separability, faithfulness, finite dimension or chosen representation is an additional input.

Conclusion

Points are representation kernels, regarded as subsets of A. Pure states with the same GNS kernel determine the same point. This is not the quotient by unitary equivalence, nor is it the family of functional kernels.

Main citations

- Exact complete definition.
- primitiveKernelFamily: the full range expression.
- NonUnitalPureState: the pure-state subtype.
- gnsRepresentation: the action of the original algebra.
- coe_toTwoSidedIdeal: the ideal-carrier guarantee.

Lean source signature (exact)

The following is the complete defining right-hand side, copied from the fixed source.

```
def PrimitiveSpectrum := {P : Set A // P ∈ primitiveKernelFamily A}
```

In the source	Mathematical meaning
Exact declaration `A` and the surrounding three instances	The complex C*-algebra with the order used for positivity.
Exact declaration `{P : Set A // ...}`	A subset of the original algebra together with membership in the indicated family.
Exact declaration `P ∈ primitiveKernelFamily A`	$P = \ker \pi_\varphi$ for some pure state φ of A.
The complete one-line RHS **Exact declaration**	No quotient of representations and no topological data are fields of this definition.

Exact surrounding source context (separate excerpts)

Source lines 22–26:

```
namespace MathlibAnnex.Analysis.CStarAlgebra

universe u v
variable (A : Type u) [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A]
open NonUnitalCStarRepresentation
```

An extreme dual functional has a pure-state unitary translate

`MathlibAnnex.Analysis.CStarAlgebra.exists_pureState_unitaryTranslate_of_mem_extremePoints_dual_closedBall`

theorem

Expresses each extreme dual functional as a unitary translate of a pure state.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity. Let A^* be the space of bounded complex-linear functionals on A, and let $B_{A^*} = \{\rho : \|\rho\| \leq 1\}$. Extremality is taken for real convex combinations. Write $\widetilde{A} = \mathbb{C} \oplus A$ for the forced C*-unitization and identify A with its canonical ideal in $\widetilde{A}$.

For a unitary $u \in \widetilde{A}$, left multiplication preserves this ideal, and defines the dual translate

$$(T_u\rho)(a) = \rho(ua) \qquad (a \in A).$$

If $\rho \in \mathrm{ext}_{\mathbb{R}}(B_{A^*})$, there are a pure state φ on A and a unitary $u \in \widetilde{A}$ such that

$$T_u\rho = \varphi, \qquad \rho = T_{u^*}\varphi. \tag{1}$$

A pure state is a positive norm-one bounded complex-linear functional that is extreme, for real convexity, in the state space. The two identities in (1) concern the same φ and u.

Assumptions

The nonzero-algebra hypothesis ensures that an extreme element of the dual unit ball has norm one. The source also fixes the compatible C*-order used to express positivity. No identity in A, separability or finite dimension is required. The unitary belongs to the forced unitization even when A happens to have an identity.

Conclusion

The same pure state φ and unitary u satisfy both functional equalities in (1). Positivity, norm one and purity concern φ on the original algebra. The assertion does not require or assert uniqueness of the unitary.

Main citations

- Exact decomposition declaration: `exists_pureState_unitaryTranslate_of_mem_extremePoints_dual_closedBall`.

- Ryotaro Tanaka, arXiv:2504.05551v1, §5: the paper's facial and pure-state comparison. The forced-unitization and original-algebra coefficient construction here follow the pinned source.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem exists_pureState_unitaryTranslate_of_mem_extremePoints_dual_closedBall
    {rho : StrongDual ℂ A}
    (hrho : rho ∈ (closedBall (0 : StrongDual ℂ A) 1).extremePoints ℝ) :
    ∃ phi : StrongDual ℂ A, IsPureNonUnitalState A phi ∧
      ∃ u : unitary (Unitization ℂ A),
        unitaryTranslate u rho = phi ∧ rho = unitaryTranslate (star u) phi
```

In the source	Mathematical meaning
Exact declaration `{A : Type u}, [NonUnitalCStarAlgebra A], [Nontrivial A]`	The nonzero complex C*-algebra. An identity is not assumed.
Exact declaration `[PartialOrder A], [StarOrderedRing A]`	The compatible C*-order needed for positive functionals and states.
Exact declaration `rho : StrongDual ℂ A`	The bounded complex-linear functional ρ on the original algebra.
Exact declaration `hrho : rho ∈ (closedBall ... 1).extremePoints ℝ`	Extremality in the entire norm-closed dual unit ball, with real convexity. It implies norm one here.
Exact declaration `∃ phi : StrongDual ℂ A, ...`	One bounded complex-linear functional φ is selected in the same dual space as ρ. Positivity is supplied by the following predicate, not by this type alone.
Exact declaration `IsPureNonUnitalState A phi`	Extremality among positive norm-one functionals, including membership in that state space. It supplies positivity and norm one for the same φ.
Exact declaration `∃ u : unitary (Unitization ℂ A), ...`	One unitary in the forced unitization is selected after the pure state; it need not be an element of A.
Exact declaration `unitaryTranslate u rho = phi`	The equality of the whole functionals $T_u\rho = \varphi$, hence equality at every $a \in A$.
The equality with **Exact declaration** `star u`	The whole inverse-translation equality $\rho = T_{u^*}\varphi$, using the same φ and u in both equalities.

Exact surrounding source context (separate excerpts)

Source lines 35–35:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Source lines 56–56:

```
variable [PartialOrder A] [StarOrderedRing A]
```

Exact declaration for the source guide

```
theorem exists_pureState_unitaryTranslate_of_mem_extremePoints_dual_closedBall
    {rho : StrongDual ℂ A}
    (hrho : rho ∈ (closedBall (0 : StrongDual ℂ A) 1).extremePoints ℝ) :
    ∃ phi : StrongDual ℂ A, IsPureNonUnitalState A phi ∧
      ∃ u : unitary (Unitization ℂ A),
        unitaryTranslate u rho = phi ∧ rho = unitaryTranslate (star u) phi := by
```

Unions of kernels of supporting functionals

`NormedSpace.supportKernelUnion`

def

Unions of kernels of supporting functionals

Statement

For a subset of a seminormed space, consider the contractive continuous linear functionals that equal 1 throughout that subset. The support-kernel union consists of the vectors annihilated by at least one of these functionals. It is a union of kernels, with no assertion that the union is a vector subspace.

Definition

Let E be a seminormed space over a real-or-complex scalar field $\mathbb{K}$, and let E^* be its continuous $\mathbb{K}$-linear dual with the operator norm. For $M \subseteq E$, put

$$\Sigma_{\mathbb{K}}(M) = \{f \in E^* : \|f\| \le 1,\ f|_M = 1\},$$

where $f|_M = 1$ means $f(x) = 1$ for every $x \in M$. Define

$$\begin{aligned} I_{\mathbb{K}}(M) &= \bigcup_{f \in \Sigma_{\mathbb{K}}(M)} \ker f \\ &= \{y \in E : \exists f \in \Sigma_{\mathbb{K}}(M),\ f(y) = 0\}. \end{aligned}$$

Thus membership requires one witnessing functional. It does not require all supporting functionals to vanish. For nonempty M, the condition $f|_M = 1$ is equivalent to $f(M) = \{1\}$. The pointwise formulation also covers $M = \varnothing$.

The cited elementary identities make the empty cases explicit:

$$\begin{aligned} \Sigma_{\mathbb{K}}(M) = \varnothing &\Longrightarrow I_{\mathbb{K}}(M) = \varnothing, \\ 0 \in I_{\mathbb{K}}(M) &\Longleftrightarrow \Sigma_{\mathbb{K}}(M) \neq \varnothing, \\ I_{\mathbb{K}}(\varnothing) &= E. \end{aligned}$$

The last equality follows because the zero functional belongs to $\Sigma_{\mathbb{K}}(\varnothing)$ and has kernel E.

Assumptions

The space E may have a seminorm: definiteness and completeness are not required. The subset M is arbitrary. It need not be nonempty, convex or contained in the unit ball, and the definition does not assume that a supporting functional exists.

Conclusion

The output is the subset $I_{\mathbb{K}}(M)$ of E. For each $x \in M$, a supporting functional has $f(x) = 1$, so it cannot also annihilate x. Consequently

$$M \cap I_{\mathbb{K}}(M) = \varnothing, \qquad M \neq \varnothing \Longrightarrow I_{\mathbb{K}}(M) \neq E.$$

This properness result, proved in the cited source, is what later permits the hull–kernel construction for maximal faces.

Main citations

- The supporting family and its nonempty image characterization: `NormedSpace.supportFunctionals`, lines 42–43; the equivalence for nonempty subsets is at lines 59–78.
- Exact union and union-of-kernels formula: `NormedSpace.supportKernelUnion`, lines 108–122.
- Disjointness, properness and the empty cases: lines 130–161.
- The real-or-complex scalar interface: Mathlib `RCLike`, at the pinned mathlib commit.
- arXiv:2504.05551v1, Definition 4.1: the support-kernel union for maximal faces of a nontrivial Banach space. The present declaration allows arbitrary subsets of seminormed spaces.

Lean source signature (exact)

The exact definition uses the scalar and space declarations reproduced separately below.

```
def supportKernelUnion (F : Set E) : Set E :=
  {x | ∃ f ∈ supportFunctionals 𝕜 F, f x = 0}
```

In the source	Mathematical meaning
Exact declaration `𝕜, [RCLike 𝕜]`	The scalar field $\mathbb{K}$ is real or complex.
Exact declaration `E, [SeminormedAddCommGroup E], [NormedSpace 𝕜 E]`	E is a seminormed vector space over $\mathbb{K}$; completeness and a nonzero vector are not assumed.
Exact declaration `F : Set E`	The arbitrary subset $M \subseteq E$, possibly empty.

In the source	Mathematical meaning
Exact declaration `supportFunctionals 𝕜 F`	$\Sigma_{\mathbb{K}}(M) = \{f \in E^* : \|f\| \le 1,\ f(x) = 1 \text{ for every } x \in M\}$, where E^* is the continuous linear dual (`StrongDual 𝕜 E`).
Exact declaration `{x | ∃ f ∈ supportFunctionals 𝕜 F, f x = 0}`	The full defining subset $I_{\mathbb{K}}(M) = \{x \in E : \exists f \in \Sigma_{\mathbb{K}}(M),\ f(x) = 0\}$. For each vector x, one functional must satisfy both supporting conditions and annihilate x.
Exact declaration `: Set E`	The output is a subset of E, the union of these kernels. It is not bundled as a linear subspace, and the supporting family need not be nonempty.

Source context (separate exact excerpt)

The surrounding declarations below are from lines 26–27; their mathematical meaning is included in the guide above.

```
variable (𝕜 : Type v) [RCLike 𝕜]
variable {E : Type u} [SeminormedAddCommGroup E] [NormedSpace 𝕜 E]
```

Exact declaration for the source guide

```
def supportKernelUnion (F : Set E) : Set E :=
  {x | ∃ f ∈ supportFunctionals 𝕜 F, f x = 0}

@[simp]
```

RCLike · fixed Mathlib definition

Fixed source, lines 60–84 · Raw UTF-8 source

```
class RCLike (K : semiOutParam Type*) extends DenselyNormedField K, StarRing K,
    NormedAlgebra ℝ K, CompleteSpace K where
  /-- The real part as an additive monoid homomorphism -/
  re : K →+ ℝ
  /-- The imaginary part as an additive monoid homomorphism -/
  im : K →+ ℝ
  /-- Imaginary unit in `K`. Meant to be set to `0` for `K = ℝ`. -/
  I : K
  I_re_ax : re I = 0
  I_mul_I_ax : I = 0 ∨ I * I = -1
  re_add_im_ax : ∀ z : K, 𝓚 (re z) + 𝓚 (im z) * I = z
  ofReal_re_ax : ∀ r : ℝ, re (𝓚 r) = r
  ofReal_im_ax : ∀ r : ℝ, im (𝓚 r) = 0
  mul_re_ax : ∀ z w : K, re (z * w) = re z * re w - im z * im w
  mul_im_ax : ∀ z w : K, im (z * w) = re z * im w + im z * re w
  conj_re_ax : ∀ z : K, re (conj z) = re z
  conj_im_ax : ∀ z : K, im (conj z) = -im z
  conj_I_ax : conj I = -I
  norm_sq_eq_def_ax : ∀ z : K, ‖z‖ ^ 2 = re z * re z + im z * im z
  mul_im_I_ax : ∀ z : K, im z * im I = im z
  /-- only an instance in the `ComplexOrder` scope -/
  [toPartialOrder : PartialOrder K]
  le_iff_re_im {z w : K} : z ≤ w ↔ re z ≤ re w ∧ im z = im w
  -- note we cannot put this in the `extends` clause
  [toDecidableEq : DecidableEq K]
```

Finite projective incidence determines a semilinear equivalence

`Projectivization.exists_semilinearEquiv_of_finite_span`

theorem

Realizes a finite-incidence-preserving projective bijection by a semilinear equivalence.

Statement

Let K, L be fields, let V be a K-vector space, and let W be an L-vector space. Write $[v] = Kv$ for $v \in V \setminus \{0\}$ and $[w] = Lw$ for $w \in W \setminus \{0\}$. These are the points of the projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(W)$, respectively. For $D \subseteq \mathbb{P}(V)$, let $\langle D \rangle_{\mathrm{p}}$ be the set of lines in the algebraic linear span of representatives of D. In particular, $\langle \varnothing \rangle_{\mathrm{p}} = \varnothing$.

Suppose $e : \mathbb{P}(V) \to \mathbb{P}(W)$ is a bijection and

$$p \in \langle D \rangle_{\mathrm{p}} \quad \Longleftrightarrow \quad e(p) \in \langle e(D) \rangle_{\mathrm{p}} \qquad (D \text{ finite},\ p \in \mathbb{P}(V)). \tag{1}$$

If the cardinal-valued algebraic dimension of V over K is at least three, there exist a field isomorphism $\sigma : K \to L$ and an additive bijection $U : V \to W$ such that

$$U(av) = \sigma(a) U(v), \qquad e([v]) = [U(v)] \quad (a \in K,\ v \in V \setminus \{0\}). \tag{2}$$

The scalar identity also holds at $v = 0$. Thus one semilinear equivalence realizes the prescribed projective bijection on every projective point.

Assumptions

Incidence is both preserved and reflected for every finite family, not merely for the chosen basis rays. The dimension assumption means that three linearly independent vectors exist; it does not assert finite dimensionality. The spaces need no norms, inner products, or topology, and the two fields need not coincide. No assumption on the characteristic is made.

Conclusion

This is a version of the fundamental theorem of projective geometry.

The same pair (σ, U) works for all nonzero representatives. The semilinear equivalence includes its σ^{-1}-semilinear inverse $U^{-1} : W \to V$. In particular,

$$e^{-1}([w]) = [U^{-1}(w)] \qquad (w \in W \setminus \{0\}). \tag{3}$$

The declaration asserts existence, not uniqueness of the choices. It imposes no involution, compatibility with an involution, or continuity condition on σ.

Main citations

The finite-incidence realization theorem gives one field isomorphism and one semilinear equivalence realizing every projective point. The semilinear equivalence construction includes the inverse over the inverse field isomorphism.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem exists_semilinearEquiv_of_finite_span
    (e : ℙ K V ≃ ℙ L W)
    (he : ∀ (T : Set (ℙ K V)), T.Finite → ∀ p,
      p ∈ Subspace.span T ↔ e p ∈ Subspace.span (e '' T))
    (hdim : 3 ≤ Module.rank K V) :
    ∃ σ : K ≃+* L, ∃ U : V ≃ₛₗ[σ.toRingHom] W,
      ∀ (v : V) (hv : v ≠ 0), e (mk K v hv) =
        mk L (U v) (by simpa only [map_zero] using U.injective.ne hv)
```

In the source	Mathematical meaning
Exact declaration `K`, `L`, `V`, `W` and their ambient instances	Two fields and their vector spaces; neither finite dimensionality nor an analytic structure is present.
Exact declaration `e : ℙ K V ≃ ℙ L W`	The prescribed projective bijection, not a lift constructed in advance.
Exact declaration `he`, with `T.Finite` and arbitrary `p`	Both directions of (1), using the algebraic projective span, for every finite $D \subseteq \mathbb{P}(V)$ and every $p \in \mathbb{P}(V)$, including $D = \varnothing$.
Exact declaration `hdim : 3 ≤ Module.rank K V`	The cardinal-valued dimension $\dim_K V$ is at least three; infinite-dimensional spaces are allowed.
Exact declaration `∃ σ : K ≃+* L, ∃ U : V ≃ₛₗ[σ.toRingHom] W`	One field isomorphism $\sigma : K \to L$ and one additive bijection $U : V \to W$ with $U(av) = \sigma(a)U(v)$. Its inverse is semilinear over σ^{-1}.
Exact declaration `∀ (v : V) (hv : v ≠ 0), e (mk K v hv) = ...`	The equality $e([v]) = [U(v)]$ for every $v \in V \setminus \{0\}$, with the same σ and U throughout. The vector $U(v)$ is nonzero, so the right-hand projective point is defined.

The following source context is separate from the declaration prefix. The later comment at lines 535–536 is not part of this declaration.

Exact surrounding source context (separate excerpts)

Source lines 38–41:

```
universe uK uL uV uW uI
variable {K : Type uK} {L : Type uL} {V : Type uV} {W : Type uW}
variable [Field K] [Field L]
variable [AddCommGroup V] [Module K V] [AddCommGroup W] [Module L W]
```

Source lines 44–47:

```
local instance (σ : K ≃+* L) : RingHomInvPair σ.toRingHom σ.symm.toRingHom :=
  RingHomInvPair.of_ringEquiv σ
local instance (σ : K ≃+* L) : RingHomInvPair σ.symm.toRingHom σ.toRingHom :=
  RingHomInvPair.of_ringEquiv_symm σ
```

Exact declaration for the source guide

```
theorem exists_semilinearEquiv_of_finite_span
    (e : ℙ K V ≃ ℙ L W)
    (he : ∀ (T : Set (ℙ K V)), T.Finite → ∀ p,
      p ∈ Subspace.span T ↔ e p ∈ Subspace.span (e '' T))
    (hdim : 3 ≤ Module.rank K V) :
    ∃ σ : K ≃+* L, ∃ U : V ≃ₛₗ[σ.toRingHom] W,
      ∀ (v : V) (hv : v ≠ 0), e (mk K v hv) =
        mk L (U v) (by simpa only [map_zero] using U.injective.ne hv) := by
```

Finite-homeomorphisms of closure spaces

`ClosureSpace.FiniteHomeomorph`

structure

Packages one bijection together with finite continuity of the map and its actual inverse.

Statement

Let (X, c) and (Y, d) be closure spaces. A finite-homeomorphism from (X, c) to (Y, d) is a bijection whose forward map and actual inverse both preserve closure on finite test subsets. It is a structure carrying the map and these two properties, rather than only a proposition asserting that some map exists.

Definition

For a function f and a subset E of its domain, write $f(E) = \{f(x) : x \in E\}$. A finite-homeomorphism consists of a bijection $e : X \to Y$, with its inverse $e^{-1} : Y \to X$, satisfying

$$e(c(F)) \subseteq d(e(F)) \qquad \text{for every finite } F \subseteq X,$$

and

$$e^{-1}(d(K)) \subseteq c(e^{-1}(K)) \qquad \text{for every finite } K \subseteq Y.$$

The inverse in the second inclusion belongs to the same bijection: $e^{-1}(e(x)) = x$ and $e(e^{-1}(y)) = y$. The first inclusion transports every point of the closure of a finite source set into the closure of its image. The second imposes the corresponding requirement in the reverse direction.

Here “finite” qualifies the test sets F and K. The sets X, Y, the image $e(X) = Y$, and the closures $c(F), d(K)$ need not be finite. The closure operations themselves still act on all subsets.

Assumptions

The two given closure operations are extensive, monotone and idempotent and preserve the empty set. The carriers may be empty and may belong to independent universes. No topology, metric, dimension bound or finite-character property of either closure is an input.

Full continuity would require the first inclusion for every subset of X, not only finite ones, and similarly for the inverse on Y. Neither full-continuity requirement is a field of this structure.

Conclusion

An element of the defined type supplies one invertible map with both finite-continuity properties. These data can be used wherever a finite-homeomorphism is required without supplying ordinary continuity.

The separately defined continuous finite-homeomorphism adds full continuity of the forward map; even that additional structure does not include inverse full continuity. This distinction concerns an extra field, not a change to the underlying closure operators.

Main citations

- Finite-homeomorphism: the complete structure, lines 157–159: the equivalence and its two finite-continuity fields.
- Full and finite continuity, lines 29–34: the arbitrary-set and finite-set tests.
- The additional forward full-continuity field, lines 163–165: `ContinuousFiniteHomeomorph`.
- Closure spaces before topology, for the underlying closure axioms; fixed source, lines 28–30 and 37–61.

Lean source signature (exact)

Exact declaration. The complete structure, including its parent and both fields is shown below.

```
structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  finitelyContinuous_toFun : c.FinitelyContinuous d toFun
  finitelyContinuous_invFun : d.FinitelyContinuous c invFun
```

In the source	Mathematical meaning
Exact declaration `{α : Type u}`, `{β : Type v}` in the surrounding scope	The arbitrary carriers X, Y, in independent universes. The nearby third carrier is unused by this declaration.
Exact declaration `(c : ClosureSpace α) (d : ClosureSpace β)`	The given closure operations on all subsets of the two carriers.
Exact declaration `extends α ≃ β`	Data of one bijection $e : X \to Y$ and its actual inverse, including the inverse identities.
Exact declaration `finitelyContinuous_toFun : c.FinitelyContinuous d toFun`	For every finite $F \subseteq X$, $e(c(F)) \subseteq d(e(F))$.
Exact declaration `finitelyContinuous_invFun : d.FinitelyContinuous c invFun`	For every finite $K \subseteq Y$, the inverse of that same e satisfies $e^{-1}(d(K)) \subseteq c(e^{-1}(K))$.

In the source	Mathematical meaning
Exact declaration `∀ s : Set α, s.Finite → ...` in the separate predicate excerpt	Finiteness belongs to the tested subset, not to the carrier or all values of the map.
Exact declaration `f '' s` in the continuity predicates	The image of a set is written $f(S)$ in the exposition.
Exact declaration `FiniteHomeomorph c d`	A type of maps with specified properties; its definition does not assert that this type is inhabited.

Exact surrounding scope and supporting definitions

The following source excerpts are separate from the declaration above. Their line ranges are not concatenated into a synthetic signature.

Source Maps.lean, lines 22–26:

```
universe u v w

namespace ClosureSpace

variable {α : Type u} {β : Type v} {γ : Type w}
```

Source Maps.lean, lines 29–34:

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)

/-- Preservation of closure, tested only on finite subsets of the domain. -/
def FinitelyContinuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, s.Finite → f '' c.closure s ⊆ d.closure (f '' s)
```

Source Maps.lean, lines 163–165:

```
structure ContinuousFiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β)
    extends FiniteHomeomorph c d where
  continuous_toFun : c.Continuous d toFun
```

Exact declaration for the source guide

```
structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  finitelyContinuous_toFun : c.FinitelyContinuous d toFun
  finitelyContinuous_invFun : d.FinitelyContinuous c invFun

/-- A finite homeomorphism whose forward map is continuous.
Ordinary continuity of the inverse is deliberately not a field. -/
```

The hull–kernel closure construction

`ClosureSpace.HullKernel.space`

def

The hull–kernel closure construction

Statement

A family of proper subsets of a set carries a natural closure structure. To close a subfamily, intersect its members and then take every member of the original family containing that intersection. The points of the resulting closure space are the subsets themselves.

Definition

Let X be a set and let $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfy $X \notin \mathcal{F}$. For $\mathscr{A} \subseteq \mathcal{F}$ and $a \subseteq X$, define the kernel and hull by

$$\begin{aligned} k(\mathscr{A}) &= \bigcap_{I \in \mathscr{A}} I, \\ h(a) &= \{I \in \mathcal{F} : a \subseteq I\}. \end{aligned}$$

The empty intersection is X. The **hull–kernel closure** on $\mathcal{F}$ is

$$c_{\mathcal{F}}(\mathscr{A}) = h(k(\mathscr{A})) = \{I \in \mathcal{F} : k(\mathscr{A}) \subseteq I\}.$$

There are two levels of subsets: a point I is a subset of X, while $\mathscr{A}$ is a set of such points. Accordingly, $k(\mathscr{A}) \subseteq X$ and $c_{\mathcal{F}}(\mathscr{A}) \subseteq \mathcal{F}$.

Both k and h reverse inclusion, so $c_{\mathcal{F}}$ is monotone. Every $I \in \mathscr{A}$ contains $k(\mathscr{A})$, giving extensivity. The cited kernel identity proves idempotence:

$$\begin{aligned} & k(h(k(\mathscr{A}))) = k(\mathscr{A}) \\ \Longrightarrow \quad & c_{\mathcal{F}}(c_{\mathcal{F}}(\mathscr{A})) = c_{\mathcal{F}}(\mathscr{A}). \end{aligned}$$

Finally,

$$c_{\mathcal{F}}(\varnothing) = h(X) = \{I \in \mathcal{F} : X \subseteq I\} = \varnothing,$$

because $X \notin \mathcal{F}$. These are the four closure-space axioms.

Assumptions

The only restriction on $\mathcal{F}$ is $X \notin \mathcal{F}$; the family itself may be empty. Its members are arbitrary proper subsets of X. No vector-space structure, topology, or closure of $\mathcal{F}$ under intersections is assumed.

Conclusion

The construction yields the closure space $(\mathcal{F}, c_{\mathcal{F}})$ with the displayed formula. A subfamily $\mathscr{A}$ is closed precisely when $h(k(\mathscr{A})) = \mathscr{A}$.

The separate separation theorem shows that this closure is T_0: distinct points have distinct singleton closures. If $\mathcal{F}$ is an antichain under inclusion, every singleton is closed, the T_1 property. The antichain condition belongs to that further theorem, not to the construction.

Main citations

- Kernel, hull and the idempotence identity: `ClosureSpace.HullKernel.kernel`, `ClosureSpace.HullKernel.hull` and `ClosureSpace.HullKernel.kernel_hull_kernel`, lines 30–66.
- Exact construction and closure formula: `ClosureSpace.HullKernel.space`, lines 69–89.
- The separation properties: `ClosureSpace.HullKernel.isT0_space` and `ClosureSpace.HullKernel.isT1_space_of_antichain`, lines 104–134.
- The four closure axioms: `ClosureSpace`, lines 28–30, with its laws at lines 44–61.
- arXiv:2504.05551v1, Definition 4.1: the later hull–kernel construction on geometric points. The generic construction here has purely set-theoretic assumptions.

Lean source signature (exact)

The carrier `X` is declared in the separate source context below.

```
def space (F : Set (Set X)) (hF : Set.univ ∉ F) : ClosureSpace F where
  toClosureOperator := ClosureOperator.mk'
    (fun s => hull F (kernel F s))
    (by
      intro s t hst
      exact hull_antitone F (kernel_antitone F hst))
    (subset_hull_kernel F)
    (by
      intro s
      rw [kernel_hull_kernel])
  map_empty' := by
    change hull F (kernel F ∅) = ∅
    rw [kernel_empty]
    apply Set.eq_empty_iff_forall_notMem.2
    intro I hI
    have hval : I.val = Set.univ := Set.Subset.antisymm (Set.subset_univ _) hI
    exact hF (hval ▸ I.property)
```

In the source	Mathematical meaning
Exact declaration `X, F : Set (Set X)`	An arbitrary set X and a family $\mathcal{F} \subseteq \mathcal{P}(X)$. A point $I \in \mathcal{F}$ is itself a subset of X (`I.val`).
Exact declaration `hF : Set.univ ∉ F`	$X \notin \mathcal{F}$: every member is proper. The family itself may be empty.
Exact declaration `kernel F s, hull F`	For `s` corresponding to $\mathscr{A} \subseteq \mathcal{F}$, $k(\mathscr{A}) = \bigcap_{I \in \mathscr{A}} I$, with empty intersection X, and $h(B) = \{I \in \mathcal{F} : B \subseteq I\}$.
Exact declaration `fun s => hull F (kernel F s)`	The whole closure operation $c_{\mathcal{F}}(\mathscr{A}) = \{I \in \mathcal{F} : \bigcap_{J \in \mathscr{A}} J \subseteq I\}$ on subfamilies of $\mathcal{F}$.
Exact declaration `ClosureOperator.mk', map_empty'`	The construction certifies monotonicity, extensivity and idempotence of this operation, and $c_{\mathcal{F}}(\varnothing) = \varnothing$ using `hF`.
Exact declaration `: ClosureSpace F`	The returned closure-space structure is on the point set $\mathcal{F}$, with all four closure axioms; a topology on X or finite-union preservation is not part of the output.

Source context (separate exact excerpt)

The surrounding declarations below are from line 27; their mathematical meaning is included in the guide above.

```
variable {X : Type u}
```

Exact declaration for the source guide

```
def space (F : Set (Set X)) (hF : Set.univ ∉ F) : ClosureSpace F where
  toClosureOperator := ClosureOperator.mk'
    (fun s => hull F (kernel F s))
    (by
      intro s t hst
      exact hull_antitone F (kernel_antitone F hst))
    (subset_hull_kernel F)
    (by
      intro s
      rw [kernel_hull_kernel])
  map_empty' := by
    change hull F (kernel F ∅) = ∅
    rw [kernel_empty]
    apply Set.eq_empty_iff_forall_notMem.2
    intro I hI
    have hval : I.val = Set.univ := Set.Subset.antisymm (Set.subset_univ _) hI
    exact hF (hval ▸ I.property)

@[simp]
```

Linked points in a closure space

`ClosureSpace.Linked`

def

Linked points in a closure space

Statement

Two points of a closure space are linked if they are equal or their two-point set is not closed. It tests closure of a pair, without requiring an ambient topology.

Definition

Let X be a set with a closure operation $c : \mathcal{P}(X) \to \mathcal{P}(X)$: the map is extensive, monotone and idempotent, with $c(\varnothing) = \varnothing$. A subset $A \subseteq X$ is closed when $c(A) = A$.

For $x, y \in X$, write $x \leftrightarrow_c y$ for the relation

$$x \leftrightarrow_c y \quad \Longleftrightarrow \quad x = y \ \text{ or } \ c(\{x, y\}) \neq \{x, y\}.$$

If $x \neq y$, this says precisely that closing the pair adds a point outside it. The equality alternative ensures $x \leftrightarrow_c x$ even when the singleton $\{x\}$ is closed.

The relation is symmetric, since $\{x, y\} = \{y, x\}$. Reflexivity and symmetry are established by the cited elementary lemmas. Transitivity is supplied later by passing to the equivalence relation generated by linked pairs.

Assumptions

The only input is a closure space (X, c) and two points $x, y \in X$. Finite-union preservation and separation properties are not assumed. The formula therefore applies even when c is not a topological closure.

Conclusion

The output is the proposition that x and y are linked. It is the generating relation for finite-chain components: a chain may move between equal points or between points whose pair is not closed. The associated component construction is defined separately by the equivalence closure of this relation.

Main citations

- Exact relation, reflexivity and symmetry: `ClosureSpace.Linked`, `ClosureSpace.linked_refl` and `ClosureSpace.linked_symm`, lines 30–46.
- The meaning of closedness: `ClosureSpace.closure`, `ClosureSpace.IsClosed` and `ClosureSpace.isClosed_iff`, lines 37–61.
- Passing to finite-chain components: `ClosureSpace.componentSetoid`, `ClosureSpace.Related` and `ClosureSpace.component`, lines 49–68. This is the explanatory next construction.
- Ryotaro Tanaka, arXiv:2504.05551v1, Definition 3.20: the same generating relation and its finite-chain extension.

Lean source signature (exact)

The namespace is `ClosureSpace`; the surrounding carrier `α` denotes the set X.

```
def Linked (c : ClosureSpace α) (x y : α) : Prop := x = y ∨ ¬ c.IsClosed {x, y}
```

In the source	Mathematical meaning
Exact declaration `α, c : ClosureSpace α`	The underlying set X and the closure structure c: an extensive, monotone, idempotent operation on $\mathcal{P}(X)$ that preserves $\varnothing$. The operation itself is `c.closure`.
Exact declaration `x y : α`	The two given points $x, y \in X$; no distinctness is required.
Exact declaration `c.IsClosed {x, y}`	The equality $c(\{x, y\}) = \{x, y\}$.
Exact declaration `x = y ∨ ¬ c.IsClosed {x, y}`	The entire defining proposition $x = y$ or $c(\{x, y\}) \neq \{x, y\}$. The disjunction retains the equality case, and the negation applies to closedness of the pair.
Exact declaration `: Prop`	The output is the linked-pair relation $x \leftrightarrow_c y$, before taking any transitive closure or component.

The specified equivalence from a dense orthogonal ideal sum

`MathlibAnnex.Analysis.CStarAlgebra.ClosedStarIdeal.c0SumEquivOfDense`

def

Identifies an algebra with the c₀-sum of a dense orthogonal family of closed ideals.

Statement

Let $J = (I_i)_{i \in \iota}$ be a family of closed $*$-ideals in a complex C*-algebra A. Assume $xy = 0$ whenever $x \in I_i$, $y \in I_j$ and $i \neq j$, and that the set of finite sums of elements of these ideals is norm dense in A. The ideal-sum map defines a $*$-isomorphism from the dependent c_0-sum of these ideals onto A.

Definition

Give each ideal I_i its inherited algebra structure, with inclusion $\iota_i : I_i \hookrightarrow A$. Use the dependent c_0-sum of these ideals. The forward map is the continuous ideal-sum map

$$S_J : \bigoplus_i^{c_0} I_i \longrightarrow A.$$

The single-coordinate and finite-truncation formulas are

$$S_J(\mathrm{single}_i\, x) = \iota_i(x), \qquad S_J(\mathrm{truncate}_F\, x) = \sum_{i \in F} \iota_i(x_i).$$

Here truncation sets coordinates outside the finite set F to zero. On a general family, S_J is the continuous extension of finite ideal summation. Its injectivity and surjectivity under density make this same map a bijection. It preserves addition, multiplication, complex scalars and involution.

Assumptions

The family consists of closed $*$-ideals satisfying the stated ordered zero-product condition. Density concerns all finite sums in A, not density of each individual ideal. No countability or nonempty-index assumption is imposed.

Conclusion

The output S_J is directed from the ideal c_0-sum to A. Its inverse gives the ideal coordinates used in the subsequent compact-operator decomposition.

Main citations

- Exact complete defining RHS.

- Fiber: the ideal.
- PairwiseOrthogonal: full ordered zero-product condition.
- finiteSums: full set definition.
- c0SumHom: the specified continuous ideal-sum construction.
- c0SumHom_single: single-coordinate guarantee.
- c0SumHom_truncate: finite-truncation guarantee.
- c0SumEquivOfDense_apply: the same forward map.

Lean source signature (exact)

Complete contiguous definition from the fixed source:

```
def c0SumEquivOfDense (hJ : PairwiseOrthogonal J)
    (hdense : Dense (finiteSums J)) :
    C0DirectSum (Fiber J) ≃⋆ₐ[ℂ] A :=
  { Equiv.ofBijective (c0SumHom J hJ)
      ⟨c0SumHom_injective J hJ, c0SumHom_surjective J hJ hdense⟩ with
    map_add' := map_add (c0SumHom J hJ)
    map_mul' := map_mul (c0SumHom J hJ)
    map_smul' := map_smul (c0SumHom J hJ)
    map_star' := map_star (c0SumHom J hJ) }
```

In the source	Mathematical meaning
Exact declaration `J : ι → ClosedStarIdeal A` and `Fiber J`	The family $J = (I_i)$; `Fiber J i` is the same ideal I_i, regarded as an algebra with its inherited operations.
Exact declaration `hJ` and `hdense`	The displayed zero-product and finite-sum density assumptions.
Exact declaration `Equiv.ofBijective (c0SumHom J hJ) ...`	A bijection with exactly the existing ideal-sum map as its forward function; named results supply the two properties.
Exact declaration `map_add'`, `map_mul'`, `map_smul'`, `map_star'`	These fields equip that same equivalence with its complex $*$-algebra structure.
Exact declaration `C0DirectSum (Fiber J) ≃⋆ₐ[ℂ] A`	The direction is from the coordinate family to the ambient algebra.

Exact surrounding source context (separate excerpts)

Source lines 44–48:

```
universe uι uA

variable {ι : Type uι} {A : Type uA} [NonUnitalCStarAlgebra A]

namespace ClosedStarIdeal
```

Source lines 63–63:

```
variable (J : ι → ClosedStarIdeal A)
```

The dependent co-sum of compact-operator algebras

`MathlibAnnex.Analysis.CStarAlgebra.CompactOperatorC0Sum`

abbrev

Names the c_0-sum of a family of compact-operator algebras.

Statement

Let $(H_i)_{i\in\iota}$ be any family of complete complex Hilbert spaces. This abbreviation denotes the c_0-sum of the compact-operator algebras $\mathcal{K}(H_i)$.

Definition

Use the dependent c_0-sum with fiber $A_i = \mathcal{K}(H_i)$, the compact operators on the given H_i. Its elements are bounded families $T = (T_i)$, $T_i \in \mathcal{K}(H_i)$, such that for every $\varepsilon > 0$ only finitely many i have $\varepsilon \le \|T_i\|$. Its norm is $\sup_i \|T_i\|$ and its algebra operations are coordinatewise. The RHS specifies precisely this family of subalgebras of bounded operators.

Assumptions

Every H_i has its own normed additive group, complex inner product and completeness. Zero and infinite-dimensional Hilbert spaces are allowed. There is no restriction on the size of the index set.

Conclusion

The output is an abbreviation for this dependent algebra. It does not denote all compact operators on a Hilbert-space sum and does not include additional off-diagonal operator entries.

Main citations

- Exact complete definition.
- C0DirectSum: the defining abbreviation.

Lean source signature (exact)

The following is the complete defining right-hand side, copied from the fixed source.

```
abbrev CompactOperatorC0Sum {ι : Type uι} (H : ι → Type uH)
    [∀ i, NormedAddCommGroup (H i)] [∀ i, InnerProductSpace ℂ (H i)]
    [∀ i, CompleteSpace (H i)] :=
  C0DirectSum
    (fun i => ContinuousLinearMap.compactOperatorStarSubalgebra ℂ (H i))
```

In the source	Mathematical meaning
Exact declaration `H : ι → Type uH`	The family of Hilbert spaces H_i; the spaces may vary with i.
The three **Exact declaration** `∀ i` instances	Each H_i has the stated norm and complex inner product and is complete.
Exact declaration `ContinuousLinearMap.compactOperatorStarSubalgebra ℂ (H i)`	The subalgebra $\mathcal{K}(H_i)$ of compact complex-linear endomorphisms of this same fiber.
Exact declaration `C0DirectSum (fun i => ...)`	The preceding dependent vanishing-at-infinity construction applied to exactly those algebras.

Exact surrounding source context (separate excerpts)

Source lines 23–25:

```
namespace MathlibAnnex.Analysis.CStarAlgebra

universe uι uA uH
```

The Hilbert space of a minimal-ideal compact model

`MathlibAnnex.Analysis.CStarAlgebra.MinimalClosedStarIdeal.CompactModelSpace`

abbrev

Names the cyclic Hilbert space chosen for a minimal closed ideal.

Statement

Let A be a complex C*-algebra with compatible order, and let $\pi : A \to \mathcal{B}(H)$ be a faithful representation with every value compact, on a complete complex Hilbert space. For a minimal nonzero closed $*$-ideal i of A, this abbreviation names the chosen cyclic Hilbert subspace $H_i \subseteq H$.

Definition

The minimal-ideal model construction chooses a pair $(p, x) \in A \times H$ with the following properties: p is a minimal projection, the closed ideal generated by p is the displayed ideal i, $x \neq 0$, $\pi(p)x = x$, and the same cyclic restriction $\sigma_{p,x} : I_p \to \mathcal{B}(H_i)$ is an exact compact-operator model: it is injective, every value is compact, and every compact operator on this same H_i is $\sigma_{p,x}(a)$ for some $a \in I_p$. Faithfulness and compact-valuedness alone do not include the last condition. These properties are guaranteed for the chosen pair; they are not additional assumptions.

Write I_p for the closed ideal generated by p. The chosen model subspace is

$$H_i = \overline{\{\pi(a)x : a \in I_p\}} \subseteq H$$

for that same pair (p, x), with its inherited norm and complex inner product. The closedness and Hilbert-space results ensure that H_i is complete and nonzero.

Assumptions

The algebra carries a partial order and compatible order. The input ideal is minimal among nonzero closed $*$-ideals. The representation π is injective and every value $\pi(a)$ is compact. There is no additional choice of Hilbert structure, no countability assumption and no global nonzero-algebra hypothesis.

Conclusion

The resulting space depends on the ideal and the chosen representation π. The compact-model isomorphism uses this same Hilbert space; different ideals need not give the same space.

Main citations

- Exact complete definition.
- MinimalClosedStarIdeal: the indexing type.

- compactModelWitness: one generator/vector pair and its full specification.
- IsCompactOperatorModel: the three conditions on one action.
- compactModelSubspace: the exact cyclic subspace.
- isClosed_compactModelSubspace and the completeness/nontriviality instances.

Lean source signature (exact)

The following is the complete defining right-hand side, copied from the fixed source.

```
abbrev CompactModelSpace (i : MinimalClosedStarIdeal A)
    (pi : NonUnitalCStarRepresentation A H) (hpi : pi.IsCompactlyRepresented) : Type v :=
  i.compactModelSubspace pi hpi
```

In the source	Mathematical meaning
Exact declaration `i : MinimalClosedStarIdeal A`	A minimal nonzero closed $*$-ideal.
Exact declaration `pi` and `hpi`	The fixed representation and its faithful compact-valued condition, used to choose the projection and cyclic vector.
Exact declaration `: Type v`	The resulting space H_i is a subspace of the given Hilbert space H.
Exact declaration `i.compactModelSubspace pi hpi`	The closed cyclic subspace associated with the chosen pair (p, x), with norm and inner product inherited from H.

Exact surrounding source context (separate excerpts)

Source lines 128–129:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A]
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```

Source lines 140–140:

```
namespace MinimalClosedStarIdeal
```

Maximal faces of the C*-algebra unit ball and their unique support

`MathlibAnnex.Analysis.CStarAlgebra.isMaximalFace_iff_exists_mem_extremePoints_dual_closedBall`

theorem

Characterizes maximal proper faces of a C*-algebra unit ball by extreme dual functionals.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity, and let $B_A = \{a \in A : \|a\| \leq 1\}$. Faces and convexity are taken over $\mathbb{R}$. We consider maximal proper faces of B_A.

Write A^* for the space of bounded complex-linear functionals on A, with the operator norm. For $\rho \in A^*$, write

$$F_\rho = \{a \in B_A : \rho(a) = 1\}.$$

For every subset $F \subseteq A$,

$$F \text{ is a maximal proper face of } B_A \iff F = F_\rho \text{ for some } \rho \in \operatorname{ext}_{\mathbb{R}}(B_{A^*}). \quad (1)$$

The supporting condition is the complex equality $\rho(a) = 1$. In the unit ball it is equivalent to the usual real supporting condition $\operatorname{Re}\rho(a) = 1$ when $\|\rho\| \leq 1$.

Assumptions

The source fixes the C*-order on A, which is used by the pure-state and interpolation inputs. Nontriviality is essential to the displayed equivalence: the zero algebra has no nonempty proper face, while its dual ball has the extreme point zero. No compactness of B_A, identity in A, or finite-dimensional assumption is imposed.

Conclusion

Equation (1) is the complete selected equivalence about subsets. The separately named theorem `supportFunctionals_unitBallSupportFace_eq_singleton` guarantees that the entire dual unit-ball support family of an extreme slice is the singleton containing that same extreme functional. `existsUnique_extreme_support_of_isMaximalFace` guarantees uniqueness. These are separate cited results, not extra conjuncts in the selected type.

Main citations

- Exact equivalence: `isMaximalFace_iff_exists_mem_extremePoints_dual_closedBall`.

- Full support-family and unique-support results in the same module: `weakStarSupportFunctionals_unitBallSupportFace_eq_singleton`, `supportFunctionals_unitBallSupportFace_eq_singleton`, and `existsUnique_extreme_support_of_isMaximalFace` give the additional support and uniqueness guarantees just distinguished from (1).

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem isMaximalFace_iff_exists_mem_extremePoints_dual_closedBall (F : Set A) :
    NormedSpace.IsMaximalFace F ↔ ∃ rho : StrongDual ℂ A,
      rho ∈ (closedBall (0 : StrongDual ℂ A) 1).extremePoints ℝ ∧
        F = NormedSpace.unitBallSupportFace ℂ rho
```

In the source	Mathematical meaning
Exact declaration `{A : Type u}`, the C*-algebra, order and nontriviality instances	The nonzero complex C*-algebra with its compatible order. The real convex structure is obtained by restricting scalars.
Exact declaration `(F : Set A)`	An arbitrary subset of the original algebra, not a set of states or a subset of the self-adjoint part.
Exact declaration `NormedSpace.IsMaximalFace F`	Nonemptiness, real convexity, the face property in the norm-closed unit ball, properness, and maximality among proper convex faces.
Exact declaration `∃ rho : StrongDual ℂ A`	One continuous complex-linear supporting functional is chosen for the face.
Exact declaration `rho ∈ (closedBall ... 1).extremePoints ℝ`	The same ρ is extreme for real convex combinations in the whole dual closed unit ball.
Exact declaration `F = NormedSpace.unitBallSupportFace ℂ rho`	Equality of the whole subset F with $\{a : \|a\| \leq 1, \rho(a) = 1\}$. Together with extremality, this is the complete right side of the equivalence.
Exact declaration `↔` and the two conjuncts	Both directions hold, with one ρ satisfying both conjuncts. Support-family uniqueness is a separately named result, not a field hidden inside `IsMaximalFace`.

Exact surrounding source context (separate excerpts)

Source lines 34–34:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Source lines 35–35:

```
variable [PartialOrder A] [StarOrderedRing A]
```

Exact declaration for the source guide

```
theorem isMaximalFace_iff_exists_mem_extremePoints_dual_closedBall (F : Set A) :
    NormedSpace.IsMaximalFace F ↔ ∃ rho : StrongDual ℂ A,
      rho ∈ (closedBall (0 : StrongDual ℂ A) 1).extremePoints ℝ ∧
        F = NormedSpace.unitBallSupportFace ℂ rho := by
```

Points of the geometric structure space

`NormedSpace.GeometricStructureSpace`

def

Points of the geometric structure space

Statement

The geometric structure space of a normed space has as its points the support-kernel unions arising from maximal proper real convex faces of the closed unit ball. A point is the resulting subset of the vector space. Faces yielding the same subset represent the same point.

Definition

Let E be a normed space over a real-or-complex scalar field $\mathbb{K}$, with closed unit ball $B_E = \{x \in E : \|x\| \le 1\}$. Let $\mathfrak{M}_E$ be the family of maximal proper real convex faces of B_E. Explicitly, a member M is a nonempty convex subset of B_E, different from B_E, such that

$$\begin{aligned} x, y \in B_E, \quad 0 < t < 1, \quad (1-t)x + ty \in M \\ \implies \quad x, y \in M, \end{aligned}$$

and every proper convex face containing M equals M. The scalar t is real, including when $\mathbb{K}$ is complex.

Write E^* for the continuous $\mathbb{K}$-linear dual. For $M \in \mathfrak{M}_E$, define

$$\begin{aligned} \Sigma_{\mathbb{K}}(M) &= \{f \in E^* : \|f\| \le 1,\ f|_M = 1\}, \\ I_{\mathbb{K}}(M) &= \bigcup_{f \in \Sigma_{\mathbb{K}}(M)} \ker f. \end{aligned}$$

Here $f|_M = 1$ means $f(x) = 1$ for every $x \in M$.

The family of geometric points is

$$\mathcal{F}_{\mathbb{K}}(E) = \{I_{\mathbb{K}}(M) : M \in \mathfrak{M}_E\}.$$

The **geometric structure space** $S_{\mathbb{K}}(E)$ has precisely these subsets as its points. Thus an element $I \in S_{\mathbb{K}}(E)$ is a subset $I \subseteq E$ with a face M satisfying $I = I_{\mathbb{K}}(M)$. Choosing a representing face does not add data to the point. The assignment $M \mapsto I_{\mathbb{K}}(M)$ is not asserted to be injective.

Assumptions

The space E is normed over $\mathbb{K}$; completeness and nontriviality are not required. Convexity and the face condition are taken over the reals. Exposedness and supporting-functional existence are not additional conditions on the maximal-face predicate.

Conclusion

The resulting point set is $S_{\mathbb{K}}(E)$. Each point is a proper subset of E: if $I = I_{\mathbb{K}}(M)$ and $x \in M$, then $f(x) = 1$ for every $f \in \Sigma_{\mathbb{K}}(M)$, so $x \notin I$. Consequently $E \notin \mathcal{F}_{\mathbb{K}}(E)$, as required by the later hull–kernel closure construction.

The cited zero-space result gives $S_{\mathbb{K}}(E) = \varnothing$ when $E = \{0\}$. The present declaration specifies the points; the closure operation is a separate construction.

Main citations

- Maximal proper real convex faces: `NormedSpace.IsMaximalFace`, lines 32–39; its zero-space exclusion, lines 45–52.
- Supporting functionals and their kernel union: `NormedSpace.supportFunctionals`, lines 42–43; `NormedSpace.supportKernelUnion`, lines 108–109, and properness, lines 130–142.
- The family and the exact point type: `NormedSpace.geometricStructureFamily` and `NormedSpace.GeometricStructureSpace`, lines 37–54; the zero-space result, lines 73–86.
- arXiv:2504.05551v1, Definition 4.1: the face-based construction. The formal source separates its carrier from its closure and allows normed spaces without completeness or nontriviality.

Lean source signature (exact)

The following definition and the related family definition share the surrounding scalar and space context reproduced below.

```
def GeometricStructureSpace : Type u := {I : Set E // I ∈ geometricStructureFamily 𝕜 E}
```

In the source	Mathematical meaning
Exact declaration `𝕜, [RCLike 𝕜]`	The real or complex scalar field $\mathbb{K}$.
Exact declaration `E, [NormedAddCommGroup E], [NormedSpace 𝕜 E]`	A normed vector space E over $\mathbb{K}$; it may be zero and need not be complete.
Exact declaration `I : Set E`	A subset $I \subseteq E$, regarded as a possible geometric point.

In the source	Mathematical meaning
Exact declaration `I ∈ geometricStructureFamily 𝕜 E`	The condition $I \in \mathcal{F}_{\mathbb{K}}(E)$: there exists a maximal proper real convex face M of the closed unit ball with $I_{\mathbb{K}}(M) = I$.
Exact declaration `{I : Set E // I ∈ geometricStructureFamily 𝕜 E}`	The complete defining type $S_{\mathbb{K}}(E)$ consists of such subsets with their membership condition. A point `p` has underlying subset `p.val`; no chosen face is stored.
Exact declaration `: Type u`	The output is the point set $S_{\mathbb{K}}(E)$, before any closure-space structure is attached.

Related definition — separate exact excerpt. The family used above is defined at lines 37–39 of the same source file.

```
def geometricStructureFamily : Set (Set E) :=
  letI : NormedSpace ℝ E := NormedSpace.restrictScalars ℝ 𝕜 E
  {I | ∃ F : Set E, IsMaximalFace F ∧ supportKernelUnion 𝕜 F = I}
```

In the source	Mathematical meaning
Exact declaration `letI ... := NormedSpace.restrictScalars ℝ 𝕜 E`	Convexity and faces use the real vector-space structure induced from the given $\mathbb{K}$-space; this is not an extra choice or assumption.
Exact declaration `∃ F : Set E, IsMaximalFace F ∧ supportKernelUnion 𝕜 F = I`	One subset M must satisfy both conditions: it is a maximal proper real convex face of the closed unit ball and its support-kernel union equals I. Here the source name `F` denotes M.
Exact declaration `: Set (Set E)`	The full RHS defines the family $\mathcal{F}_{\mathbb{K}}(E)$ of resulting subsets. Equal subsets give the same member even when their witnessing faces differ; no face is chosen as part of the output.

Source context (separate exact excerpt)

The surrounding declarations below are from lines 32–33; their mathematical meaning is included in the guide above.

```
variable (𝕜 : Type v) [RCLike 𝕜]
variable (E : Type u) [NormedAddCommGroup E] [NormedSpace 𝕜 E]
```

Exact declaration for the source guide

```
def GeometricStructureSpace : Type u := {I : Set E // I ∈ geometricStructureFamily 𝕜 E}

/-- The hull-kernel closure on `S(E)`. -/
```

Pairwise linked subsets of a closure space

`ClosureSpace.IsLinkedSet`

def

Defines subsets whose every pair of points is linked in the ambient closure space.

Statement

Let (X, c) be a closure space and let $S \subseteq X$. The linked-set predicate requires every two points of S to be linked with respect to the ambient closure c. It includes the empty set and singletons.

Definition

A set $A \subseteq X$ is closed when $c(A) = A$. The ambient linked relation is

$$x \sim_c y \quad \Longleftrightarrow \quad x = y \text{ or } c(\{x, y\}) \neq \{x, y\}.$$

Thus the linked-set condition is

$$\forall x \in S,\ \forall y \in S, \qquad x \sim_c y.$$

Both points are chosen from the same set S, while the closure of their pair is taken in the given ambient space X. Distinct points must form a nonclosed two-point set. When the two points coincide, the equality alternative already makes them linked.

This is a direct pairwise condition. The separate relation of belonging to the same component is generated by finite chains of linked pairs. A chain may pass through intermediate points; it does not replace the pairwise test in this definition.

Assumptions

The inputs are the closure space (X, c) and an arbitrary subset S. Nonemptiness, two or more elements, closedness and maximality are not required. No finite-character or finite-union property is imposed on the closure.

Conclusion

The definition returns a proposition about all pairs in S, with their membership conditions explicitly quantified. It neither chooses a distinguished point nor asserts that S is a whole component or a maximal linked subset.

The cited empty-set and singleton lemmas are separate consequences, not extra clauses of the definition. For the empty set there is no $x \in S$ to test. For $S = \{a\}$, any two members both equal a, so the equality alternative of the linked relation applies. These two arguments explain why the

definition admits these cases without a nonemptiness convention.

Main citations

- Linked-set predicate: the complete definition, lines 30–31.
- The ambient linked relation and its reflexivity, lines 30–32; see also the existing Linked points Card.
- Empty and singleton suppliers, lines 34–41, for the two separate consequences explained above.
- Closedness as a closure equality, lines 60–61.
- The generated relation and its components, lines 48–52 and 67–68; the existing Closure components Card treats this different notion.

Lean source signature (exact)

Exact declaration. The complete defining right-hand side is shown below.

```
def IsLinkedSet (c : ClosureSpace α) (S : Set α) : Prop :=
  ∀ x ∈ S, ∀ y ∈ S, c.Linked x y
```

In the source	Mathematical meaning
Exact declaration `{α : Type u}` in the surrounding scope	The arbitrary ambient carrier X. The nearby carrier `β` and universe `v` are unused by this declaration.
Exact declaration `(c : ClosureSpace α)`	The given ambient closure; the pair is not tested in a newly chosen topology or subspace closure.
Exact declaration `(S : Set α)`	The one subset S whose pairs are tested.
Exact declaration `∀ x ∈ S, ∀ y ∈ S`	For every x with $x \in S$, and every y with $y \in S$; equal points are included. No nonemptiness or distinctness condition is inserted.
Exact declaration `c.Linked x y`	The equality-or-nonclosed-pair condition in the separate complete `Linked` definition below.
Exact declaration `: Prop :=` and the whole right-hand side	A predicate asserting this condition for all pairs, not a data structure choosing points, a maximal set or a component.

Exact surrounding scope and supporting definitions

The following source excerpts are separate from the declaration above. Their line ranges are not concatenated into a synthetic signature.

Source LinkedSet.lean, lines 23–27:

```
universe u v

namespace ClosureSpace

variable {α : Type u} {β : Type v}
```

Source Components.lean, lines 23–32:

```
universe u v

namespace ClosureSpace

variable {α : Type u} {β : Type v}

/-- Equality or a nonclosed two-point set. -/
def Linked (c : ClosureSpace α) (x y : α) : Prop := x = y ∨ ¬ c.IsClosed {x, y}

theorem linked_refl (c : ClosureSpace α) (x : α) : c.Linked x x := Or.inl rfl
```

Source LinkedSet.lean, lines 34–41:

```
theorem isLinkedSet_empty (c : ClosureSpace α) : c.IsLinkedSet ∅ := by
  intro x hx
  exact hx.elim

/-- Reflexivity of `Linked` makes every singleton an R-set. -/
theorem isLinkedSet_singleton (c : ClosureSpace α) (x : α) : c.IsLinkedSet {x} := by
  rintro y rfl z rfl
  exact c.linked_refl _
```

Source Basic.lean, lines 60–61:

```
theorem isClosed_iff (c : ClosureSpace α) (s : Set α) :
    c.IsClosed s ↔ c.closure s = s := c.toClosureOperator.isClosed_iff
```

Source Components.lean, lines 48–52:

```
/-- The equivalence relation generated by linked pairs. -/
def componentSetoid (c : ClosureSpace α) : Setoid α := Relation.EqvGen.setoid c.Linked

/-- Belonging to the same finite-chain component. -/
abbrev Related (c : ClosureSpace α) (x y : α) : Prop := (c.componentSetoid).r x y
```

Source Components.lean, lines 67–68:

```
/-- The equivalence class as an ambient subset. -/
def component (c : ClosureSpace α) (x : α) : Set α := {y | c.Related x y}
```

Exact declaration for the source guide

```
def IsLinkedSet (c : ClosureSpace α) (S : Set α) : Prop :=
  ∀ x ∈ S, ∀ y ∈ S, c.Linked x y

/-- The empty set is an R-set, with no nonemptiness convention added. -/
```

Components generated by linked pairs

`ClosureSpace.component`

def

Components generated by linked pairs

Statement

The component of a point in a closure space consists of all points reachable from it by a finite chain of linked pairs. It is represented as an actual subset of the original set.

Definition

Let X be a set with an extensive, monotone and idempotent closure operation $c : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying $c(\varnothing) = \varnothing$. A subset A is closed when $c(A) = A$. Define

$$x \leftrightarrow_c y \quad \Longleftrightarrow \quad x = y \ \text{ or } \ c(\{x, y\}) \neq \{x, y\}.$$

Let $x \sim_c y$ mean that there is an integer $n \geq 0$ and a finite chain

$$x = x_0 \leftrightarrow_c x_1 \leftrightarrow_c \cdots \leftrightarrow_c x_n = y.$$

The case $n = 0$ is the chain with a single point, so it gives $x \sim_c x$. The **component of** x is

$$C_c(x) = \{y \in X : x \sim_c y\}.$$

The source defines $\sim_c$ as the equivalence relation generated by the linked relation. This agrees with the finite-chain description: the linked relation is symmetric, reversing a chain gives symmetry, and concatenating two chains gives transitivity. The empty chain of steps supplies reflexivity. These are exactly the operations in the cited equivalence-closure construction.

Assumptions

The only data are a closure space (X, c) and a point $x \in X$. No topology or separation condition is required. If X is empty, there is no point at which to evaluate the component function.

Conclusion

The result is the subset $C_c(x) \subseteq X$, which contains x. The separately cited equality criterion gives

$$C_c(x) = C_c(y) \quad \Longleftrightarrow \quad x \sim_c y.$$

Thus choosing another point in the same component leaves the component set unchanged. Its closure $c(C_c(x))$ is a separate object, defined separately in the source. The definition of $C_c(x)$ does not assert that this subset is closed or that it is a connected component in a topological sense.

Main citations

- The generating relation and equivalence closure: `ClosureSpace.Linked`, `ClosureSpace.componentSetoid` and `ClosureSpace.Related`, lines 30–65.
- Exact component, membership and the separate component closure: `ClosureSpace.component`, `ClosureSpace.mem_component`, `ClosureSpace.component_eq_of_related` and `ClosureSpace.componentClosure`, lines 68–84.
- Equality of component sets: `ClosureSpace.component_eq_iff_related`, lines 30–37.
- The finite equivalence-closure constructors: Mathlib `Relation.EqvGen`, lines 349–353, at the pinned mathlib commit. Its constructors insert relation steps, reflexivity, symmetry and transitivity.
- Ryotaro Tanaka, arXiv:2504.05551v1, Definition 3.20, Proposition 3.23 and Definition 3.24: the finite-chain relation; the component notation continues on page 11.

Lean source signature (exact)

The namespace is `ClosureSpace`; the surrounding carrier `α` denotes the set X.

```
def component (c : ClosureSpace α) (x : α) : Set α := {y | c.Related x y}
```

In the source	Mathematical meaning
Exact declaration `α, c : ClosureSpace α`	An arbitrary set X equipped with the extensive, monotone, idempotent closure c preserving $\varnothing$; finite-union preservation is not required.
Exact declaration `x : α`	The given starting point $x \in X$.
Exact declaration `c.Related x y`	$x \sim_c y$: the equivalence relation generated by linked pairs, or equivalently the finite-chain relation defined above, including a chain with no steps.
Exact declaration `{y | c.Related x y}`	The complete RHS is $C_c(x) = \{y \in X : x \sim_c y\}$. It ranges over all possible endpoints y while keeping x fixed.
Exact declaration `: Set α`	The returned object is this subset of X, not a quotient point or its closure $c(C_c(x))$. The latter is the separate declaration `componentClosure`.

Relation.EqvGen · fixed Mathlib definition

Fixed source, lines 346–353 · Raw UTF-8 source

```
variable (r) in
/-- `EqvGen r`: equivalence closure of `r`. -/
@[mk_iff]
inductive EqvGen : α → α → Prop
  | rel x y : r x y → EqvGen x y
  | refl x : EqvGen x x
  | symm x y : EqvGen x y → EqvGen y x
  | trans x y z : EqvGen x y → EqvGen y z → EqvGen x z
```

Compact coordinates from the inverse ideal-sum map

`MathlibAnnex.Analysis.CStarAlgebra.ClosedStarIdeal.compactC0SumEquivOfDense`

def

Expresses an algebra in compact-operator coordinates through its orthogonal ideal decomposition.

Statement

Let $J = (I_i)_{i \in \iota}$ be a pairwise orthogonal family of closed $*$-ideals whose finite sums are norm dense in a complex C*-algebra A. For each ideal I_i, fix a $*$-isomorphism $E_i : I_i \to \mathcal{K}(H_i)$, where H_i is the specified complete complex Hilbert space. This definition gives a $*$-isomorphism from A to the compact-operator c_0-sum.

Definition

Let $S_J : \bigoplus_i^{c_0} I_i \to A$ be the specified ideal-sum equivalence. Let E_* apply the given E_i in every coordinate, with the same index retained. The definition is precisely

$$D = E_* \circ S_J^{-1} : A \longrightarrow \bigoplus_i^{c_0} \mathcal{K}(H_i).$$

The target is the compact-operator c_0-sum. The coordinate-evaluation theorem gives

$$(D(a))_i = E_i\big((S_J^{-1}(a))_i\big).$$

The inverse identification, single-coordinate formula, and finite-truncation formula give

$$D^{-1}(\mathrm{single}_i\, T) = \iota_i(E_i^{-1}(T)), \qquad D^{-1}(\mathrm{truncate}_F\, T) = \sum_{i \in F} \iota_i(E_i^{-1}(T_i)).$$

Here ι_i is the ideal inclusion; truncation retains precisely the finite coordinate set F. The cited evaluation theorems give these formulas.

Assumptions

Orthogonality means that products from distinct ideals vanish. Density concerns their finite sums in the ambient algebra. The coordinate maps E_i are given equivalences onto all compact operators on their respective H_i, not merely faithful compact-valued actions. No countability or nontriviality is added.

Conclusion

Thus D first resolves an element into its ideal coordinates and then applies E_i in each coordinate. Its target contains no off-diagonal operator entries and is not asserted to be the full compact-operator algebra of a Hilbert sum.

Main citations

- Exact complete defining RHS.
- c0SumEquivOfDense: complete ideal-sum equivalence.
- eval_compactC0SumEquivOfDense: coordinate formula.
- compactC0SumEquivOfDense_symm_apply: inverse identification.
- compactC0SumEquivOfDense_symm_single: one coordinate.
- compactC0SumHom_truncate: finite coordinates.

Lean source signature (exact)

Complete contiguous definition from the fixed source:

```
def compactC0SumEquivOfDense (hdense : Dense (finiteSums J)) :
    A ≃⋆ₐ[ℂ] CompactOperatorC0Sum H :=
  (c0SumEquivOfDense J hJ hdense).symm.trans
    (C0DirectSum.equiv (Fiber J)
      (fun i => ContinuousLinearMap.compactOperatorStarSubalgebra ℂ (H i))
      (Equiv.refl ι) E)
```

In the source	Mathematical meaning
Exact declaration `(c0SumEquivOfDense J hJ hdense).symm`	The inverse S_J^{-1}, directed from A to the ideal-coordinate c_0-sum.
Exact declaration `.trans (C0DirectSum.equiv ... (Equiv.refl ι) E)`	First use S_J^{-1}, then apply E_i coordinatewise, retaining the same index i.
Exact declaration `Fiber J i` and `compactOperatorStarSubalgebra ℂ (H i)`	The ideal I_i in the family $J = (I_i)_{i\in\iota}$ and the full compact-operator algebra on H_i, respectively.
Exact declaration `hdense : Dense (finiteSums J)`	The density assumption used by the ideal-sum equivalence, not a new conclusion of this definition.

Exact surrounding source context (separate excerpts)

Source lines 51–58:

```
variable {ι : Type uι} {A : Type uA} [NonUnitalCStarAlgebra A]
variable (J : ι → ClosedStarIdeal A) (hJ : PairwiseOrthogonal J)
variable (H : ι → Type uH)
variable [∀ i, NormedAddCommGroup (H i)] [∀ i, InnerProductSpace ℂ (H i)]
variable [∀ i, CompleteSpace (H i)]
variable (E : ∀ i, Fiber J i ≃⋆ₐ[ℂ]
  ContinuousLinearMap.compactOperatorStarSubalgebra ℂ (H i))
```

The geometric structure family is the family of extreme-functional kernels

`MathlibAnnex.Analysis.CStarAlgebra.geometricStructureFamily_eq_image_ker_extremePoints_dual_closedBall`

theorem

Identifies the geometric structure family with kernels of extreme dual functionals.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, with no identity required. Write A^* for the space of bounded complex-linear functionals on A and B_{A^*} for its closed unit ball. For a maximal proper face F of the algebra unit ball $B_A = \{a \in A : \|a\| \leq 1\}$, let

$$\Sigma(F) = \{\rho \in A^* : \|\rho\| \leq 1,\ \rho(a) = 1 \text{ for all } a \in F\}, \qquad I_F = \bigcup_{\rho \in \Sigma(F)} \ker \rho.$$

For these nonempty unit-ball faces, the support-functional norm theorem identifies $\Sigma(F)$ with the norm-one linear functionals supporting F. The original geometric structure family is $\mathcal{S}_A = \{I_F : F \text{ is a maximal proper face}\}$, a family of subsets of A. Then

$$\mathcal{S}_A = \{\ker \rho : \rho \in \operatorname{ext}_{\mathbb{R}}(B_{A^*})\}. \qquad (1)$$

Thus the points of the geometric structure space $S(A)$, originally defined from maximal faces and all their supporting kernels, are exactly these extreme-functional kernels.

Assumptions

Faces use the underlying real convex structure, while every functional and every kernel in (1) is complex-linear on the full algebra A. The source uses the compatible C*-order. Nontriviality excludes the zero algebra, where the original face family is empty but the dual ball has the extreme point zero. No change to the original definition of $S(A)$ is part of the assertion.

Conclusion

Formula (1) is equality of families of subsets, with both inclusions. Repeated kernels represent one point. No injectivity of the kernel parametrization, and no equality between a functional kernel and a representation kernel, is asserted.

Main citations

- Exact family equality: `geometricStructureFamily_eq_image_ker_extremePoints_dual_closedBall`.
- Maximal faces and their full support singleton: the separate facial and support statements.

- The original family definition and supporting-kernel union: the objects on the left side of (1).
- ExtremeKernel.lean, `supportKernelUnion_unitBallSupportFace_eq_ker`: the separate union-of-supporting-kernels identity.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem geometricStructureFamily_eq_image_ker_extremePoints_dual_closedBall :
    NormedSpace.geometricStructureFamily ℂ A =
      (fun rho : StrongDual ℂ A => (rho.ker : Set A)) ''
        (closedBall (0 : StrongDual ℂ A) 1).extremePoints ℝ
```

In the source	Mathematical meaning

Algebra and order
`variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]`
`variable [PartialOrder A] [StarOrderedRing A]`
	The nonzero complex C*-algebra A and its compatible order. No identity is required.

Original geometric family
`theorem geometricStructureFamily_eq_image_ker_extremePoints_dual_closedBall :`
`NormedSpace.geometricStructureFamily ℂ A =`
	The left side is the previously defined family $\mathcal{S}_A$ of subsets of A, obtained from maximal proper faces and all their supporting kernels.

Kernel map and its image
`(fun rho : StrongDual ℂ A => (rho.ker : Set A)) ''`
`(closedBall (0 : StrongDual ℂ A) 1).extremePoints ℝ := by`
	The function sends a bounded complex-linear functional $\rho \in A^*$ to its kernel as a subset of A. Its image is taken over the real extreme points of the closed dual unit ball:
	$$\{\ker \rho : \rho \in \operatorname{ext}_{\mathbb{R}}(B_{A^*})\}.$$
	The equality in (1) identifies this entire family with $\mathcal{S}_A$; it does not assert injectivity of the kernel map.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Exact surrounding source:

```
variable [PartialOrder A] [StarOrderedRing A]
```

Related exact source for the reading guide

Algebra and order · Fixed source

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
variable [PartialOrder A] [StarOrderedRing A]
```

Kernel map and its image · Fixed source

```
      (fun rho : StrongDual ℂ A => (rho.ker : Set A)) ''
        (closedBall (0 : StrongDual ℂ A) 1).extremePoints ℝ := by
```

The coefficient-kernel normal part

`MathlibAnnex.Analysis.CStarAlgebra.normalPart`

def

Defines the normal part by vector coefficient kernels that are geometric points.

Statement

Let H be a complete complex Hilbert space and $A \subseteq \mathcal{B}(H)$ a complex $*$-subalgebra, not required to contain an identity. Give A the inherited operator norm. Its **normal part** selects those points of the geometric structure space of A whose underlying subsets are kernels of vector coefficient functionals

$$f_{x,y} : A \longrightarrow \mathbb{C}, \qquad f_{x,y}(a) = \langle y, a(x) \rangle, \quad x, y \neq 0. \tag{1}$$

The inner product is conjugate-linear in its first variable. This definition itself assumes neither norm-closedness nor irreducibility of A.

Definition

For each maximal proper face M of the closed unit ball of A, take the union of the kernels of the norm-one bounded complex-linear functionals supporting M. The geometric structure space $S(A)$ is the family of these subsets, with repetitions identified. Faces use real convexity. The norm-one description follows from the support-functional norm theorem, since M is a nonempty subset of the unit ball.

The normal part is the following subset of that already defined space:

$$N(A) = \left\{ I \in S(A) : \begin{matrix} \text{there exist } x, y \in H \text{ with } x \neq 0,\ y \neq 0, \\ I = \ker f_{x,y} \end{matrix} \right\}. \tag{2}$$

In (2), the same pair x, y satisfies both nonzero conditions and the kernel equality. With the stated inner-product convention, $f_{x,y}$ is complex-linear in a.

The set (2) consists of points of $S(A)$ that are coefficient kernels. For a general algebra A, the kernel of a nonzero vector pair need not belong to $S(A)$. In particular, nonzero vectors alone need not make the restricted coefficient nonzero when the algebra is reducible. Under the additional closedness and irreducibility hypotheses in the component theorem, every nonzero pair does give a geometric point and the normal part is one entire component.

Assumptions

Only the complete complex Hilbert structure on H and the complex $*$-subalgebra structure on A are required. Completeness or norm-closedness of A, nonzero action, irreducibility, separability, and the existence of a unit in A are not assumed.

Conclusion

The result is a subset of the geometric structure space, defined by literal equality with a vector coefficient's zero set. It is not a new quotient of pairs of vectors and does not yet assert a component equality, a projective parametrization, or a homeomorphism. Those statements require the additional hypotheses in the cited results.

Main citations

- Exact normal-part definition: the set (2), before the later closedness binder.
- The ambient geometric structure space: the original face-and-support construction used by the type in (2).
- A coefficient-kernel component equals the normal part: the stronger conclusion for a closed irreducible concrete algebra.
- arXiv:2504.05551v1, Definition 6.4: the normal part in the irreducible concrete setting. The general source definition is deliberately stated before those additional hypotheses.

Lean source signature (exact)

The complete definition below is copied from the pinned source, including its defining right-hand side.

```
def normalPart (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) :
    Set (NormedSpace.GeometricStructureSpace ℂ S) :=
  {I | ∃ x y : H, x ≠ 0 ∧ y ≠ 0 ∧
    {a : S | inner ℂ y ((a : H →L[ℂ] H) x) = 0} = I.val}
```

In the source, `S` denotes the algebra A used here.

In the source	Mathematical meaning
Exact declaration `{H}` and the three Hilbert-space instances	The complete complex Hilbert space H of (1). These conditions are on H, not on the normed subalgebra A.
Exact declaration `S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)`	The algebra A in the text: a complex linear subspace of bounded complex-linear operators, closed under multiplication and adjoint; no unit-preservation or norm-closedness condition is part of this parameter.

In the source	Mathematical meaning
Exact declaration `Set (NormedSpace.GeometricStructureSpace ℂ S)`	The normal part is a subset of the existing geometric space $S(A)$, whose points have underlying subsets of A.
Exact declaration `∃ x y : H, x ≠ 0 ∧ y ≠ 0 ∧ ...`	One pair of vectors satisfies both nonzero conditions and the following equality. Separate pairs are not chosen for separate conjuncts.
Exact declaration `inner ℂ y ((a : H →L[ℂ] H) x) = 0` and `= I.val`	The entire coefficient zero set is the underlying subset of the same geometric point I. The operator acts on x and is paired against y; this is exactly (2).
The complete set-builder right-hand side **Exact declaration**	Selects only kernels that are already geometric points; component and projective claims require separate hypotheses and results.

No closedness or irreducibility binder is appended. The line immediately following this definition in the retained block is not part of its scope.

Exact surrounding source context (separate excerpts)

Source lines 32–32:

```
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```

The geometric structure closure

`NormedSpace.geometricStructureClosure`

def

The geometric structure closure

Statement

The geometric structure closure equips the set of geometric points with the hull–kernel rule. It closes a collection of points by intersecting them as subsets of the original vector space and then taking all geometric points containing that intersection.

Definition

Let E be a normed space over a real-or-complex scalar field $\mathbb{K}$, and let E^* denote its continuous $\mathbb{K}$-linear dual. Write $B_E = \{x : \|x\| \le 1\}$. A maximal proper real convex face M of B_E is a nonempty convex subset, different from B_E, that contains both endpoints of any segment in B_E whose interior meets M, and is maximal among proper convex faces.

Put

$$\begin{aligned} \Sigma_{\mathbb{K}}(M) &= \{f \in E^* : \|f\| \le 1,\ f|_M = 1\}, \\ I_{\mathbb{K}}(M) &= \bigcup_{f \in \Sigma_{\mathbb{K}}(M)} \ker f. \end{aligned}$$

Here $f|_M = 1$ means $f(x) = 1$ for every $x \in M$.

Let $S_{\mathbb{K}}(E)$ be the set of resulting subsets $I_{\mathbb{K}}(M)$ as M ranges over these faces. Equal subsets are the same geometric point.

For $\mathscr{A} \subseteq S_{\mathbb{K}}(E)$, define

$$\begin{aligned} k_E(\mathscr{A}) &= \bigcap_{J \in \mathscr{A}} J, \\ c_E(\mathscr{A}) &= \{I \in S_{\mathbb{K}}(E) : k_E(\mathscr{A}) \subseteq I\}. \end{aligned}$$

The intersection is taken inside E, with $k_E(\varnothing) = E$, while $c_E(\mathscr{A})$ is a set of geometric points. Each geometric point is proper: any x in a representing face satisfies $f(x) = 1$ for all its supporting functionals and hence lies outside their kernel union. The cited hull–kernel construction therefore applies to this family and proves, for all $\mathscr{A}, \mathscr{B} \subseteq S_{\mathbb{K}}(E)$,

$$\mathscr{A} \subseteq c_E(\mathscr{A}), \qquad \mathscr{A} \subseteq \mathscr{B} \Longrightarrow c_E(\mathscr{A}) \subseteq c_E(\mathscr{B}),$$
$$c_E(c_E(\mathscr{A})) = c_E(\mathscr{A}), \qquad c_E(\varnothing) = \varnothing.$$

Assumptions

The space E is normed over $\mathbb{K}$, without a completeness or nontriviality hypothesis. Properness of every geometric point follows from the defining faces and is supplied internally to the hull–kernel construction.

Conclusion

The result is the closure space $(S_{\mathbb{K}}(E), c_E)$. A collection of geometric points is closed when it is fixed by the displayed operation. This operation acts on collections of points; it is distinct from norm closure inside E.

The source separately proves T_0 (distinct points have distinct singleton closures) and later T_1 (every singleton is closed), the latter using incomparability of distinct geometric points under inclusion. Finite-union preservation is not asserted here. For the zero space, the carrier is empty and the same construction applies.

Main citations

- Faces, functionals and the point family: `NormedSpace.IsMaximalFace`; the family and its properness, `NormedSpace.geometricStructureFamily` and `NormedSpace.univ_notMem_geometricStructureFamily`, lines 37–54.
- Exact closure and its membership formula: `NormedSpace.geometricStructureClosure`, lines 57–65; T_0 and the zero-space results, lines 68–86.
- The construction proving the four axioms: `ClosureSpace.HullKernel.space`, lines 69–89.
- Incomparability and singleton closedness: `NormedSpace.GeometricStructureSpace.eq_of_subset` and `NormedSpace.isT1_geometricStructureClosure`, lines 52–72.
- arXiv:2504.05551v1, Definition 4.1: the same hull–kernel formula on geometric points; the source retains the weaker normed-space assumptions stated here.

Lean source signature (exact)

The scalar and normed-space declarations used by this definition are reproduced separately below.

```
def geometricStructureClosure : ClosureSpace (GeometricStructureSpace 𝕜 E) :=
  ClosureSpace.HullKernel.space (geometricStructureFamily 𝕜 E)
    (univ_notMem_geometricStructureFamily 𝕜 E)
```

In the source	Mathematical meaning
Exact declaration `𝕜, [RCLike 𝕜]`	The real or complex scalar field $\mathbb{K}$.
Exact declaration `E, [NormedAddCommGroup E], [NormedSpace 𝕜 E]`	A normed vector space E over $\mathbb{K}$; it may be zero and need not be complete.
Exact declaration `GeometricStructureSpace 𝕜 E, geometricStructureFamily 𝕜 E`	The point set $S_{\mathbb{K}}(E)$ and its underlying family of subsets $\mathcal{F}_{\mathbb{K}}(E)$, defined by maximal-face support-kernel unions.
Exact declaration `ClosureSpace.HullKernel.space (geometricStructureFamily 𝕜 E)`	The defining construction gives $c_E(\mathscr{A}) = \{I \in S_{\mathbb{K}}(E) : \bigcap_{J \in \mathscr{A}} J \subseteq I\}$ for every subfamily $\mathscr{A}$; an empty intersection is E.
Exact declaration `univ_notMem_geometricStructureFamily 𝕜 E`	The cited properness theorem supplies $E \notin \mathcal{F}_{\mathbb{K}}(E)$. It is a proved input to the constructor, not an additional hypothesis on E.
Exact declaration `: ClosureSpace (GeometricStructureSpace 𝕜 E)`	The returned object is the operation c_E together with extensivity, monotonicity, idempotence and $c_E(\varnothing) = \varnothing$. Finite-union preservation is not asserted.

Source context (separate exact excerpt)

The surrounding declarations below are from lines 32–33; their mathematical meaning is included in the guide above.

```
variable (𝕜 : Type v) [RCLike 𝕜]
variable (E : Type u) [NormedAddCommGroup E] [NormedSpace 𝕜 E]
```

Exact declaration for the source guide

```
def geometricStructureClosure : ClosureSpace (GeometricStructureSpace 𝕜 E) :=
  ClosureSpace.HullKernel.space (geometricStructureFamily 𝕜 E)
    (univ_notMem_geometricStructureFamily 𝕜 E)

/-- The closure formula exposes the intersection, including the empty intersection. -/
```

A full homeomorphism to a compact-operator GSS

`ClosureSpace.HasCompactOperatorHomeomorph`

def

Requires a full homeomorphism to the geometric structure space of a compact-operator algebra.

Statement

Let (X, c) be a closure space. The compact-operator homeomorphism condition holds when (X, c) is fully homeomorphic to $S(\mathcal{K}(H))$ for some nonzero complex Hilbert space H.

Definition

Here $\mathcal{K}(H)$ denotes the algebra of compact operators on H. For a complex normed space D, $S(D)$ consists of the unions of kernels of norm-one bounded complex-linear functionals supporting each maximal proper face of its closed unit ball. Faces use real convexity. The support-functional norm theorem justifies the norm-one description for these nonempty faces. Its closure is $c_D(\mathcal{U}) = \{I \in S(D) : \bigcap_{J \in \mathcal{U}} J \subseteq I\}$.

The condition requires a nonzero complex Hilbert space H and a closure-space homeomorphism

$$e : (X, c) \longrightarrow S(\mathcal{K}(H)).$$

This is a bijection with full continuity in both directions. Explicitly, the required inclusions are $e(c(E)) \subseteq c_{\mathcal{K}(H)}(e(E))$ for every $E \subseteq X$, and $e^{-1}(c_{\mathcal{K}(H)}(F)) \subseteq c(e^{-1}(F))$ for every target subset F. The inverse is the inverse of the same bijection. Existence is required; no preferred homeomorphism is selected.

Assumptions

Only the input closure space is given. The Hilbert space and its structures are existential data. Nonzero H is required; separability and finite dimension are not. The definition uses arbitrary-set closure and does not replace it by finite closure.

Conclusion

This condition requires a full geometric model, with both directions referring to the same bijection. A continuous finite comparison is a separate condition; it does not contain the inverse full-continuity field required here. No statement about an irreducible representation image is included in this definition.

Main citations

- Exact definition.

- Continuous finite compact comparison.

Lean source signature (exact)

Exact source and declaration. The complete defining right-hand side is included.

```
def HasCompactOperatorHomeomorph.{uH, uX} {X : Type uX}
    (c : ClosureSpace X) : Prop :=
  ∃ (H : Type uH) (instN : NormedAddCommGroup H),
    letI := instN
    ∃ instI : InnerProductSpace ℂ H,
      letI := instI
      ∃ instC : CompleteSpace H,
        letI := instC
        ∃ instT : Nontrivial H,
          letI := instT
          Nonempty (Homeomorph c
            (NormedSpace.geometricStructureClosure ℂ
              (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))
```

In the source	Mathematical meaning

Input closure
`def HasCompactOperatorHomeomorph.{uH, uX} {X : Type uX}`
`(c : ClosureSpace X) : Prop :=`
	The input c is the closure on X. The following existential statement defines the requested model condition; it does not select a preferred model or map.

One normed model space
`∃ (H : Type uH) (instN : NormedAddCommGroup H),`
`letI := instN`
	There must exist one normed additive group H. The first `letI` installs that chosen normed structure on H for the inner-product type that follows.

Complex inner product on that model
`∃ instI : InnerProductSpace ℂ H,`
`letI := instI`
	The same H must have a compatible complex inner product. This `letI` installs it for the later Hilbert-space and operator-algebra types.

Completeness of that model
`∃ instC : CompleteSpace H,`
`letI := instC`
	The same normed space must be complete. The next `letI` installs this completeness, used in the compact-operator algebra on this Hilbert space.

A nonzero model
`∃ instT : Nontrivial H,`
`letI := instT`
	The same H must be nonzero. Its `letI` makes this nontriviality available in the remaining conditions; it does not introduce another space.

Existence of a homeomorphism to the same model
`Nonempty (Homeomorph c`
`(NormedSpace.geometricStructureClosure ℂ`
`(ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))`
	`Nonempty` requires an element of the displayed homeomorphism type: one bijection
	$$e : X \to S(\mathcal{K}(H))$$
	with continuity in both directions. The complete target argument is the geometric closure $c_{\mathcal{K}(H)}$ on the compact-operator algebra of the same H used in all preceding conditions.

In the source	Mathematical meaning

The homeomorphism and its inverse
`structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where`
`  continuous_toFun : c.Continuous d toFun`
`  continuous_invFun : d.Continuous c invFun`
	The inherited equivalence gives e and its inverse e^{-1}. For every $E \subseteq X$ and $F \subseteq S(\mathcal{K}(H))$, the two fields require, respectively,
	$$e(c(E)) \subseteq c_{\mathcal{K}(H)}(e(E)),$$
	$$e^{-1}(c_{\mathcal{K}(H)}(F)) \subseteq c(e^{-1}(F)).$$
	Both use the same model and the same map; the definition does not choose separate forward and inverse maps.

Continuity for all subsets
`def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=`
`  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`
	For $r : (Y, d) \to (Z, k)$, the definition requires
	$$r(d(E)) \subseteq k(r(E)) \quad (E \subseteq Y).$$
	There is no finiteness restriction in this condition.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
namespace ClosureSpace

open MathlibAnnex.Analysis.CStarAlgebra
attribute [local instance] ContinuousLinearMap.isClosed_compactOperatorStarSubalgebra
```

The model-space level `uH` and the input-space level `uX` are independent fixed parameters. All model structures and conditions above refer to the same quantified H.

Related exact source for the reading guide

Continuity for all subsets · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

The homeomorphism and its inverse · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```

A continuous finite comparison with a compact-operator GSS

`ClosureSpace.HasContinuousFiniteCompactComparison`

def

Requires a finite homeomorphism to a compact-operator geometric space whose forward map is fully continuous.

Statement

Let (X, c) be a closure space. A continuous finite compact comparison is the existence of a nonzero complex Hilbert space H and a finite homeomorphism from (X, c) to the geometric structure closure space $S(\mathcal{K}(H))$ whose forward map is also fully continuous.

Definition

Here $\mathcal{K}(H)$ denotes the algebra of compact operators on H. For a complex normed space D, $S(D)$ consists of the unions of kernels of norm-one bounded complex-linear functionals supporting each maximal proper face of its closed unit ball. Faces use real convexity. The support-functional norm theorem justifies the norm-one description for these nonempty faces. Its closure is $c_D(\mathcal{U}) = \{I \in S(D) : \bigcap_{J \in \mathcal{U}} J \subseteq I\}$.

For closure spaces $(X, c), (Y, d)$, full continuity of $e : X \to Y$ means $e(c(E)) \subseteq d(e(E))$ for every $E \subseteq X$. Finite continuity requires this only for finite E. The condition holds when there exists a bijection $e : X \to S(\mathcal{K}(H))$ such that e and e^{-1} are finitely continuous, and additionally

$$e(c(E)) \subseteq c_{\mathcal{K}(H)}(e(E)) \quad (E \subseteq X).$$

The norm and inner product belong to the same nonzero Hilbert space H. Full continuity of e^{-1} is not required by this condition.

Assumptions

The input is an arbitrary closure space. It need not already be presented as a GSS or an operator-algebra normal part. There is no representation of a C*-algebra in the input or conclusion. The model Hilbert space is nonzero; no separability or finite-dimensionality is required.

Conclusion

This is an existence property of a closure space, with forward full continuity added to finite continuity of a bijection and its inverse. No representation range condition is part of this definition.

Main citations

- Exact defining expression.
- The full-homeomorphism condition, which requires full continuity in both directions.

Lean source signature (exact)

Exact source and declaration. The complete defining right-hand side is included.

```
def HasContinuousFiniteCompactComparison.{uH, uX} {X : Type uX}
    (c : ClosureSpace X) : Prop :=
  ∃ (H : Type uH) (instN : NormedAddCommGroup H),
    letI := instN
    ∃ instI : InnerProductSpace ℂ H,
      letI := instI
      ∃ instC : CompleteSpace H,
        letI := instC
        ∃ instT : Nontrivial H,
          letI := instT
          ∃ e : FiniteHomeomorph c
              (NormedSpace.geometricStructureClosure ℂ
                (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)),
            c.Continuous (NormedSpace.geometricStructureClosure ℂ
              (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)) e.toEquiv
```

In the source	Mathematical meaning

Input closure
`def HasContinuousFiniteCompactComparison.{uH, uX} {X : Type uX}`
`(c : ClosureSpace X) : Prop :=`
	The input c is the closure on X. The following existential statement defines the requested model condition; it does not select a preferred model or map.

One normed model space
`∃ (H : Type uH) (instN : NormedAddCommGroup H),`
`letI := instN`
	There must exist one normed additive group H. The first `letI` installs that chosen normed structure on H for the inner-product type that follows.

Complex inner product on that model
`∃ instI : InnerProductSpace ℂ H,`
`letI := instI`
	The same H must have a compatible complex inner product. This `letI` installs it for the later Hilbert-space and operator-algebra types.

Completeness of that model
`∃ instC : CompleteSpace H,`
`letI := instC`
	The same normed space must be complete. The next `letI` installs this completeness, used in the compact-operator algebra on this Hilbert space.

A nonzero model
`∃ instT : Nontrivial H,`
`letI := instT`
	The same H must be nonzero. Its `letI` makes this nontriviality available in the remaining conditions; it does not introduce another space.

In the source	Mathematical meaning
Finite comparison with the compact-operator GSS `∃ e : FiniteHomeomorph c (NormedSpace.geometricStructureClosure ℂ (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)),`	There must exist one finite homeomorphism $$e : X \to S(\mathcal{K}(H)).$$ The target closure is $c_{\mathcal{K}(H)}$, the geometric closure of the compact-operator algebra on this same H, not a closure on H. For every finite $F \subseteq X$, $$e(c(F)) = c_{\mathcal{K}(H)}(e(F)).$$
Full forward continuity of the same map `c.Continuous (NormedSpace.geometricStructureClosure ℂ (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)) e.toEquiv`	`e.toEquiv` is the bijection underlying that same finite homeomorphism. The remaining condition is, for every $E \subseteq X$, $$e(c(E)) \subseteq c_{\mathcal{K}(H)}(e(E)).$$ It imposes no full-continuity condition on e^{-1}.
Finite continuity in both directions `structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where finitelyContinuous_toFun : c.FinitelyContinuous d toFun finitelyContinuous_invFun : d.FinitelyContinuous c invFun`	The inherited equivalence supplies e and e^{-1}. The two fields require finite continuity of these maps in opposite directions. Together they give the finite closure equality above; full continuity of the forward map is the extra condition in the preceding row.
Continuity for all subsets `def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop := ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`	For $r : (Y, d) \to (Z, k)$, the definition requires $$r(d(E)) \subseteq k(r(E)) \quad (E \subseteq Y).$$ There is no finiteness restriction in this condition.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
namespace ClosureSpace

open MathlibAnnex.Analysis.CStarAlgebra
attribute [local instance] ContinuousLinearMap.isClosed_compactOperatorStarSubalgebra
```

The model-space level `uH` and the input-space level `uX` are independent fixed parameters. All model structures and conditions above refer to the same quantified H.

Related exact source for the reading guide

Continuity for all subsets · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

Finite continuity in both directions · Fixed source

```
structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  finitelyContinuous_toFun : c.FinitelyContinuous d toFun
  finitelyContinuous_invFun : d.FinitelyContinuous c invFun
```

Continuous finite conjugation relative to a class of source spaces

`ClosureSpace.HasContinuousFiniteConjugation`

def

Defines when finite conjugation is continuous in both directions.

Statement

This property requires a conjugate fgf^{-1} and its inverse to be fully continuous whenever f and g satisfy the finite-continuity and forward full-continuity conditions below. The requirement ranges over every component of every closure space in a specified class.

A closure operator here is extensive, monotone, idempotent, and sends the empty set to the empty set. It need not satisfy the finite-union axiom of topological closure. A map $h : (X, c) \to (Y, d)$ is fully continuous when

$$h(c(E)) \subseteq d(h(E)) \qquad (E \subseteq X). \tag{1}$$

Finite continuity requires (1) only for finite E. A finite-homeomorphism is a bijection for which both the map and its inverse are finitely continuous; neither is thereby assumed fully continuous.

Definition

Let (X, c) be a closure space, and let D be a class of source closure spaces in a fixed universe. For an eligible source L and $t \in L$, let $M = \mathrm{Comp}_L(t)$, equipped with its **subspace closure**. Components are generated by pairs which are not already closed, together with equality; they are not assumed to be topological connected components or closed subsets.

The property $\mathrm{FC}_D(X, c)$ holds when the following universally quantified requirement holds:

$$\begin{aligned} &\text{for every } L \in D, \quad t \in L, \quad M = \mathrm{Comp}_L(t), \\ &\text{for every finite-homeomorphism } f : M \longrightarrow X \text{ with } f \text{ fully continuous}, \\ &\text{for every finite-homeomorphism } g : M \longrightarrow M \text{ with } g \text{ fully continuous}, \\ &h = f \circ g \circ f^{-1} \text{ and } h^{-1} = f \circ g^{-1} \circ f^{-1} \text{ are fully continuous on } (X, c). \end{aligned} \tag{2}$$

Both conclusions are part of the defining conjunction. Explicitly, for every $E \subseteq X$ they require

$$\begin{aligned} h(c(E)) &\subseteq c(h(E)), \\ h^{-1}(c(E)) &\subseteq c(h^{-1}(E)). \end{aligned} \tag{3}$$

The finite-conjugation construction uses the same outer maps f and f^{-1} in the inverse, not independently chosen maps.

The hypotheses in (2) impose full continuity on f and g only. They do not impose full continuity on f^{-1} or g^{-1}. Both full-continuity conclusions in (3) are required by the definition.

Assumptions

To form this property one supplies the target closure operator c and the eligible class D. In each test, L is any member of D, t is any point, and f, g are any finite-homeomorphisms satisfying the two stated forward full-continuity conditions. No eligible source, point, or map is existentially selected by the definition. If there are no qualifying tests, the universal requirement holds vacuously; the definition asserts no realization or existence theorem.

If $j : M \hookrightarrow L$ denotes the inclusion, the source closure used in both map hypotheses is

$$c_M(E) = j^{-1}\big(c_L(j(E))\big) \qquad (E \subseteq M). \qquad (4)$$

It is not replaced by a closure generated only by finite sets, and M is not equipped with an independently chosen topology.

Conclusion

The property $\mathrm{FC}_D(X, c)$ is defined by all of (2), including both lines of (3); it need not hold for every closure space. When it holds, the full conjugation homeomorphism uses exactly fgf^{-1} and its displayed inverse.

Main citations

Full versus finite continuity supplies (1). The finite conjugate and its inverse supplies the two maps in (2). The whole defining right-hand side is at lines 41–50; packaging that same conjugate is separately at lines 54–91. Subspace closure gives (4), and the separate C^*-realizability predicate specifies the eligible class in the application.

Lean source signature (exact)

For this definition the exact excerpt includes the **entire right-hand side**. The following declaration is separate.

```
def HasContinuousFiniteConjugation (c : ClosureSpace α) (D : ClosureCat.{s} → Prop) : Prop :=
  ∀ (L : ClosureCat.{s}), D L → ∀ t : L,
    ∀ f : FiniteHomeomorph (L.str.subspace (L.str.component t)) c,
      (L.str.subspace (L.str.component t)).Continuous c f.toEquiv →
      ∀ g : FiniteHomeomorph (L.str.subspace (L.str.component t))
          (L.str.subspace (L.str.component t)),
        (L.str.subspace (L.str.component t)).Continuous
          (L.str.subspace (L.str.component t)) g.toEquiv →
        c.Continuous c (f.conjugate g).toEquiv ∧
          c.Continuous c (f.conjugate g).toEquiv.symm
```

In the source	Mathematical meaning
Target closure and source class `def HasContinuousFiniteConjugation (c : ClosureSpace α) (D : ClosureCat.{s} → Prop) : Prop :=`	The input `c` is the closure on X (the source calls its underlying type `α`). The other input specifies the class D of source closure spaces. This defines the condition $\mathrm{FC}_D(X, c)$; it does not assert that the condition holds.
All source spaces and all components `∀ (L : ClosureCat.{s}), D L → ∀ t : L,`	Every closure space $L \in D$ and every point $t \in L$ is tested. The component used below is $M = \mathrm{Comp}_L(t)$ with the closure induced from L, as in (4). No particular source space or component is selected.

The comparison and its forward continuity
`∀ f : FiniteHomeomorph (L.str.subspace (L.str.component t)) c,`
`  (L.str.subspace (L.str.component t)).Continuous c f.toEquiv →`
	The first map is any finite homeomorphism $f : M \to X$. Both occurrences of `L.str.subspace (L.str.component t)` are the same induced closure c_M. Separately, for every $E \subseteq M$,
	$$f(c_M(E)) \subseteq c(f(E)).$$
	Thus the assumption is forward full continuity of this same f, not of f^{-1}.

The source automorphism and its forward continuity
`∀ g : FiniteHomeomorph (L.str.subspace (L.str.component t))`
`  (L.str.subspace (L.str.component t)),`
`  (L.str.subspace (L.str.component t)).Continuous`
`  (L.str.subspace (L.str.component t)) g.toEquiv →`
	The second map is any finite homeomorphism $g : M \to M$ on that same component with that same induced closure. It must additionally satisfy, for every $E \subseteq M$,
	$$g(c_M(E)) \subseteq c_M(g(E)).$$
	Full continuity of g^{-1} is not assumed.

Continuity of the conjugate `c.Continuous c (f.conjugate g).toEquiv ∧`	The equivalence `f.conjugate g` has underlying map $h = f \circ g \circ f^{-1} : X \to X$. The first conjunct requires, for every $E \subseteq X$, $$h(c(E)) \subseteq c(h(E)).$$
Continuity of the same conjugate inverse `c.Continuous c (f.conjugate g).toEquiv.symm`	`symm` gives the inverse of that same equivalence, namely $h^{-1} = f \circ g^{-1} \circ f^{-1}$. The second conjunct requires, for every $E \subseteq X$, $$h^{-1}(c(E)) \subseteq c(h^{-1}(E)).$$ The two conjuncts together are (3).

Finite-homeomorphism fields
`structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where`
`  finitelyContinuous_toFun : c.FinitelyContinuous d toFun`
`  finitelyContinuous_invFun : d.FinitelyContinuous c invFun`
	The inherited equivalence supplies a bijection and its inverse. Both fields impose closure continuity only on finite subsets, as defined in the Statement. These fields are used for $f : M \to X$ and $g : M \to M$; their additional full-continuity assumptions are separate.

In the source	Mathematical meaning

The conjugate and its actual inverse
`def FiniteHomeomorph.conjugate (f : FiniteHomeomorph c d)`
`    (g : FiniteHomeomorph c c) : FiniteHomeomorph d d :=`
`  f.symm.trans (g.trans f)`
	The conjugate is formed from the same finite homeomorphisms f and g. Composition sends $x \in X$ first by f^{-1}, then by g, then by f. Its inverse uses g^{-1} between those same outer maps, as displayed in (2).

Exact surrounding source context (separate excerpts)

These are independent ambient excerpts.

```
universe u v w s t
```

```
variable {α : Type u} {β : Type v} {γ : Type w}
variable {c : ClosureSpace α} {d : ClosureSpace β} {k : ClosureSpace γ}
```

The selected definition uses the carrier `α` and explicitly supplies its own `c`, together with source universe `s`. The surrounding `β`, `γ`, `d`, and `k` serve later transport declarations and are not additional inputs to this definition. No following declaration's local hypotheses are imported into its scope.

Related exact source for the reading guide

The conjugate and its actual inverse · Fixed source

```
def FiniteHomeomorph.conjugate (f : FiniteHomeomorph c d)
    (g : FiniteHomeomorph c c) : FiniteHomeomorph d d :=
  f.symm.trans (g.trans f)
```

Finite-homeomorphism fields · Fixed source

```
structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  finitelyContinuous_toFun : c.FinitelyContinuous d toFun
  finitelyContinuous_invFun : d.FinitelyContinuous c invFun
```

A finite-dimensional compact model with a dimension condition

`ClosureSpace.HasFiniteDimensionalCompactHomeomorph`

def

Requires a finite-dimensional compact model satisfying a specified condition on its dimension.

Statement

Let (X, c) be a closure space and let $P(n)$ be a condition on a natural number n. The finite-dimensional compact-homeomorphism condition holds when there exist a nonzero finite-dimensional complex Hilbert space H satisfying $P(\dim_{\mathbb{C}} H)$ and a full homeomorphism $(X, c) \to S(\mathcal{K}(H))$.

Definition

Here $\mathcal{K}(H)$ denotes the algebra of compact operators on H. For a complex normed space D, $S(D)$ consists of the unions of kernels of norm-one bounded complex-linear functionals supporting each maximal proper face of its closed unit ball. Faces use real convexity. The support-functional norm theorem justifies the norm-one description for these nonempty faces. Its closure is $c_D(\mathcal{U}) = \{I \in S(D) : \bigcap_{J \in \mathcal{U}} J \subseteq I\}$.

All the following data refer to the same H: its norm, complex inner product and completeness, nontriviality, finite-dimensionality, the truth of $P(\dim_{\mathbb{C}} H)$, and a full closure-space homeomorphism

$$e : (X, c) \longrightarrow S(\mathcal{K}(H)).$$

Full continuity applies to arbitrary subsets in both directions. Finite-dimensionality is required separately from the condition on its finite complex dimension.

Assumptions

The input consists only of c and P. The condition requires the existence of the Hilbert space, its finite-dimensionality, and the stated dimension condition. Nonzero H is explicit in the literal right-hand side. No separability hypothesis is added. Here the natural number dimension is finite complex vector-space dimension; later infinite-dimensional comparisons use Hilbert-basis cardinality separately.

Conclusion

One finite-dimensional Hilbert space must satisfy both the dimension condition and the homeomorphism requirement. Taking $P(m)$ to be $m = n$ specifies an exact dimension; taking it to be $m \leq n$ specifies an upper bound. Neither choice replaces the explicit finite-dimensionality and

nonzero-model conditions.

Main citations

- Exact definition.
- Finite-comparison counterpart.

Lean source signature (exact)

Exact source and declaration. The complete defining right-hand side is included.

```
def HasFiniteDimensionalCompactHomeomorph.{uH, uX} {X : Type uX}
    (c : ClosureSpace X) (P : ℕ → Prop) : Prop :=
  ∃ (H : Type uH) (instN : NormedAddCommGroup H),
    letI := instN
    ∃ instI : InnerProductSpace ℂ H,
      letI := instI
      ∃ instC : CompleteSpace H,
        letI := instC
        ∃ instT : Nontrivial H,
          letI := instT
          FiniteDimensional ℂ H ∧ P (Module.finrank ℂ H) ∧
            Nonempty (Homeomorph c
              (NormedSpace.geometricStructureClosure ℂ
                (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))
```

In the source	Mathematical meaning

Input closure and dimension condition
`def HasFiniteDimensionalCompactHomeomorph.{uH, uX} {X : Type uX}`
`(c : ClosureSpace X) (P : ℕ → Prop) : Prop :=`
	The input c is the closure on X. The second input is the condition $P(n)$ on a natural number n. The following existential statement defines the requested model condition; it does not select a preferred model or map.

One normed model space
`∃ (H : Type uH) (instN : NormedAddCommGroup H),`
`letI := instN`
	There must exist one normed additive group H. The first `letI` installs that chosen normed structure on H for the inner-product type that follows.

Complex inner product on that model
`∃ instI : InnerProductSpace ℂ H,`
`letI := instI`
	The same H must have a compatible complex inner product. This `letI` installs it for the later Hilbert-space and operator-algebra types.

Completeness of that model
`∃ instC : CompleteSpace H,`
`letI := instC`
	The same normed space must be complete. The next `letI` installs this completeness, used in the compact-operator algebra on this Hilbert space.

A nonzero model
`∃ instT : Nontrivial H,`
`letI := instT`
	The same H must be nonzero. Its `letI` makes this nontriviality available in the remaining conditions; it does not introduce another space.

Finite-dimensionality and its dimension condition `FiniteDimensional ℂ H ∧ P (Module.finrank ℂ H) ∧`	The same H must be finite-dimensional, and its finite complex dimension must satisfy $P(\dim_{\mathbb{C}} H)$. These are two conditions. The finite-dimensionality requirement is not replaced by a condition on `finrank`.

In the source	Mathematical meaning
Existence of a homeomorphism to the same model `Nonempty (Homeomorph c (NormedSpace.geometricStructureClosure ℂ (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))`	`Nonempty` requires an element of the displayed homeomorphism type: one bijection $$e : X \to S(\mathcal{K}(H))$$ with continuity in both directions. The complete target argument is the geometric closure $c_{\mathcal{K}(H)}$ on the compact-operator algebra of the same H used in all preceding conditions.

The homeomorphism and its inverse
`structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where`
`  continuous_toFun : c.Continuous d toFun`
`  continuous_invFun : d.Continuous c invFun`
	The inherited equivalence gives e and its inverse e^{-1}. For every $E \subseteq X$ and $F \subseteq S(\mathcal{K}(H))$, the two fields require, respectively, $$e(c(E)) \subseteq c_{\mathcal{K}(H)}(e(E)),$$ $$e^{-1}(c_{\mathcal{K}(H)}(F)) \subseteq c(e^{-1}(F)).$$ Both use the same model and the same map; the definition does not choose separate forward and inverse maps.

Continuity for all subsets
`def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=`
`  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`
	For $r : (Y, d) \to (Z, k)$, the definition requires $$r(d(E)) \subseteq k(r(E)) \quad (E \subseteq Y).$$ There is no finiteness restriction in this condition.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
namespace ClosureSpace

attribute [local instance] ContinuousLinearMap.isClosed_compactOperatorStarSubalgebra
```

The model-space level `uH` and the input-space level `uX` are independent fixed parameters. All model structures and conditions above refer to the same quantified H.

Related exact source for the reading guide

Continuity for all subsets · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

The homeomorphism and its inverse · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```

The canonical compact c0-decomposition

`MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.compactC0Decomposition`

def

Defines the compact-operator c_0-decomposition associated with a faithful compact-valued representation.

Statement

Let $\pi : A \to \mathcal{B}(H)$ be an injective compact-valued representation of a complex C*-algebra on a complete complex Hilbert space H, and write $\mathcal{K}(L)$ for the compact operators on a Hilbert space L. Let $M(A)$ be the set of minimal nonzero closed $*$-ideals of A, and write $I_i \subseteq A$ for the ideal indexed by i. Set $J = (I_i)_{i \in M(A)}$. For each $i \in M(A)$, take the chosen cyclic Hilbert space H_i and compact model $E_i : I_i \to \mathcal{K}(H_i)$ produced from this same π. The definition constructs one $*$-isomorphism

$$D_\pi : A \longrightarrow \bigoplus_{i \in M(A)}^{c_0} \mathcal{K}(H_i).$$

Definition

The index set $M(A)$ is the family of all minimal nonzero closed $*$-ideals. The Hilbert space H_i is the compact model space chosen from this same π and ideal i, with the corresponding compact-model isomorphism $E_i : I_i \to \mathcal{K}(H_i)$.

The target compact-operator c_0-sum is the subalgebra of the bounded product given by

$$\bigoplus_{i \in M(A)}^{c_0} \mathcal{K}(H_i) = \{(T_i) : \forall \varepsilon > 0,\ \{i : \|T_i\| \geq \varepsilon\} \text{ is finite}\}.$$

Its algebra operations are coordinatewise and its norm is $\sup_i \|T_i\|$. See the dependent c_0-sum for this same vanishing condition and inherited norm.

Let $S_J : \bigoplus_i^{c_0} I_i \to A$ be the specified ideal-sum equivalence. Its forward map is the continuous ideal-sum map, whose finite-support values are the corresponding sums in A. Let $E_* : \bigoplus_i^{c_0} I_i \to \bigoplus_i^{c_0} \mathcal{K}(H_i)$ act by the same coordinate models E_i. The RHS uses the compact-coordinate equivalence, namely

$$D_\pi = E_* \circ S_J^{-1}.$$

The coordinate-evaluation theorem gives

$$(D_\pi(a))_i = E_i\big((S_J^{-1}(a))_i\big).$$

The single-coordinate formula, finite-truncation formula, and inverse identification specify

$$D_\pi^{-1}(\mathrm{single}_i\, T) = \iota_i(E_i^{-1}(T)), \qquad D_\pi^{-1}(\mathrm{truncate}_F\, T) = \sum_{i\in F} \iota_i(E_i^{-1}(T_i)).$$

Here $\iota_i : I_i \hookrightarrow A$ is the inclusion, F is finite, and truncation retains T_i on F and is zero elsewhere. The cited evaluation theorems give these formulas for D_π^{-1}.

The defining expression uses the canonical ideal family, the orthogonality theorem for the minimal ideal family, the spaces H_i and maps E_i, and the density theorem for finite sums of these ideals. These theorems guarantee orthogonality and density for the specified ideal family; neither is an additional assumption.

Assumptions

Only π and its faithfulness and pointwise compactness are inputs beyond the algebra and Hilbert structures. The cited theorems guarantee orthogonality, density, nonzero Hilbert spaces, and compact models for this family. The index can be empty. No unit, nonzero algebra, countability, separability, or finite-dimensionality is imposed.

Conclusion

The output is a $*$-isomorphism using the same ideal family, representation and dependent model choices throughout. The definition permits an empty index family and does not assume a nonzero algebra. It does not identify the block algebra with all compact operators on a Hilbert sum.

Main citations

- Exact defining right-hand side and companion formulas.
- The ideal family, density theorem, and chosen Hilbert spaces.
- compactC0Decomposition_symm_apply: exact inverse assembly.
- compactC0SumEquivOfDense: complete composition RHS.
- eval_compactC0SumEquivOfDense: coordinate evaluation.
- compactC0SumEquivOfDense_symm_apply: the same inverse map.
- compactC0SumEquivOfDense_symm_single: inverse of one coordinate.
- compactC0SumHom_truncate: the finite-truncation value.

Lean source signature (exact)

Exact source and declaration. The complete defining right-hand side is included.

```
def compactC0Decomposition :
    letI : PartialOrder A := _root_.CStarAlgebra.spectralOrder A
    letI : StarOrderedRing A := _root_.CStarAlgebra.spectralOrderedRing A
    A ≃⋆ₐ[ℂ] CompactOperatorC0Sum
      (fun i : MinimalClosedStarIdeal A => i.CompactModelSpace pi hpi) := by
  letI : PartialOrder A := _root_.CStarAlgebra.spectralOrder A
  letI : StarOrderedRing A := _root_.CStarAlgebra.spectralOrderedRing A
  exact ClosedStarIdeal.compactC0SumEquivOfDense
    (minimalClosedStarIdealFamily (A := A))
    pairwiseOrthogonal_minimalClosedStarIdealFamily
    (fun i : MinimalClosedStarIdeal A => i.CompactModelSpace pi hpi)
    (fun i => i.compactModelEquiv pi hpi)
    (dense_finiteSums_minimalClosedStarIdealFamily pi hpi)
```

In the source	Mathematical meaning
Exact declaration `pi`, `hpi` in the surrounding context	The fixed faithful compact-valued representation π, used for all the models.
The result-type **Exact declaration** `letI` expressions	The compatible spectral order on A used in this construction.
Exact declaration `minimalClosedStarIdealFamily (A := A)`	The family $J = (I_i)_{i \in M(A)}$ of all minimal nonzero closed $*$-ideals of A.
Exact declaration `pairwiseOrthogonal_minimalClosedStarIdealFamily`	The guarantee that $xy = 0$ for $x \in I_i$, $y \in I_j$ and $i \neq j$. It is not an extra hypothesis.
Exact declaration `i.CompactModelSpace pi hpi`	The chosen cyclic Hilbert space H_i for the ideal I_i and the same representation π.
Exact declaration `i.compactModelEquiv pi hpi`	The $*$-isomorphism $E_i : I_i \to \mathcal{K}(H_i)$ using that ideal's chosen projection and cyclic vector.
Exact declaration `dense_finiteSums_minimalClosedStarIdealFamily pi hpi`	The guarantee that finite sums of elements of the ideals I_i are norm dense in A. This too follows from the cited result, rather than being an extra input.
Exact declaration `compactC0SumEquivOfDense`	The map $D_\pi = E_* \circ S_J^{-1}$: first take ideal coordinates using the inverse of the specified ideal-sum map, then apply the same E_i in each coordinate.
Exact declaration `A ≃⋆ₐ[ℂ] CompactOperatorC0Sum ...`	A $*$-isomorphism from A onto the displayed c_0-sum of compact-operator algebras.

Exact surrounding source context (separate excerpts)

Source lines 53-60:

```
universe u v

variable {A : Type u} [NonUnitalCStarAlgebra A]
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable (pi : NonUnitalCStarRepresentation A H) (hpi : pi.IsCompactlyRepresented)

attribute [local instance] _root_.CStarAlgebra.spectralOrder
  _root_.CStarAlgebra.spectralOrderedRing
```

A representation component is the full coefficient-kernel family

`MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.component_matrixCoefficientKernel_eq_family`

theorem

Describes each irreducible representation component by all of its nonzero vector coefficients.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, with no identity required, and let $\pi : A \to \mathcal{B}(H)$ be a nonzero irreducible $*$-representation on a complete complex Hilbert space. The inner product is conjugate-linear in its first argument. For nonzero $p, q \in H$, put

$$f^{\pi}_{p,q}(a) = \langle q, \pi(a)(p) \rangle, \qquad \mathcal{F}_{\pi} = \{\ker f^{\pi}_{p,q} : p, q \neq 0\}.$$

Write $S(A)$ for the geometric structure space of A. In this C*-algebra setting its points are precisely $\ker \rho$, where ρ is extreme, for real convexity, in the closed unit ball of the Banach dual; the extreme-kernel identification guarantees this description from the original face definition. For $\mathcal{T} \subseteq S(A)$, its closure is

$$c_A(\mathcal{T}) = \left\{ I \in S(A) : \bigcap_{J \in \mathcal{T}} J \subseteq I \right\}.$$

Two points are linked if they are equal or their pair is not closed. Let $C(I)$ be the equivalence class of I under finite chains of linked points. Then, for every nonzero $x, y \in H$,

$$C(\ker f^{\pi}_{x,y}) = \mathcal{F}_{\pi}.$$

Assumptions

The source uses the compatible C*-order on A. Irreducibility includes nonzero action and excludes proper nonzero closed reducing subspaces. No faithfulness, finite dimension or separability is required. Under this irreducibility hypothesis, $\ker f^{\pi}_{p,q}$ is a point of $S(A)$ for every $p, q \neq 0$. The component is defined by the closure relation above, without assuming that c_A is a topological closure.

Conclusion

Every point of the entire component, including one initially presented without reference to π, is a coefficient kernel of this fixed representation. Conversely, every pair of nonzero coefficient vectors gives a point of the component. Thus the choice of the initial pair (x, y) has no effect on this

component. This is an equality of subsets of $S(A)$; it does not identify an individual coefficient kernel with $\ker \pi$.

Main citations

- Exact whole-component theorem: `MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.component_matrixCoefficientKernel_eq_family`.
- Component–representation correspondence: the equivalence between finite-chain components and unitary equivalence.
- The coefficient-kernel family and its same-component inclusion: `matrixCoefficientKernelFamily` and `matrixCoefficientKernelFamily_subset_component`.
- Ryotaro Tanaka, arXiv:2504.05551v1, Lemma 5.10. The present declaration gives the component formula for an arbitrary irreducible representation; the pure-state specialization and the intersection calculation have separate explanations.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem component_matrixCoefficientKernel_eq_family
    (pi : NonUnitalCStarRepresentation A H) (hpi : pi.IsIrreducible)
    (x y : H) (hx : x ≠ 0) (hy : y ≠ 0) :
    (NormedSpace.geometricStructureClosure ℂ A).component
      (matrixCoefficientKernel pi hpi x y hx hy) = matrixCoefficientKernelFamily pi hpi
```

In the source	Mathematical meaning

Algebra and order
`variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]`
`variable [PartialOrder A] [StarOrderedRing A]`
	The nonzero complex C*-algebra A with its compatible order; no identity is required.

Representation space `variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`	The complex Hilbert space H is complete.
One irreducible representation `(pi : NonUnitalCStarRepresentation A H) (hpi : pi.IsIrreducible)`	The same fixed nonzero irreducible representation $\pi : A \to \mathcal{B}(H)$ supplies all coefficient kernels. It need not be faithful.
Starting vectors `(x y : H) (hx : x ≠ 0) (hy : y ≠ 0) :`	The starting pair is $x, y \in H \setminus \{0\}$, without a normalization assumption.

Component and full family
`(NormedSpace.geometricStructureClosure ℂ A).component`
`  (matrixCoefficientKernel pi hpi x y hx hy) = matrixCoefficientKernelFamily pi hpi := by`
	The component is taken in the ambient closure c_A, starting at $\ker f^{\pi}_{x,y}$. The complete equality is
	$$C(\ker f^{\pi}_{x,y}) = \mathcal{F}_{\pi}.$$
	It concerns all points of that component, not just kernels chosen in advance.

In the source	Mathematical meaning
The coefficient-kernel family `def matrixCoefficientKernelFamily (pi : NonUnitalCStarRepresentation A H) (hpi : pi.IsIrreducible) : Set (NormedSpace.GeometricStructureSpace ℂ A) := {I | ∃ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0), matrixCoefficientKernel pi hpi x y hx hy = I}`	A point $I \in S(A)$ belongs to $\mathcal{F}_\pi$ exactly when there are $p, q \in H \setminus \{0\}$ with $I = \ker f^\pi_{p,q}$. The vectors may depend on I; the representation is the same π throughout.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Exact surrounding source:

```
variable [PartialOrder A] [StarOrderedRing A]
```

Exact surrounding source:

```
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```

Related exact source for the reading guide

The coefficient-kernel family · Fixed source

```
def matrixCoefficientKernelFamily (pi : NonUnitalCStarRepresentation A H) (hpi :
pi.IsIrreducible) :
    Set (NormedSpace.GeometricStructureSpace ℂ A) :=
  {I | ∃ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0), matrixCoefficientKernel pi hpi x y hx hy = I}
```

Algebra and order · Fixed source

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
variable [PartialOrder A] [StarOrderedRing A]
```

Component and full family · Fixed source

```
    (NormedSpace.geometricStructureClosure ℂ A).component
      (matrixCoefficientKernel pi hpi x y hx hy) = matrixCoefficientKernelFamily pi hpi := by
```

Components of coefficient kernels encode unitary equivalence

`MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.related_matrixCoefficientKernel_iff_unitaryEquivalent`

theorem

Characterizes equality of coefficient-kernel components by unitary equivalence of representations.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra and let $\pi : A \to \mathcal{B}(H)$ and $\sigma : A \to \mathcal{B}(K)$ be nonzero irreducible $*$-representations on complete complex Hilbert spaces. No identity in A is required. Put

$$f^{\pi}_{x,y} : A \longrightarrow \mathbb{C}, \qquad f^{\pi}_{x,y}(a) = \langle y, \pi(a)(x)\rangle \quad (x, y \neq 0),$$

and define $f^{\sigma}_{z,w}$ in the same way. The inner products are linear in the second slot. The geometric structure space $S(A)$ consists, by the extreme-kernel identification, of kernels $\ker \rho \subseteq A$ of bounded complex-linear functionals $\rho : A \to \mathbb{C}$ that are extreme points, for real convexity, of the entire closed dual unit ball B_{A^*}. Its closure is

$$c_A(\mathcal{T}) = \left\{ I \in S(A) : \bigcap_{J \in \mathcal{T}} J \subseteq I \right\}.$$

Write $I \leftrightarrow J$ when $I = J$ or $c_A(\{I, J\}) \neq \{I, J\}$, and write $I \sim J$ for the equivalence relation generated by these links. For nonzero $x, y \in H$ and $z, w \in K$,

$$\ker f^{\pi}_{x,y} \sim \ker f^{\sigma}_{z,w} \quad \Longleftrightarrow \quad \pi \simeq_u \sigma.$$

Here $\pi \simeq_u \sigma$ means that π and σ are unitarily equivalent, that is, there is one surjective complex-linear isometry $U : H \to K$ satisfying $U\pi(a) = \sigma(a)U$ for every $a \in A$.

Assumptions

The source carries the compatible C*-order on A. The Hilbert spaces may have different dimensions, and neither representation is assumed faithful. Each of the four vectors is required to be nonzero. The relation on the left is the pre-existing geometric finite-chain relation, not a relation defined by representation equivalence.

Conclusion

The assertion concerns the full finite-chain relation and one simultaneous intertwiner of the two original representations. Neither direction replaces that relation by direct linking of every pair in a component. Faithfulness is not a conclusion.

Main citations

- Exact equivalence and its finite-chain converse:
  `related_matrixCoefficientKernel_iff_unitaryEquivalent`.
- Ryotaro Tanaka, arXiv:2504.05551v1, Theorem 5.8. This declaration states the result directly for arbitrary irreducible representations and nonzero coefficient vectors.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem related_matrixCoefficientKernel_iff_unitaryEquivalent
    (pi : NonUnitalCStarRepresentation A H) (sigma : NonUnitalCStarRepresentation A K)
    (hpi : pi.IsIrreducible) (hsigma : sigma.IsIrreducible)
    (x y : H) (z w : K) (hx : x ≠ 0) (hy : y ≠ 0) (hz : z ≠ 0) (hw : w ≠ 0) :
    (NormedSpace.geometricStructureClosure ℂ A).Related
      (matrixCoefficientKernel pi hpi x y hx hy) (matrixCoefficientKernel sigma hsigma z w hz
hw) ↔
        pi.UnitaryEquivalent sigma
```

In the source	Mathematical meaning

Algebra and order
`variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]`
`variable [PartialOrder A] [StarOrderedRing A]`
	The nonzero complex C*-algebra A with its compatible order is the domain of both representations.

Hilbert spaces
`variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`
`variable {K : Type w} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]`
	The complex Hilbert spaces H, K are complete. Their universe levels may differ.

Representations and irreducibility
`(pi : NonUnitalCStarRepresentation A H) (sigma : NonUnitalCStarRepresentation A K)`
`(hpi : pi.IsIrreducible) (hsigma : sigma.IsIrreducible)`
	The representations $\pi : A \to \mathcal{B}(H)$ and $\sigma : A \to \mathcal{B}(K)$ are nonzero and irreducible. Faithfulness is not assumed.

Coefficient vectors `(x y : H) (z w : K) (hx : x ≠ 0) (hy : y ≠ 0) (hz : z ≠ 0) (hw : w ≠ 0) :`	The four vectors satisfy $x, y \in H \setminus \{0\}$ and $z, w \in K \setminus \{0\}$. They are not required to have unit norm.

Finite-chain relation of the two kernels
`(NormedSpace.geometricStructureClosure ℂ A).Related`
`  (matrixCoefficientKernel pi hpi x y hx hy)`
`(matrixCoefficientKernel sigma hsigma z w hz hw) ↔`
	The ambient closure is c_A on $S(A)$. The complete two arguments are the coefficient-kernel points of the displayed representations and vectors. The left side of the equivalence is
	$$\ker f^{\pi}_{x,y} \sim \ker f^{\sigma}_{z,w}.$$
	This is finite-chain relatedness, including equality, rather than direct linking alone.

In the source	Mathematical meaning
Unitary equivalence `pi.UnitaryEquivalent sigma :=`	The right side says that there is one surjective complex-linear isometry $U : H \to K$ with $$U\pi(a) = \sigma(a)U \qquad (a \in A).$$ The equivalence joins this single simultaneous intertwiner condition to the preceding finite-chain relation.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Exact surrounding source:

```
variable [PartialOrder A] [StarOrderedRing A]
```

Exact surrounding source:

```
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```

Exact surrounding source:

```
variable {K : Type w} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
```

Related exact source for the reading guide

Algebra and order · Fixed source

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
variable [PartialOrder A] [StarOrderedRing A]
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable {K : Type w} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
```

Unitary equivalence · Fixed source

```
        pi.UnitaryEquivalent sigma :=
```

Two-sided continuity is exact operator equality

`MathlibAnnex.Analysis.CStarAlgebra.continuous_normalPartLinearMap_and_inverse_iff`

theorem

Characterizes two-sided continuity of a normal-part equivalence by equality of operator sets.

Statement

Let H, K be complex Hilbert spaces. Let $A \subseteq \mathcal{B}(H)$ and $B \subseteq \mathcal{B}(K)$ be nonzero norm-closed complex $*$-subalgebras acting irreducibly on H and K, respectively. Neither algebra is assumed unital. For $x, y \in H \setminus \{0\}$ and $z, w \in K \setminus \{0\}$, define

$$\begin{aligned} f^A_{x,y} &: A \to \mathbb{C}, \qquad f^A_{x,y}(a) = \langle y, a(x)\rangle, \\ f^B_{z,w} &: B \to \mathbb{C}, \qquad f^B_{z,w}(b) = \langle w, b(z)\rangle. \end{aligned}$$

Both inner products are conjugate-linear in the first variable. The normal parts are

$$\begin{aligned} N(A) &= \Big\{\ker f^A_{x,y} : x, y \in H \setminus \{0\}\Big\}, \\ N(B) &= \big\{\ker f^B_{z,w} : z, w \in K \setminus \{0\}\big\}. \end{aligned}$$

Let $U, V : H \to K$ be bounded complex-linear isomorphisms with bounded inverses. The maps on normal points are

$$\begin{aligned} F &: N(A) \longrightarrow N(B), \qquad F(\ker f^A_{x,y}) = \ker f^B_{Ux,Vy}, \\ F^{-1} &: N(B) \longrightarrow N(A), \qquad F^{-1}(\ker f^B_{z,w}) = \ker f^A_{U^{-1}z,V^{-1}w}. \end{aligned} \tag{1}$$

The normal-part equivalence guarantees that these are inverse bijections, and its point-evaluation theorem guarantees the formulas for every representative. For the inherited hull–kernel closures,

$$F \text{ and } F^{-1} \text{ are fully continuous} \iff V^*BU = A. \tag{2}$$

Here $V^* : K \to H$ is the adjoint of V, and $V^*BU = \{V^*bU : b \in B\}$. For $D \subseteq N(A)$, write $c_A(D) = \{I \in N(A) : \bigcap_{J \in D} J \subseteq I\}$. For $E \subseteq N(B)$, write $c_B(E) = \{I \in N(B) : \bigcap_{J \in E} J \subseteq I\}$. Full continuity of F means

$$F(c_A(D)) \subseteq c_B(F(D)) \qquad \text{for every } D \subseteq N(A),$$

and full continuity of its inverse means

$$F^{-1}(c_B(E)) \subseteq c_A(F^{-1}(E)) \qquad \text{for every } E \subseteq N(B).$$

Assumptions

Boundedness of the vector maps and their inverses is part of the input, making their adjoints bounded operators in the indicated directions. They are not assumed unitary or assumed to intertwine A, B. The map F^{-1} is the inverse arising from the **same** U, V, not an independently chosen map. The equality on the right of (2) is a conclusion equivalent to the two continuities, not an additional hypothesis on both sides.

Conclusion

Two-sided full continuity is equivalent to equality of concrete operator sets in $\mathcal{B}(H)$. It is stronger than the finite-homeomorphism property obtained from the algebraic vector changes alone. No assertion is made that the operator transport $b \mapsto V^* b U$ is multiplicative or preserves adjoints.

Main citations

- One-direction full continuity: the related inclusion criterion for the same specified forward map.
- Exact two-sided bounded criterion: forward and inverse maps use the same bounded equivalences.
- arXiv:2504.05551v1, Theorem 6.9: the full-homeomorphism image condition.
- The bounded operator transport: `adjointSandwich U V b` is exactly the composite $V^* \circ b \circ U$, from H to H.
- Normal-part point evaluation and its inverse equivalence: the same vector changes define the map and its inverse.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem continuous_normalPartLinearMap_and_inverse_iff (U V : H ≃L[ℂ] K) :
    (((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Continuous
        ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T))
        (normalPartSemilinearMap S T hT U.toLinearEquiv V.toLinearEquiv) ∧
      ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)).Continuous
        ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))
        (normalPartSemilinearMap T S hS U.symm.toLinearEquiv V.symm.toLinearEquiv)) ↔
      InnerProductSpace.adjointSandwich U.toContinuousLinearMap V.toContinuousLinearMap ''
        (T : Set (K →L[ℂ] K)) = (S : Set (H →L[ℂ] H))
```

In the source, `S` and `T` denote the algebras A and B used here, respectively.

In the source	Mathematical meaning

Hilbert spaces and irreducible algebras
`variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`
`variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]`
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))`
`variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]`
`variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
	The spaces H, K are complex Hilbert spaces. The source algebras `S`,`T` are $A \subseteq \mathcal{B}(H)$ and $B \subseteq \mathcal{B}(K)$. Closedness and the two inclusion representations express the nonzero norm-closed irreducible actions in the Statement.

Bounded linear isomorphisms `theorem continuous_normalPartLinearMap_and_inverse_iff (U V : H ≃L[ℂ] K) :`	The same two bounded complex-linear isomorphisms $U, V : H \to K$, with bounded inverses, define F and F^{-1} in (1).

Forward closure continuity
`(((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Continuous`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T))`
`  (normalPartSemilinearMap S T hT U.toLinearEquiv V.toLinearEquiv) ∧`
	The `subspace` arguments give the relative closures on $N(A)$ and $N(B)$. The underlying linear equivalences are the same U, V, with their boundedness forgotten only in this map constructor. For every $D \subseteq N(A)$,
	$$F(c_A(D)) \subseteq c_B(F(D)).$$

Inverse closure continuity
`((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)).Continuous`
`  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))`
`  (normalPartSemilinearMap T S hS U.symm.toLinearEquiv V.symm.toLinearEquiv)) ↔`
	The second map uses the actual inverses U^{-1}, V^{-1}, with the algebras exchanged. For every $E \subseteq N(B)$,
	$$F^{-1}(c_B(E)) \subseteq c_A(F^{-1}(E)).$$
	These two conditions, joined by conjunction, are the left side of (2).

Adjoint composition
`def adjointSandwich (U V : H →L[𝕜] K) (b : K →L[𝕜] K) : H →L[𝕜] H :=`
`  V.adjoint.comp (b.comp U)`
	For $b \in \mathcal{B}(K)$ this is the bounded operator $V^* b U \in \mathcal{B}(H)$, in exactly that order. The adjoint $V^* : K \to H$ is taken for the same Hilbert structures.

Equality of operator sets
`InnerProductSpace.adjointSandwich U.toContinuousLinearMap V.toContinuousLinearMap ''`
`  (T : Set (K →L[ℂ] K)) = (S : Set (H →L[ℂ] H)) :=`
	The image operation applies $b \mapsto V^* b U$ to B. The right side is
	$$V^* B U = \{V^* b U : b \in B\} = A.$$
	The theorem makes this equivalent to both continuity conditions for the same F, F^{-1}. It does not assert that $b \mapsto V^* b U$ preserves multiplication or the involution.

The two displayed maps are inverse maps induced by the same bounded equivalences; the later bundled full-homeomorphism definition is not included.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))
variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
S))
variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
T))
```

The effective source context uses independent universes for H and K. The adjoint in V^*bU is the bounded Hilbert-space adjoint from K to H. The literal definition is `V.adjoint.comp (b.comp U)`; its order is the same as the operator formula above.

Related exact source for the reading guide

Equality of operator sets · Fixed source

```
      InnerProductSpace.adjointSandwich U.toContinuousLinearMap V.toContinuousLinearMap ''
        (T : Set (K →L[ℂ] K)) = (S : Set (H →L[ℂ] H)) :=
```

Adjoint composition · Fixed source

```
def adjointSandwich (U V : H →L[𝕜] K) (b : K →L[𝕜] K) : H →L[𝕜] H :=
  V.adjoint.comp (b.comp U)
```

Continuity is an operator-inclusion condition

`MathlibAnnex.Analysis.CStarAlgebra.continuous_normalPartLinearMap_iff`

theorem

Characterizes continuity of a normal-part map by an inclusion of operator sets.

Statement

Let H, K be complex Hilbert spaces. Let $A \subseteq \mathcal{B}(H)$ and $B \subseteq \mathcal{B}(K)$ be nonzero norm-closed complex $*$-subalgebras acting irreducibly on H and K, respectively. Neither algebra is assumed unital. For $x, y \in H \setminus \{0\}$ and $z, w \in K \setminus \{0\}$, define

$$\begin{aligned} f^A_{x,y} &: A \to \mathbb{C}, \qquad f^A_{x,y}(a) = \langle y, a(x) \rangle, \\ f^B_{z,w} &: B \to \mathbb{C}, \qquad f^B_{z,w}(b) = \langle w, b(z) \rangle. \end{aligned}$$

Both inner products are conjugate-linear in the first variable. The normal parts are

$$\begin{aligned} N(A) &= \left\{ \ker f^A_{x,y} : x, y \in H \setminus \{0\} \right\}, \\ N(B) &= \left\{ \ker f^B_{z,w} : z, w \in K \setminus \{0\} \right\}. \end{aligned}$$

For $D \subseteq N(A)$, the inherited hull–kernel closure is

$$c_A(D) = \left\{ I \in N(A) : \bigcap_{J \in D} J \subseteq I \right\},$$

and c_B is defined analogously on subsets of $N(B)$. Let $U, V : H \to K$ be bounded complex-linear isomorphisms with bounded inverses, and let

$$F : N(A) \longrightarrow N(B), \qquad F(\ker f^A_{x,y}) = \ker f^B_{Ux,Vy}.$$

The point-evaluation theorem guarantees this formula for every representative of a normal point. This map satisfies

$$F \text{ is fully continuous} \iff V^* B U \subseteq A, \qquad (1)$$

where $V^* : K \to H$ is the adjoint of V and $V^* B U = \{ V^* b U : b \in B \}$. Here full continuity means

$$F(c_A(D)) \subseteq c_B(F(D)) \qquad \text{for every } D \subseteq N(A).$$

Assumptions

The maps U, V and their inverses are bounded. Neither map is assumed unitary, and no intertwining relation is imposed. The operator inclusion in (1) is equivalent to continuity, not an additional hypothesis.

Conclusion

The criterion concerns closure preservation for all subsets of $N(A)$, not just finite subsets. Its right-hand side is an inclusion of subsets of $\mathcal{B}(H)$.

Main citations

- Exact bounded continuity criterion: `continuous_normalPartLinearMap_iff`.
- General coefficient-transport criterion: `continuous_normalPartSemilinearMap_iff`, specialized to the identity scalar homomorphism.
- arXiv:2504.05551v1, Theorem 6.9: the full-continuity criterion for bounded linear changes of coefficient vectors.
- The bounded operator transport: `adjointSandwich U V b` is exactly the composite $V^* \circ b \circ U$, from H to H.
- Normal-part point evaluation and its inverse equivalence: the same vector changes define the map and its inverse.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem continuous_normalPartLinearMap_iff (U V : H ≃L[ℂ] K) :
    ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Continuous
      ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T))
      (normalPartSemilinearMap S T hT U.toLinearEquiv V.toLinearEquiv) ↔
      ∀ b : T, InnerProductSpace.adjointSandwich U.toContinuousLinearMap
V.toContinuousLinearMap
        (b : K →L[ℂ] K) ∈ S
```

In the source, `S` and `T` denote the algebras A and B used here, respectively.

In the source	Mathematical meaning

Ambient spaces and algebras
`variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`
`variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]`
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))`
`variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]`
`variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
	The complex Hilbert spaces H, K and the nonzero norm-closed complex $*$-subalgebras $A \subseteq \mathcal{B}(H)$, $B \subseteq \mathcal{B}(K)$, acting irreducibly. The source names S, T denote A, B.

Source irreducibility parameter `include hS in`	The irreducibility of the action of A is retained in this theorem, as well as that of B.
Chosen bounded maps `theorem continuous_normalPartLinearMap_iff (U V : H ≃L[ℂ] K) :`	The same bounded complex-linear isomorphisms $U, V : H \to K$, with bounded inverses, used to define F.

Closures and the specified map
`((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Continuous`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T))`
`  (normalPartSemilinearMap S T hT U.toLinearEquiv V.toLinearEquiv) ↔`
	The relative closures c_A on $N(A)$ and c_B on $N(B)$ are the source and target of full continuity. Passing to the underlying linear maps does not change U, V or the induced map F. Thus the left side of (1) is
	$$F(c_A(D)) \subseteq c_B(F(D)) \quad (D \subseteq N(A)).$$

Full continuity
`def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=`
`  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`
	For arbitrary closure spaces and a map, the image of the closure of every subset lies in the closure of its image. Applied here, the map is F and the closures are precisely c_A, c_B, not ambient closures on all geometric points.

Operator inclusion
`∀ b : T, InnerProductSpace.adjointSandwich U.toContinuousLinearMap V.toContinuousLinearMap`
`  (b : K →L[ℂ] K) ∈ S :=`
	For every $b \in B$, the composite is an element of A:
	$$V^* b U \in A \quad (b \in B).$$
	Equivalently,
	$$V^* B U \subseteq A.$$
	Together with the preceding equivalence sign, this is exactly (1).

Adjoint sandwich
`def adjointSandwich (U V : H →L[𝕜] K) (b : K →L[𝕜] K) : H →L[𝕜] H :=`
`  V.adjoint.comp (b.comp U)`
	The bounded operator $V^* \circ b \circ U : H \to H$. The arguments are the same U, V as functions and the given operator $b : K \to K$; their order is unchanged.

The include directive shown separately applies to this theorem alone. It accounts for the source irreducibility parameter in the retained type.

Exact surrounding source context (separate excerpts)

Exact ambient excerpt:

```
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))
variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
S))
variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
T))
```

Exact ambient excerpt:

```
include hS in
```

The effective source context uses independent universes for H and K. The adjoint in V^*bU is the bounded Hilbert-space adjoint from K to H. The literal definition is `V.adjoint.comp (b.comp U)`; its order is the same as the operator formula above.

Related exact source for the reading guide

Operator inclusion · Fixed source

```
      ∀ b : T, InnerProductSpace.adjointSandwich U.toContinuousLinearMap
V.toContinuousLinearMap
        (b : K →L[ℂ] K) ∈ S :=
```

Adjoint sandwich · Fixed source

```
def adjointSandwich (U V : H →L[𝕜] K) (b : K →L[𝕜] K) : H →L[𝕜] H :=
  V.adjoint.comp (b.comp U)
```

Full continuity · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

The geometric structure family is the family of pure-state translate kernels

`MathlibAnnex.Analysis.CStarAlgebra.geometricStructureFamily_eq_setOf_exists_pureState_unitaryTranslate`

theorem

Identifies geometric points with kernels of unitary translates of pure states.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity. Let $S(A)$ be its original geometric structure space: its points are the sets obtained as unions of the kernels of all norm-one bounded linear functionals supporting maximal proper faces of the closed unit ball. Write $\mathcal{P}(A)$ for the pure states of A, that is, positive norm-one bounded complex-linear functionals $A \to \mathbb{C}$ that are extreme in that state space for real convexity.

For a unitary u in the forced unitization $\widetilde{A} = \mathbb{C} \oplus A$, define $(T_u\varphi)(a) = \varphi(ua)$. Then

$$S(A) = \{\ker(T_u\varphi) : \varphi \in \mathcal{P}(A),\ u \in U(\widetilde{A})\}. \qquad (1)$$

Here ua denotes the unitization action on the original ideal A; the functional and its kernel in this formula are on A.

Assumptions

The source requires the nonzero complex C*-algebra and its compatible order. Positivity and purity refer to functionals on A; the translating unitary is allowed to lie in its forced unitization. The kernel is complex-linear, even though the face and extreme-point conditions use real convexity. No positivity of the translated functional $T_u\varphi$ is required.

Conclusion

Every original geometric-structure point can be written using one pure state and one unitary, and every such pair gives a point. Formula (1) is a family equality, so two pairs with the same kernel give the same point. It does not assert uniqueness of a state–unitary pair or replace the original family by a new definition.

Main citations

- Exact pure-state translate family equality: `geometricStructureFamily_eq_setOf_exists_pureState_unitaryTranslate`.

- Extreme-kernel family and pure-state unitary decomposition: the related family equality and translation identities for the same pure state and unitary.
- Ryotaro Tanaka, arXiv:2504.05551v1, Lemma 5.3: the paper's pure-state description of the geometric structure space.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem geometricStructureFamily_eq_setOf_exists_pureState_unitaryTranslate :
    NormedSpace.geometricStructureFamily ℂ A =
      {I : Set A | ∃ phi : StrongDual ℂ A, IsPureNonUnitalState A phi ∧
        ∃ u : unitary (Unitization ℂ A), ((unitaryTranslate u phi).ker : Set A) = I}
```

In the source	Mathematical meaning
Exact declaration `{A : Type*}`, the nonunital C*-algebra, order and nontriviality instances	The common algebra and hypotheses of the two family descriptions.
Exact declaration `NormedSpace.geometricStructureFamily ℂ A`	The family $S(A)$ of geometric-structure points, each a subset of A obtained from a maximal face and its supporting functionals.
Exact declaration `{I : Set A | ...}`	A family described by a property of the entire subset $I \subseteq A$.
Exact declaration `∃ phi : StrongDual ℂ A, IsPureNonUnitalState A phi ∧ ...`	A pure state $\varphi \in \mathcal{P}(A)$, with positivity, norm one and extremality as specified above.
Exact declaration `∃ u : unitary (Unitization ℂ A)`	One unitary of the forced unitization, for that same state. The state is the same one already quantified; the unitary is not required to belong to the original algebra.
Exact declaration `((unitaryTranslate u phi).ker : Set A) = I`	The equality $\ker(T_u\varphi) = I$, with the translated functional defined on A.
The complete equality **Exact declaration**	Equality of families of subsets: every left member has the displayed witnesses, and every such pair gives a left member. No unique parametrization is asserted.

Exact surrounding source context (separate excerpts)

Source lines 25–25:

```
variable {A : Type*} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Source lines 26–26:

```
variable [PartialOrder A] [StarOrderedRing A]
```

Exact declaration for the source guide

```
theorem geometricStructureFamily_eq_setOf_exists_pureState_unitaryTranslate :
    NormedSpace.geometricStructureFamily ℂ A =
      {I : Set A | ∃ phi : StrongDual ℂ A, IsPureNonUnitalState A phi ∧
        ∃ u : unitary (Unitization ℂ A), ((unitaryTranslate u phi).ker : Set A) = I} := by
```

Linked normal kernels share a projective coordinate

`MathlibAnnex.Analysis.CStarAlgebra.linked_normalPartPoint_iff_span_eq_or_span_eq`

theorem

Characterizes linked normal kernels by agreement of one projective coordinate.

Statement

Let H be a complex Hilbert space, and let $A \subseteq \mathcal{B}(H)$ be a nonzero norm-closed complex $*$-subalgebra acting irreducibly on H. An identity is not required. For $x, y \in H \setminus \{0\}$, define the bounded linear functional

$$f_{x,y} : A \to \mathbb{C}, \qquad f_{x,y}(a) = \langle y, a(x) \rangle.$$

The inner product is conjugate-linear in its first variable. The normal part is $N(A) = \{\ker f_{x,y} : x, y \in H \setminus \{0\}\}$. For $D \subseteq N(A)$, its inherited closure is

$$c(D) = \left\{ I \in N(A) : \bigcap_{J \in D} J \subseteq I \right\}.$$

Two points are **linked** when they are equal or their two-point set is not closed for this closure. For $x_1, x_2, y_1, y_2 \in H \setminus \{0\}$,

$$\ker f_{x_1,y_1} \text{ and } \ker f_{x_2,y_2} \text{ are linked} \iff \mathbb{C}x_1 = \mathbb{C}x_2 \text{ or } \mathbb{C}y_1 = \mathbb{C}y_2. \quad (1)$$

Assumptions

All four vectors are nonzero, but no unit-length or linear-independence assumption is imposed. Under these hypotheses, the coefficient-kernel membership theorem guarantees that every nonzero pair defines a point of the normal part. The closure in (1) is restricted to $N(A)$, not to arbitrary points of the ambient geometric structure space.

Conclusion

One shared projective coordinate characterizes a linked pair. In contrast, the coefficient-kernel equality theorem identifies equality of the two points with agreement of **both** projective coordinates. Linkedness here is the original pair relation, not its finite-chain transitive closure.

Main citations

- Linked-pair theorem: the equivalence (1), for the inherited normal-part closure.
- arXiv:2504.05551v1, Lemma 6.10: the linkedness criterion in the normal part.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem linked_normalPartPoint_iff_span_eq_or_span_eq
    (x₁ x₂ y₁ y₂ : H) (hx₁ : x₁ ≠ 0) (hx₂ : x₂ ≠ 0)
    (hy₁ : y₁ ≠ 0) (hy₂ : y₂ ≠ 0) :
    ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Linked
      (normalPartPoint S hS x₁ y₁ hx₁ hy₁) (normalPartPoint S hS x₂ y₂ hx₂ hy₂) ↔
      Submodule.span ℂ {x₁} = Submodule.span ℂ {x₂} ∨
      Submodule.span ℂ {y₁} = Submodule.span ℂ {y₂}
```

In the source, `S` denotes the algebra A used here.

In the source	Mathematical meaning

Hilbert space and irreducible algebra
`variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable [IsClosed (S : Set (H →L[ℂ] H))]`
`variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
	The complex Hilbert space H and the nonzero norm-closed complex $*$-subalgebra $A \subseteq \mathcal{B}(H)$ acting irreducibly. The source calls the algebra `S`.

Four nonzero vectors `(x₁ x₂ y₁ y₂ : H) (hx₁ : x₁ ≠ 0) (hx₂ : x₂ ≠ 0) (hy₁ : y₁ ≠ 0) (hy₂ : y₂ ≠ 0) :`	The vectors x_1, x_2, y_1, y_2 belong to $H \setminus \{0\}$. There is no normalization or independence condition.
Linked coefficient kernels `((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Linked (normalPartPoint S hS x₁ y₁ hx₁ hy₁) (normalPartPoint S hS x₂ y₂ hx₂ hy₂) ↔`	The two points are $\ker f_{x_1,y_1}$ and $\ker f_{x_2,y_2}$ in $N(A)$. The outer `subspace` gives the relative closure c on $N(A)$. Linkedness means that the points are equal or that their two-point set is not closed; it does not mean a finite chain of links.
Projective source or target equality `Submodule.span ℂ {x₁} = Submodule.span ℂ {x₂} ∨ Submodule.span ℂ {y₁} = Submodule.span ℂ {y₂} := by`	The two alternatives in (1) are $$\mathbb{C}x_1 = \mathbb{C}x_2 \quad \text{or} \quad \mathbb{C}y_1 = \mathbb{C}y_2.$$ Each span is the complex line generated by its nonzero vector. The disjunction classifies linkedness; equality of the two kernels requires both line equalities, as stated in the separately cited result.

Linkedness includes equal points. The theorem concerns one linked pair, not a finite chain.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable [IsClosed (S : Set (H →L[ℂ] H))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
S))
```

Related exact source for the reading guide

Projective source or target equality · Fixed source

```
      Submodule.span ℂ {x₁} = Submodule.span ℂ {x₂} v
      Submodule.span ℂ {y₁} = Submodule.span ℂ {y₂} := by
```

The normal part as a product of projective spaces

`MathlibAnnex.Analysis.CStarAlgebra.normalPartProjectivizationEquiv`

def

Parametrizes the normal part by an ordered pair of complex projective lines.

Statement

Let $A \subseteq \mathcal{B}(H)$ be a norm-closed complex $*$-subalgebra with nonzero irreducible inclusion representation on the complete complex Hilbert space H. No identity is required. Let $\mathbb{P}(H)$ denote its set of one-dimensional complex linear subspaces and $[x] = \mathbb{C}x$ for $x \neq 0$. For $x, y \neq 0$, define $f_{x,y} : A \to \mathbb{C}$ by $f_{x,y}(a) = \langle y, a(x) \rangle$, with inner product conjugate-linear in the first variable, and let $N(A)$ be the coefficient-kernel normal part of $S(A)$.

The definition gives a bijection

$$e : \mathbb{P}(H) \times \mathbb{P}(H) \xrightarrow{\sim} N(A), \qquad e([x],[y]) = \ker f_{x,y}. \tag{1}$$

This is an equivalence of sets. It does not assert a homeomorphism for the usual product topology on the two projective spaces.

Definition

For two projective lines p, q, choose their nonzero representatives $p^{\mathrm{rep}}, q^{\mathrm{rep}}$. The defining map is

$$(p, q) \longmapsto \ker f_{p^{\mathrm{rep}}, q^{\mathrm{rep}}}.$$

The representative-evaluation theorem gives the value $\ker f_{x,y}$ for any nonzero representatives x, y. The injectivity and surjectivity theorems make this map the bijection (1) under the stated hypotheses.

The inverse-evaluation theorem gives

$$e^{-1}(\ker f_{x,y}) = ([x],[y]) \qquad (x, y \neq 0).$$

Both formulas describe the same equivalence e.

Assumptions

The algebra is closed in operator norm and its inclusion is nonzero irreducible. These hypotheses supply geometric coefficient points and recoverability of the two lines. No normalization, finite-dimensionality, separability, or dimension-at-least-two assumption is used. Each projective line is represented by a nonzero vector by definition.

Conclusion

A normal kernel determines one source line and one target line, and every pair of such lines determines one normal kernel. This gives coordinates on the existing normal part. Closure and linkedness remain separate structures; the linked-pair theorem identifies linkedness with agreement of one coordinate, not both.

Main citations

- Projective equivalence and its defining results: the defining right-hand side and the preceding map, injectivity, and surjectivity results.
- Equality of coefficient kernels and both vector lines: the separate irreducible-representation kernel equality result.
- normalPartProjectivizationMap_mk: evaluation on any nonzero representatives.
- normalPartProjectivizationMap_injective: injectivity.
- normalPartProjectivizationMap_surjective: surjectivity.
- normalPartProjectivizationEquiv_symm_point: inverse evaluation.

Lean source signature (exact)

The complete definition below is copied from the pinned source, including its defining right-hand side.

```
def normalPartProjectivizationEquiv : (ℙ ℂ H) × (ℙ ℂ H) ≃ normalPart S :=
  Equiv.ofBijective (normalPartProjectivizationMap S hS)
    ⟨normalPartProjectivizationMap_injective S hS,
      normalPartProjectivizationMap_surjective S hS⟩
```

In the source, `S` denotes the algebra A used here.

In the source	Mathematical meaning
Exact declaration `S`, its closedness instance, and `hS`	The closed concrete algebra A in the text and nonzero irreducibility of its inclusion. These conditions ensure that all pairs used below yield normal kernels and that both lines are recoverable.
Exact declaration `(ℙ ℂ H) × (ℙ ℂ H)`	The ordered pair of source and target complex projective lines. This expression alone supplies no ordinary product-topology claim.
Exact declaration `≃ normalPart S`	A set-level bijection into the normal-part subtype, with an inverse on every such point.

In the source	Mathematical meaning
Exact declaration `normalPartProjectivizationMap S hS`	The representative map defining e in (1); its evaluation on any nonzero x, y is $\ker f_{x,y}$, not merely a formal label.
Exact declaration `Equiv.ofBijective` with `..._injective` and `..._surjective`	Injectivity and surjectivity of this very map, giving its inverse on every normal kernel.

The next comment and attribute introduce a later evaluation theorem; they are not part of this defining right-hand side.

Exact surrounding source context (separate excerpts)

Source lines 27–30:

```
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable [IsClosed (S : Set (H →L[ℂ] H))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))
```

Semilinear vector changes preserve finite normal-part closures

`MathlibAnnex.Analysis.CStarAlgebra.normalPartSemilinearFiniteHomeomorph`

def

Defines a finite homeomorphism of normal parts from two semilinear vector isomorphisms.

Statement

Let $A_1 \subseteq \mathcal{B}(H)$ and $A_2 \subseteq \mathcal{B}(K)$ be norm-closed complex $*$-subalgebras with nonzero irreducible inclusion representations on complete complex Hilbert spaces. Neither algebra needs an identity. Use inner products linear in their second variable. For nonzero vectors, write

$$f_{x,y}^{A_1}(a) = \langle y, a(x)\rangle \quad (a \in A_1), \qquad f_{z,w}^{A_2}(b) = \langle w, b(z)\rangle \quad (b \in A_2).$$

Let $N(A_j) \subseteq S(A_j)$ be the coefficient-kernel normal part, with inherited closure

$$c_{A_j}(D) = \left\{ I \in N(A_j) : \bigcap_{J \in D} J \subseteq I \right\}.$$

Suppose $\sigma : \mathbb{C} \to \mathbb{C}$ is a unital ring homomorphism with $\sigma^2 = \mathrm{id}$ and $\sigma(\overline{c}) = \overline{\sigma(c)}$ for every c, and $U, V : H \to K$ are algebraic σ-semilinear isomorphisms. The definition equips

$$F : N(A_1) \longrightarrow N(A_2), \qquad F(\ker f_{x,y}^{A_1}) = \ker f_{Ux,Vy}^{A_2} \tag{1}$$

with finite continuity in both directions. Thus F is a finite-homeomorphism: a bijection whose map and inverse preserve membership in closures of finite sets.

Definition

The underlying equivalence sends the coefficient kernel represented by (x, y) to the kernel represented by (Ux, Vy). Its inverse is

$$F^{-1} : N(A_2) \longrightarrow N(A_1), \qquad F^{-1}(\ker f_{z,w}^{A_2}) = \ker f_{U^{-1}z,V^{-1}w}^{A_1}. \tag{2}$$

The semilinear kernel bijection has the forward evaluation for every nonzero representing pair. Its inverse uses the inverses of those same U, V.

The other two fields are the finite-continuity requirements

$$\begin{aligned} F(c_{A_1}(D)) &\subseteq c_{A_2}(F(D)) && (D \subseteq N(A_1) \text{ finite}), \\ F^{-1}(c_{A_2}(E)) &\subseteq c_{A_1}(F^{-1}(E)) && (E \subseteq N(A_2) \text{ finite}). \end{aligned} \tag{3}$$

The finite-continuity theorem guarantees these inclusions for U, V and their inverses, respectively. Both inverse maps use the same self-inverse scalar homomorphism σ.

Assumptions

The same scalar homomorphism governs both vector maps. Its inverse is itself, and it commutes with complex conjugation. U, V are additive bijections with $U(cx) = \sigma(c)Ux$ and $V(cy) = \sigma(c)Vy$. They are not assumed bounded, isometric, unitary, or algebra intertwiners. Closedness and irreducibility belong to the two concrete algebras; they guarantee the coefficient-kernel realization and the finite-continuity theorem.

Conclusion

The output is a bijection of the normal parts with finite continuity of the forward map and its inverse, for all finite subsets including the empty set. Full continuity for arbitrary subsets is not a field of this construction.

Main citations

- Exact finite-homeomorphism construction: `normalPartSemilinearFiniteHomeomorph` and its three defining fields.
- arXiv:2504.05551v1, Theorem 6.9: the algebraic finite-homeomorphism layer, separated from its bounded full-continuity layer.
- normalPartSemilinearMap_point: representative-independent evaluation.
- normalPartSemilinearEquiv: the specified bijection.
- finitelyContinuous_normalPartSemilinearMap: finite continuity.

Lean source signature (exact)

The complete definition below is copied from the pinned source, including its defining right-hand side.

```
def normalPartSemilinearFiniteHomeomorph
    (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))
    (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))
    (hσ : ∀ c : ℂ, σ (star c) = star (σ c)) (U V : H ≃ₛₗ[σ] K) :
    ClosureSpace.FiniteHomeomorph
      ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))
      ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)) where
  toEquiv := normalPartSemilinearEquiv S T hS hT hσ U V
  finitelyContinuous_toFun := finitelyContinuous_normalPartSemilinearMap S T hS hT hσ U V
  finitelyContinuous_invFun :=
    finitelyContinuous_normalPartSemilinearMap T S hT hS hσ U.symm V.symm
```

In the source, S and T denote the algebras A_1 and A_2 used here, respectively.

In the source	Mathematical meaning

First algebra and Hilbert space
`variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable [IsClosed (S : Set (H →L[ℂ] H))]`
	The complex Hilbert space H and the norm-closed complex $*$-subalgebra $A_1 \subseteq \mathcal{B}(H)$. The source calls this algebra S.

Second algebra and scalar map
`variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]`
`variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))`
`variable [IsClosed (T : Set (K →L[ℂ] K))]`
`variable {σ : ℂ →+* ℂ} [RingHomInvPair σ σ]`
	The complex Hilbert space K and the norm-closed algebra $A_2 \subseteq \mathcal{B}(K)$, called T. The unital scalar homomorphism $\sigma : \mathbb{C} \to \mathbb{C}$ satisfies $\sigma \circ \sigma = \mathrm{id}$; the inverse vector maps therefore use the same scalar map.

Irreducible actions and semilinear maps
`(hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`(hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
`(hσ : ∀ c : ℂ, σ (star c) = star (σ c)) (U V : H ≃ₛₗ[σ] K) :`
	The two algebras act nontrivially and irreducibly on their respective spaces. The maps $U, V : H \to K$ are additive bijections satisfying $U(cx) = \sigma(c)U(x)$ and $V(cx) = \sigma(c)V(x)$. The scalar condition is $\sigma(\bar{c}) = \overline{\sigma(c)}$ for every $c \in \mathbb{C}$. No boundedness is assumed.

Domain and target closures
`ClosureSpace.FiniteHomeomorph`
`  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)) where`
	The output is a finite homeomorphism $F : N(A_1) \to N(A_2)$. Each subspace restricts the geometric closure to the indicated normal part; these are the relative closures c_{A_1}, c_{A_2} used in (3).

Underlying bijection `toEquiv := normalPartSemilinearEquiv S T hS hT hσ U V`	This field uses the bijection induced by the same U, V: $$F(\ker f^{A_1}_{x,y}) = \ker f^{A_2}_{Ux,Vy}.$$ Here $x, y \in H \setminus \{0\}$. Its inverse is the map in (2).

Forward and inverse maps
`def normalPartSemilinearEquiv`
`    (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`    (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
`    (hσ : ∀ c : ℂ, σ (star c) = star (σ c)) (U V : H ≃ₛₗ[σ] K) :`
`    normalPart S ≃ normalPart T where`
`  toFun := normalPartSemilinearMap S T hT U V`
`  invFun := normalPartSemilinearMap T S hS U.symm V.symm`
	The equivalence has forward map $F : N(A_1) \to N(A_2)$ and inverse $F^{-1} : N(A_2) \to N(A_1)$. The inverse uses U^{-1}, V^{-1}, so for $z, w \in K \setminus \{0\}$,
	$$F^{-1}(\ker f^{A_2}_{z,w}) = \ker f^{A_1}_{U^{-1}z,V^{-1}w}.$$
	This excerpt specifies the two maps; the inverse-law proof fields remain in the full source.

In the source	Mathematical meaning
Finite forward continuity `finitelyContinuous_toFun := finitelyContinuous_normalPartSemilinearMap S T hS hT hσ U V`	The cited result supplies, for every finite $D \subseteq N(A_1)$, $$F(c_{A_1}(D)) \subseteq c_{A_2}(F(D)).$$ It applies to the same U, V and scalar condition.
Finite inverse continuity `finitelyContinuous_invFun := finitelyContinuous_normalPartSemilinearMap T S hT hS hσ U.symm V.symm`	The algebras and irreducibility hypotheses are exchanged, and `symm` takes the actual inverses of U, V. For every finite $E \subseteq N(A_2)$, $$F^{-1}(c_{A_2}(E)) \subseteq c_{A_1}(F^{-1}(E)).$$ Neither field asserts continuity on arbitrary subsets.

The declaration retains the full three-field right-hand side. The self-inverse scalar condition is in the separately displayed ambient context.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable [IsClosed (S : Set (H →L[ℂ] H))]
```

Exact surrounding source:

```
variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))
variable [IsClosed (T : Set (K →L[ℂ] K))]
variable {σ : ℂ →+* ℂ} [RingHomInvPair σ σ]
```

Related exact source for the reading guide

Forward and inverse maps · Fixed source

```
def normalPartSemilinearEquiv
    (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))
    (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))
    (hσ : ∀ c : ℂ, σ (star c) = star (σ c)) (U V : H ≃ₛₗ[σ] K) :
    normalPart S ≃ normalPart T where
  toFun := normalPartSemilinearMap S T hT U V
  invFun := normalPartSemilinearMap T S hS U.symm V.symm
```

The geometric spectrum

`NormedSpace.GeometricSpectrum`

def

The geometric spectrum

Statement

The geometric spectrum of a normed space is the set of finite-chain components of its geometric structure space. Each spectrum point is an actual component subset. Two choices of a geometric point in the same component give one spectrum point.

Definition

Let E be a normed space over a real-or-complex scalar field $\mathbb{K}$, with closed unit ball $B_E = \{x \in E : \|x\| \le 1\}$ and continuous $\mathbb{K}$-linear dual E^*. For each nonempty maximal proper real convex face M of B_E, put

$$\begin{aligned} \Sigma_{\mathbb{K}}(M) &= \{f \in E^* : \|f\| \le 1,\ f|_M = 1\}, \\ I_{\mathbb{K}}(M) &= \bigcup_{f \in \Sigma_{\mathbb{K}}(M)} \ker f. \end{aligned}$$

Here $f|_M = 1$ means $f(x) = 1$ for every $x \in M$.

Here a face is a convex subset containing both endpoints of any segment in the ball whose interior meets it; maximality is among proper convex faces. The geometric point set $S_{\mathbb{K}}(E)$ consists of the resulting subsets $I_{\mathbb{K}}(M)$, with equal subsets regarded as the same point. Its hull–kernel closure is

$$\begin{aligned} c_E(\mathscr{A}) = \Bigg\{ I \in S_{\mathbb{K}}(E) : \bigcap_{J \in \mathscr{A}} J \subseteq I \Bigg\}, \\ \mathscr{A} \subseteq S_{\mathbb{K}}(E), \end{aligned}$$

where the empty intersection is E.

For $I, J \in S_{\mathbb{K}}(E)$, write $I \leftrightarrow_{c_E} J$ when $I = J$ or $c_E(\{I, J\}) \neq \{I, J\}$. Write $I \sim_{c_E} J$ when they can be joined by a finite chain of these linked pairs, allowing a chain with no steps. Define

$$\begin{aligned} C_E(I) &= \{J \in S_{\mathbb{K}}(E) : I \sim_{c_E} J\}, \\ \mathrm{GS}_{\mathbb{K}}(E) &= \{C_E(I) : I \in S_{\mathbb{K}}(E)\}. \end{aligned}$$

The second set is the **geometric spectrum**. Its elements are subsets of $S_{\mathbb{K}}(E)$, while the members of each such subset are themselves subsets of E. Equality of spectrum points means equality of component sets.

Assumptions

The space E is normed over $\mathbb{K}$. Completeness and nontriviality are not required. The component relation is formed using c_E; it does not assume that this closure preserves finite unions.

Conclusion

The declaration supplies the point set $\mathrm{GS}_{\mathbb{K}}(E)$. Every member is nonempty, because $I \in C_E(I)$. The separately cited projection

$$q_E : S_{\mathbb{K}}(E) \longrightarrow \mathrm{GS}_{\mathbb{K}}(E), \qquad q_E(I) = C_E(I),$$

is surjective, and $C_E(I) = C_E(J)$ exactly when $I \sim_{c_E} J$. This makes explicit how the same component is independent of its chosen starting point.

For the zero space, the geometric spectrum is empty. Its closure structure is defined separately from this carrier; it uses the intersections of component members. The component $C_E(I)$ itself is not replaced by its closure $c_E(C_E(I))$.

Main citations

- The geometric points and their closure: `NormedSpace.geometricStructureFamily`, `NormedSpace.GeometricStructureSpace` and `NormedSpace.geometricStructureClosure`, lines 37–65; the face predicate, `NormedSpace.IsMaximalFace`, lines 32–39.
- Linked pairs and finite-chain components: `ClosureSpace.Linked`, `ClosureSpace.Related` and `ClosureSpace.component`, lines 30–71.
- The component family and equality criterion: `ClosureSpace.component_eq_iff_related` and `ClosureSpace.componentFamily`, lines 30–44.
- Exact spectrum carrier and the projection: `NormedSpace.geometricSpectrumFamily` and `NormedSpace.GeometricSpectrum`, lines 36–41; `NormedSpace.toGeometricSpectrum`, lines 57–58, and its surjectivity, lines 81–82.
- Separate closure and zero-space results: `NormedSpace.geometricSpectrumClosure`, lines 114–115; emptiness of the carrier, lines 169–192.
- Ryotaro Tanaka, arXiv:2504.05551v1, Definition 4.6: the component-set realization of the geometric spectrum. The formal source allows the weaker normed-space assumptions above.

Lean source signature (exact)

The scalar and normed-space declarations shared by this definition are reproduced separately below.

```
def GeometricSpectrum : Type u :=
  {C : Set (GeometricStructureSpace 𝕜 E) // C ∈ geometricSpectrumFamily 𝕜 E}
```

In the source	Mathematical meaning
Exact declaration `𝕜, [RCLike 𝕜]`	The real or complex scalar field $\mathbb{K}$.
Exact declaration `E, [NormedAddCommGroup E], [NormedSpace 𝕜 E]`	A normed vector space E over $\mathbb{K}$, possibly zero and not necessarily complete.
Exact declaration `GeometricStructureSpace 𝕜 E`	The geometric point set $S_{\mathbb{K}}(E)$ of maximal-face support-kernel unions, equipped in the component construction with c_E.
Exact declaration `C : Set (GeometricStructureSpace 𝕜 E)`	A possible component subset $C \subseteq S_{\mathbb{K}}(E)$; its members are themselves subsets of E.
Exact declaration `C ∈ geometricSpectrumFamily 𝕜 E`	There exists $I \in S_{\mathbb{K}}(E)$ such that $C = C_E(I)$, where $C_E(I)$ is the finite-chain component for c_E.
Exact declaration `{C : Set ... // C ∈ geometricSpectrumFamily 𝕜 E}`	The full defining type $\mathrm{GS}_{\mathbb{K}}(E)$ contains exactly those component subsets with their membership condition. The underlying component is `C.val`; no representative I is stored.
Exact declaration `: Type u`	The output is the spectrum carrier. The spectrum closure and the surjective map q_E are separately defined or proved, as cited above.

Source context (separate exact excerpt)

The surrounding declarations below are from lines 32–33; their mathematical meaning is included in the guide above.

```
variable (𝕜 : Type v) [RCLike 𝕜]
variable (E : Type u) [NormedAddCommGroup E] [NormedSpace 𝕜 E]
```

Exact declaration for the source guide

```
def GeometricSpectrum : Type u :=
  {C : Set (GeometricStructureSpace 𝕜 E) // C ∈ geometricSpectrumFamily 𝕜 E}

/-- The intersection `J_C` of the geometric-structure points in a component. -/
```

The primitive geometric spectrum

`NormedSpace.PrimitiveGeometricSpectrum`

def

The primitive geometric spectrum

Statement

The primitive geometric spectrum records the common intersections of the points belonging to each component of the geometric structure space. A primitive geometric point is an actual subset of the original normed space. Repeated intersections give one point, even if they arise from different components.

Definition

Let E be a normed space over a real-or-complex scalar field $\mathbb{K}$, with closed unit ball $B_E = \{x \in E : \|x\| \le 1\}$ and continuous $\mathbb{K}$-linear dual E^*. For each nonempty maximal proper real convex face M of B_E, define

$$\begin{aligned} \Sigma_{\mathbb{K}}(M) &= \{f \in E^* : \|f\| \le 1,\ f|_M = 1\}, \\ I_{\mathbb{K}}(M) &= \bigcup_{f \in \Sigma_{\mathbb{K}}(M)} \ker f. \end{aligned}$$

Here $f|_M = 1$ means $f(x) = 1$ for every $x \in M$.

A face is a convex subset containing both endpoints of every segment in the ball whose interior meets it; maximality is among proper convex faces. Let $S_{\mathbb{K}}(E)$ be the family of the resulting subsets $I_{\mathbb{K}}(M)$, equipped with the following operation on $\mathscr{A} \subseteq S_{\mathbb{K}}(E)$:

$$c_E(\mathscr{A}) = \left\{ I \in S_{\mathbb{K}}(E) : \bigcap_{J \in \mathscr{A}} J \subseteq I \right\}.$$

The empty intersection is E. Two geometric points I, J are linked when $I = J$ or $c_E(\{I, J\}) \neq \{I, J\}$. Their finite-chain equivalence relation is denoted by $\sim_{c_E}$, and

$$C_E(I) = \{J \in S_{\mathbb{K}}(E) : I \sim_{c_E} J\}$$

is the component of I. For a component set C, put

$$J_C = \bigcap_{I \in C} I.$$

Define the **primitive geometric spectrum** by

$$\mathrm{PS}_{\mathbb{K}}(E) = \{J_{C_E(I)} : I \in S_{\mathbb{K}}(E)\}.$$

This is a family of subsets of E. A vector x belongs to J_C precisely when it belongs to every geometric point in C. All intersections are set intersections; no subspace or ideal structure is assumed.

Assumptions

The space E is normed over $\mathbb{K}$, without a completeness or nontriviality hypothesis. The component construction uses the stated hull–kernel closure. Neither closedness of the component sets nor injectivity of the assignment $C \mapsto J_C$ is assumed.

Conclusion

The resulting carrier contains exactly the component intersections. The cited properness result follows from

$$I \in C_E(I) \quad \Longrightarrow \quad J_{C_E(I)} \subseteq I \neq E.$$

Thus no primitive geometric point is all of E. This permits the separately defined hull–kernel closure: for $\mathscr{T} \subseteq \mathrm{PS}_{\mathbb{K}}(E)$,

$$d_E(\mathscr{T}) = \left\{ J \in \mathrm{PS}_{\mathbb{K}}(E) : \bigcap_{J' \in \mathscr{T}} J' \subseteq J \right\}.$$

The present declaration supplies the carrier; the cited closure declaration supplies d_E. For the zero space the carrier is empty. The empty-intersection convention therefore does not introduce an extra primitive point in that case.

Main citations

- Geometric points and their closure: `NormedSpace.geometricStructureFamily` and `NormedSpace.geometricStructureClosure`, lines 37–65; maximal faces, `NormedSpace.IsMaximalFace`, lines 32–39.
- Finite-chain components: `ClosureSpace.component` and its generating relation, lines 30–71.
- Intersections and properness: `ClosureSpace.HullKernel.componentKernel_ne_univ`, lines 35–50; the intersection family is defined at lines 72–75.
- Exact carrier and range description: `NormedSpace.PrimitiveGeometricSpectrum`, lines 48–54; the range characterization, lines 93–101.

- Closure and emptiness: `NormedSpace.primitiveGeometricSpectrumClosure`, lines 109–121; the zero-space case, lines 179–199.
- Ryotaro Tanaka, arXiv:2504.05551v1, Definition 4.6 and Theorem 4.7: component intersections and their closure for nontrivial Banach spaces.

Lean source signature (exact)

The scalar and normed-space declarations shared by this definition are reproduced separately below.

```
def PrimitiveGeometricSpectrum : Type u :=
  {J : Set E // J ∈ primitiveGeometricSpectrumFamily 𝕜 E}
```

In the source	Mathematical meaning
Exact declaration `𝕜, [RCLike 𝕜]`	The real or complex scalar field $\mathbb{K}$.
Exact declaration `E, [NormedAddCommGroup E], [NormedSpace 𝕜 E]`	A normed vector space E over $\mathbb{K}$, possibly zero and not necessarily complete.
Exact declaration `J : Set E`	A subset $J \subseteq E$, rather than a subset of the geometric point set.
Exact declaration `J ∈ primitiveGeometricSpectrumFamily 𝕜 E`	There exists a geometric point $I \in S_{\mathbb{K}}(E)$ with $J = J_{C_E(I)} = \bigcap_{K \in C_E(I)} K$. The component $C_E(I)$ is formed using c_E.
Exact declaration `{J : Set E // J ∈ primitiveGeometricSpectrumFamily 𝕜 E}`	The full RHS defines $\mathrm{PS}_{\mathbb{K}}(E)$ as exactly these intersection subsets with their membership condition. The underlying subset is `J.val`; neither a representative point nor a component witness is stored.
Exact declaration `: Type u`	The output is the primitive geometric carrier. Equality is equality of subsets of E, so equal intersections give one point; no ideal structure or injectivity of the component-intersection assignment is asserted.

Source context (separate exact excerpt)

The surrounding declarations below are from lines 32–33; their mathematical meaning is included in the guide above.

```
variable (𝕜 : Type v) [RCLike 𝕜]
variable (E : Type u) [NormedAddCommGroup E] [NormedSpace 𝕜 E]
```

Exact declaration for the source guide

```
def PrimitiveGeometricSpectrum : Type u :=
  {J : Set E // J ∈ primitiveGeometricSpectrumFamily 𝕜 E}

/-- Send a geometric-structure point to its component. -/
```

The closure-space homeomorphism induced by an orthogonality equivalence

`NormedSpace.geometricStructureHomeomorphOfBirkhoffJames`

def

Statement

Let E and F be normed spaces over real-or-complex scalar fields $\mathbb{K}$ and $\mathbb{L}$, and let $e : E \to F$ be a bijection preserving and reflecting their Birkhoff–James orthogonality relations. Define the induced closure-space homeomorphism

$$\Phi_e : S_{\mathbb{K}}(E) \longrightarrow S_{\mathbb{L}}(F), \qquad \Phi_e(I) = e(I),$$

with inverse $J \mapsto e^{-1}(J)$. Here $e(I) = \{e(x) : x \in I\}$ is a direct image and $e^{-1}(J) = \{x : e(x) \in J\}$ is a preimage. Well-definedness and closure preservation are supplied by the separately cited transport theorems.

Definition

Let E and F be normed spaces over real-or-complex scalar fields $\mathbb{K}$ and $\mathbb{L}$, respectively; the fields may differ. Define

$$x \perp_{\mathbb{K}} y \quad \Longleftrightarrow \quad \|x\| \le \|x + ay\| \text{ for every } a \in \mathbb{K}.$$

The same formula with $\mathbb{L}$ defines orthogonality on F. Let $e : E \to F$ be a bijection such that, for every $x, y \in E$,

$$e(x) \perp_{\mathbb{L}} e(y) \quad \Longleftrightarrow \quad x \perp_{\mathbb{K}} y.$$

Here is the geometric point set on which the induced map acts. Write E^* for the continuous $\mathbb{K}$-linear dual and $B_E = \{x \in E : \|x\| \le 1\}$. For a nonempty maximal proper real convex face M of B_E, put

$$\Sigma_{\mathbb{K}}(M) = \{f \in E^* : \|f\| \le 1,\ f|_M = 1\},$$
$$I_{\mathbb{K}}(M) = \bigcup_{f \in \Sigma_{\mathbb{K}}(M)} \ker f.$$

Here $f|_M = 1$ means $f(x) = 1$ for every $x \in M$.

A face is a convex subset containing both endpoints of each segment in the ball whose interior meets it; maximality is among proper convex faces. The set $S_{\mathbb{K}}(E)$ consists of the resulting subsets $I_{\mathbb{K}}(M)$, with equal subsets regarded as the same point. Its closure is

$$c_E(\mathscr{A}) = \left\{ I \in S_{\mathbb{K}}(E) : \bigcap_{J \in \mathscr{A}} J \subseteq I \right\}, \qquad \mathscr{A} \subseteq S_{\mathbb{K}}(E).$$

The empty intersection is E. Define $S_{\mathbb{L}}(F)$ and c_F by the same formulas for F. Define the point map by

$$\Phi_e : S_{\mathbb{K}}(E) \longrightarrow S_{\mathbb{L}}(F), \qquad \Phi_e(I) = e(I).$$

Its inverse sends $J \subseteq F$ to $e^{-1}(J)$, the preimage under e.

The geometric-point transport theorem makes this point map well-defined; the same theorem for the inverse bijection supplies the inverse. The incidence identity and hull–kernel closure-transport theorem supply its homeomorphism structure.

Assumptions

The spaces are normed, with no completeness or nontriviality assumption. Their real-or-complex scalar fields may differ. The given bijection preserves and reflects the displayed ordered orthogonality relation for every pair of vectors; additivity, homogeneity and norm continuity are not assumed.

Conclusion

The output is the specified point bijection Φ_e, together with closure continuity of it and its inverse. Its underlying map and inverse are exactly those displayed above. A closure-space homeomorphism is a bijection continuous in both directions for the given closure operators; the operators need not be topological closures. The cited transport theorem supplies the identity

$$\Phi_e\big(c_E(\mathscr{A})\big) = c_F\big(\Phi_e(\mathscr{A})\big) \qquad (\mathscr{A} \subseteq S_{\mathbb{K}}(E)).$$

Main citations

- Geometric points remain geometric under the given bijection: `NormedSpace.image_mem_geometricStructureFamily`.
- The point equivalence and its incidence identity: `NormedSpace.geometricStructureEquivOfBirkhoffJames` and `NormedSpace.mem_geometricStructureEquivOfBirkhoffJames`.
- Incidence-preserving equivalences commute with hull–kernel closure: `ClosureSpace.HullKernel.image_closure`. The following `homeomorphOfMemIff` definition packages this theorem.
- The geometric point family and closure; maximal proper real convex faces.
- Closure continuity and closure-space homeomorphisms and the homeomorphism structure.

Lean source signature (exact)

The construction uses the scalar, space and bijection hypotheses reproduced separately below.

```
def geometricStructureHomeomorphOfBirkhoffJames :
    ClosureSpace.Homeomorph (geometricStructureClosure 𝕜 E) (geometricStructureClosure 𝕝 F) :=
  ClosureSpace.HullKernel.homeomorphOfMemIff
    (univ_notMem_geometricStructureFamily 𝕜 E) (univ_notMem_geometricStructureFamily 𝕝 F)
    e (geometricStructureEquivOfBirkhoffJames 𝕜 𝕝 e h)
    (mem_geometricStructureEquivOfBirkhoffJames 𝕜 𝕝 e h)
```

In the source	Mathematical meaning

Scalar fields and normed spaces
`variable (𝕜 : Type v) (𝕝 : Type z) [RCLike 𝕜] [RCLike 𝕝]`
`variable {E : Type u} {F : Type w}`
`variable [NormedAddCommGroup E] [NormedSpace 𝕜 E]`
`variable [NormedAddCommGroup F] [NormedSpace 𝕝 F]`
	The fields $\mathbb{K}, \mathbb{L}$ may each be real or complex. The spaces E, F carry their respective norms and scalar actions; neither completeness nor nontriviality is required.

Bijection and orthogonality
`variable (e : E ≃ F)`
`variable (h : ∀ x y : E, BirkhoffJamesOrthogonal 𝕝 (e x) (e y) ↔`
`BirkhoffJamesOrthogonal 𝕜 x y)`
`include h`
	The given set bijection $e : E \to F$ preserves and reflects Birkhoff–James orthogonality for every ordered pair. The included hypothesis is the same h used in this construction; linearity of e is not assumed.

Two closure spaces
`def geometricStructureHomeomorphOfBirkhoffJames :`
`ClosureSpace.Homeomorph`
`(geometricStructureClosure 𝕜 E)`
`(geometricStructureClosure 𝕝 F) :=`
	The result is a homeomorphism from $(S_{\mathbb{K}}(E), c_E)$ to $(S_{\mathbb{L}}(F), c_F)$. These two complete closure arguments specify its domain and codomain.

Point map and incidence input
`ClosureSpace.HullKernel.homeomorphOfMemIff`
`(univ_notMem_geometricStructureFamily 𝕜 E)`
`(univ_notMem_geometricStructureFamily 𝕝 F)`
`e (geometricStructureEquivOfBirkhoffJames 𝕜 𝕝 e h)`
`(mem_geometricStructureEquivOfBirkhoffJames 𝕜 𝕝 e h)`
	The constructor receives the properness of both geometric families, the ambient bijection e, the point bijection Φ_e, and the incidence identity $e(x) \in \Phi_e(I) \iff x \in I$. These are the same maps as in the Statement.

Forward and inverse point maps
`def geometricStructureEquivOfBirkhoffJames :`
`GeometricStructureSpace 𝕜 E ≃`
`GeometricStructureSpace 𝕝 F where`
`toFun I := ⟨e '' I.val,`
`image_mem_geometricStructureFamily 𝕜 𝕝 e h`
`I.property⟩`
`invFun J := ⟨e.symm '' J.val,`
`image_mem_geometricStructureFamily 𝕝 𝕜 e.symm`
`(birkhoffJamesOrthogonal_symm_equiv 𝕜 𝕝 e h)`
`J.property⟩`
	The underlying point map is $\Phi_e(I) = e(I)$. Its inverse sends J to $e^{-1}(J)$; the image under the inverse bijection in the code is this same preimage. The cited geometric-point transport result supplies membership in the target family.

In the source	Mathematical meaning

Homeomorphism fields
`structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where`
`  continuous_toFun : c.Continuous d toFun`
`  continuous_invFun : d.Continuous c invFun`
	The bundle contains one bijection and continuity of that map and its inverse. Applied here, the two conditions are
	$$\Phi_e(c_E(\mathscr{A})) \subseteq c_F(\Phi_e(\mathscr{A})),$$
	$$\Phi_e^{-1}(c_F(\mathscr{B})) \subseteq c_E(\Phi_e^{-1}(\mathscr{B}))$$
	for every $\mathscr{A} \subseteq S_{\mathbb{K}}(E)$ and $\mathscr{B} \subseteq S_{\mathbb{L}}(F)$.

Closure continuity
`def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=`
`  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`
	Continuity of a map $r : (Y, c_Y) \to (Z, c_Z)$ means
	$$r(c_Y(E)) \subseteq c_Z(r(E)) \qquad (E \subseteq Y).$$
	In the preceding row this is used first for $r = \Phi_e$, then for its same inverse.

Exact surrounding source context (separate excerpts)

The surrounding declarations below are from the ambient source declarations; their mathematical meaning is included in the guide above.

```
variable (𝕜 : Type v) (𝕝 : Type z) [RCLike 𝕜] [RCLike 𝕝]
variable {E : Type u} {F : Type w}
variable [NormedAddCommGroup E] [NormedSpace 𝕜 E]
variable [NormedAddCommGroup F] [NormedSpace 𝕝 F]
variable (e : E ≃ F)
variable (h : ∀ x y : E, BirkhoffJamesOrthogonal 𝕝 (e x) (e y) ↔
  BirkhoffJamesOrthogonal 𝕜 x y)
include h
```

Related exact source for the reading guide

Forward and inverse point maps · Fixed source

```
def geometricStructureEquivOfBirkhoffJames :
    GeometricStructureSpace 𝕜 E ≃ GeometricStructureSpace 𝕝 F where
  toFun I := ⟨e '' I.val, image_mem_geometricStructureFamily 𝕜 𝕝 e h I.property⟩
  invFun J := ⟨e.symm '' J.val, image_mem_geometricStructureFamily 𝕝 𝕜 e.symm
    (birkhoffJamesOrthogonal_symm_equiv 𝕜 𝕝 e h) J.property⟩
```

Closure continuity · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

Homeomorphism fields · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```

Existence of a faithful compact representation

`MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.exists_isCompactlyRepresented_iff`

theorem

Characterizes existence of a faithful compact action by compact irreducible images and discrete primitive spectrum.

Statement

Let A be a complex C*-algebra. For a complex Hilbert space L, let $\mathcal{K}(L)$ be the algebra of compact operators on L. Write $\mathrm{Prim}(A)$ for the primitive ideal spectrum: its points are the kernels of pure-state GNS representations. Its hull–kernel topology has closure

$$c_{\mathrm{Prim}(A)}(E) = \left\{ P \in \mathrm{Prim}(A) : \bigcap_{Q \in E} Q \subseteq P \right\}, \qquad E \subseteq \mathrm{Prim}(A).$$

The following conditions are equivalent:

- There is a complex Hilbert space H and a faithful $*$-representation $\pi : A \to \mathcal{B}(H)$ such that $\pi(A) \subseteq \mathcal{K}(H)$.
- Every irreducible $*$-representation $\rho : A \to \mathcal{B}(K)$ on a complex Hilbert space satisfies $\rho(A) = \mathcal{K}(K)$, and $\mathrm{Prim}(A)$ is discrete: every singleton is open.

Assumptions

Hilbert spaces are complex and complete. Representations are $*$-homomorphisms, without a requirement of unitality; irreducibility includes a nonzero action. Zero algebras, zero representation spaces, and empty primitive spectra are allowed. No separability or countability assumption is made.

Conclusion

The two conditions characterize the existence of a faithful representation by compact operators. The cited reduced-atomic characterization gives such a representation when they hold. This does not require an arbitrary faithful representation to have compact values, or require its image to be all compact operators on its Hilbert space.

Main citations

The faithful compact-representation criterion is the stated equivalence. The fixed reduced-atomic characterization separates compactness of all irreducible images from discreteness of the existing spectrum. The existential representation comparison gives the same canonical reduced-atomic

witness; transport from an action on an independent Hilbert space does not shrink the original space.

Lean source signature (exact)

Exact source and declaration. Only the literal signature is displayed here; the linked source retains the proof.

```
theorem exists_isCompactlyRepresented_iff :
    (∃ (K : Type u) (normed : NormedAddCommGroup K),
      letI : NormedAddCommGroup K := normed
      ∃ inner : InnerProductSpace ℂ K,
        letI : InnerProductSpace ℂ K := inner
        ∃ complete : CompleteSpace K,
          letI : CompleteSpace K := complete
          ∃ pi : NonUnitalCStarRepresentation A K, pi.IsCompactlyRepresented) ↔
    (∀ (K : Type u) [NormedAddCommGroup K] [InnerProductSpace ℂ K]
      [CompleteSpace K] (rho : NonUnitalCStarRepresentation A K),
      rho.IsIrreducible →
        (NonUnitalStarAlgHom.range rho) = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ
K) ∧
    DiscreteTopology (PrimitiveSpectrum A)
```

The principal equivalence fixes the universe of A and quantifies its Hilbert spaces in that same universe. The independently specified Hilbert space of the separately cited transport theorem is not an all-universe quantifier in this equivalence.

In the source	Mathematical meaning
Algebra and positivity `attribute [local instance] _root_.CStarAlgebra.spectralOrder _root_.CStarAlgebra.spectralOrderedRing`	The C*-algebra A uses its spectral order to define positive functionals and pure states. This is the order underlying the primitive spectrum in the theorem.
Ambient algebra `variable {A : Type u} [NonUnitalCStarAlgebra A]`	The input is an arbitrary complex C*-algebra A, including the zero algebra. The type does not require a unit, and does not require the algebra to lack a unit.
Representation space and norm `(∃ (K : Type u) (normed : NormedAddCommGroup K), letI : NormedAddCommGroup K := normed`	There exists a space, called `K` here and H in the first condition, with a normed additive group structure. The `letI` makes that same norm available when specifying its inner product.
Inner product and completeness `∃ inner : InnerProductSpace ℂ K, letI : InnerProductSpace ℂ K := inner ∃ complete : CompleteSpace K, letI : CompleteSpace K := complete`	The existential quantifiers require a compatible complex inner product and completeness on the same H. The first `letI` installs the chosen inner product; the second installs completeness. Together they make H a complex Hilbert space on which the next representation is defined.

In the source	Mathematical meaning
Faithful compact representation `∃ pi : NonUnitalCStarRepresentation A K, pi.IsCompactlyRepresented) ↔`	There exists a $*$-representation $\pi : A \to \mathcal{B}(H)$ that is faithful and satisfies $\pi(A) \subseteq \mathcal{K}(H)$. This is the left side of the equivalence, not an equality of the two operator algebras.
Meaning of compact representability `def NonUnitalCStarRepresentation.IsCompactlyRepresented (pi : NonUnitalCStarRepresentation A H) : Prop := Function.Injective pi ∧ ∀ a : A, IsCompactOperator (pi a)`	The first conjunct is injectivity of the same π, that is, faithfulness. The second says that $\pi(a)$ is compact for every $a \in A$. Together they express exactly the first condition in the Statement.
All irreducible representations `(∀ (K : Type u) [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K] (rho : NonUnitalCStarRepresentation A K), rho.IsIrreducible → (NonUnitalStarAlgHom.range rho) = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ K) ∧`	For every complex Hilbert space K and every irreducible $\rho : A \to \mathcal{B}(K)$, the operator-algebra image is $$\rho(A) = \mathcal{K}(K).$$ Here `range rho` and `compactOperatorStarSubalgebra` are subalgebras of the same $\mathcal{B}(K)$. This is stronger than merely having compact values.

Primitive points
`def primitiveKernelFamily : Set (Set A) := Set.range fun phi : NonUnitalPureState A => (kernelIdeal phi.gnsRepresentation : Set A)`
 	
`/-- Actual primitive ideal carriers, not geometric primitive-spectrum points. -/ def PrimitiveSpectrum := {P : Set A // P ∈ primitiveKernelFamily A}`
	The points $P \in \mathrm{Prim}(A)$ are kernels of pure-state GNS representations, regarded as subsets of A. The subtype records membership in precisely this family; it is not the geometric structure space $S(A)$.

Hull–kernel topology `def primitiveSpectrumClosure : ClosureSpace (PrimitiveSpectrum A) := ClosureSpace.HullKernel.space (primitiveKernelFamily A) (univ_notMem_primitiveKernelFamily A)`	The first argument is the family of underlying sets of primitive ideals of A. The second is the established fact that A itself, written `Set.univ`, is not a member of this family; it is the required input for constructing the hull–kernel closure on that family. For every $E \subseteq \mathrm{Prim}(A)$, $$c_{\mathrm{Prim}(A)}(E) = \left\{ P \in \mathrm{Prim}(A) : \bigcap_{Q \in E} Q \subseteq P \right\}.$$
Discreteness `DiscreteTopology (PrimitiveSpectrum A) :=`	The second conjunct says that $\mathrm{Prim}(A)$ has the discrete topology. It accompanies the assertion about every irreducible image; it is not a substitute for that assertion. The empty spectrum is allowed.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
attribute [local instance] _root_.CStarAlgebra.spectralOrder
  _root_.CStarAlgebra.spectralOrderedRing
```

Exact surrounding source:

```
universe u v

variable {A : Type u} [NonUnitalCStarAlgebra A]
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```

Related exact source for the reading guide

Meaning of compact representability · Fixed source

```
def NonUnitalCStarRepresentation.IsCompactlyRepresented
    (pi : NonUnitalCStarRepresentation A H) : Prop :=
  Function.Injective pi ∧ ∀ a : A, IsCompactOperator (pi a)
```

Discreteness · Fixed source

```
    DiscreteTopology (PrimitiveSpectrum A) :=
```

Primitive points · Fixed source

```
def primitiveKernelFamily : Set (Set A) :=
  Set.range fun phi : NonUnitalPureState A => (kernelIdeal phi.gnsRepresentation : Set A)

/-- Actual primitive ideal carriers, not geometric primitive-spectrum points. -/
def PrimitiveSpectrum := {P : Set A // P ∈ primitiveKernelFamily A}
```

Hull–kernel topology · Fixed source

```
def primitiveSpectrumClosure : ClosureSpace (PrimitiveSpectrum A) :=
  ClosureSpace.HullKernel.space (primitiveKernelFamily A) (univ_notMem_primitiveKernelFamily A)
```

The geometric structure of a representation image as a component closure

`MathlibAnnex.Analysis.CStarAlgebra.NonUnitalCStarRepresentation.imageGeometricStructureHomeomorph`

def

Identifies the geometric structure of a representation image with a component closure.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity, and let $\pi : A \to \mathcal{B}(H)$ be a nonzero $*$-representation on a complete complex Hilbert space. Let $I_0 \in S(A)$ be a geometric-structure point whose component satisfies

$$\bigcap_{I \in C(I_0)} I = \ker \pi. \tag{1}$$

With the inherited C*-algebra structure on $\pi(A)$, there is a specified closure-space homeomorphism

$$E_{\pi,I_0} : S(\pi(A)) \longrightarrow c_A(C(I_0)). \tag{2}$$

The target carries the subspace closure from $S(A)$. The exact declaration requires nonzero action and the explicit equality (1); it does not require irreducibility of π in addition to them.

Definition

For any algebra D here, $S(D)$ denotes kernels of extreme points of the closed unit ball in the Banach dual $D \to \mathbb{C}$, using real convexity. The geometric identification is supplied by the extreme-functional kernel characterization. Its closure is $c_D(\mathcal{U}) = \{I \in S(D) : \bigcap_{J \in \mathcal{U}} J \subseteq I\}$, including the empty family.

Regard π as a map from A onto its image $\pi(A)$. The defining map and its inverse are

$$E_{\pi,I_0}(J) = \pi^{-1}(J), \qquad E^{-1}_{\pi,I_0}(I) = \pi(I).$$

Here $J \in S(\pi(A))$ and $I \in c_A(C(I_0))$. The target is the ambient closure of the whole specified component, equipped with subspace closure. It is not the component alone.

The surjective preimage construction supplies the homeomorphism onto the geometric points containing $\ker \pi$. The component-closure identification uses (1) to identify that target with $c_A(C(I_0))$, without changing the point map. The image's nontriviality and compatible order are supplied by the image construction, not additional hypotheses.

Assumptions

The displayed scope requires a nonzero complex C*-algebra A with its compatible order and a complete complex Hilbert space H. The representation π is nonzero, and the component of the specified point I_0 satisfies (1). Neither faithfulness nor irreducibility is an additional hypothesis. The cited image construction equips $\pi(A)$ with its C*-algebra structure and compatible order.

Conclusion

The output bundles the displayed preimage map and its image-map inverse, with full closure continuity in both directions. The target is exactly $c_A(C(I_0))$ with the inherited closure; it is not a claim that $C(I_0)$ is closed.

Main citations

- Exact full definition: `NonUnitalCStarRepresentation.imageGeometricStructureHomeomorph`.
- SurjectivePreimage.lean, `geometricStructurePreimageHomeomorph`: the first stage of the construction.
- ImageSubspace.lean, `closure_component_eq_hull_ker` and the value and closure lemmas following the definition: the target identification and the concrete action.
- Ryotaro Tanaka, arXiv:2504.05551v1, Theorem 5.17. This card uses the quotient of the original algebra and its representation image.
- Surjective preimage construction: the exact map and inverse.
- Actual C*-image structure.

Lean source signature (exact)

The full definition below is copied from the pinned source.

```
def imageGeometricStructureHomeomorph (hpi : pi.IsNonzero)
    (I : NormedSpace.GeometricStructureSpace ℂ A)
    (hI : NormedSpace.geometricSpectrumKernel ℂ A (NormedSpace.toGeometricSpectrum ℂ A I) =
      ((imageSubspaceContinuousMap pi).ker : Set A)) :
    ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ (NonUnitalStarAlgHom.range pi))
      ((NormedSpace.geometricStructureClosure ℂ A).subspace
        ((NormedSpace.geometricStructureClosure ℂ A).closure
          ((NormedSpace.geometricStructureClosure ℂ A).component I))) := by
  letI : Nontrivial (NonUnitalStarAlgHom.range pi) := nontrivial_image_of_isNonzero pi hpi
  letI : PartialOrder (NonUnitalStarAlgHom.range pi) := _root_.CStarAlgebra.spectralOrder
(NonUnitalStarAlgHom.range pi)
  letI : StarOrderedRing (NonUnitalStarAlgHom.range pi) := _root_.CStarAlgebra.spectralOrderedRing
(NonUnitalStarAlgHom.range pi)
  let q := (NonUnitalStarAlgHom.rangeRestrict pi)
  have hq : Function.Surjective q := rangeRestrict_surjective pi
  have hker : ((imageSubspaceContinuousMap q).ker : Set A) =
      ((imageSubspaceContinuousMap pi).ker : Set A) := by
    ext a
    change q a = 0 ↔ pi a = 0
    constructor
    · intro h
      exact congrArg Subtype.val h
    · intro h
      apply Subtype.ext
      change pi a = 0
      exact h
  have htarget : ClosureSpace.HullKernel.hull (NormedSpace.geometricStructureFamily ℂ A)
      ((imageSubspaceContinuousMap q).ker : Set A) =
        (NormedSpace.geometricStructureClosure ℂ A).closure
          ((NormedSpace.geometricStructureClosure ℂ A).component I) := by
    rw [hker, closure_component_eq_hull_ker pi I hI]
  exact (geometricStructurePreimageHomeomorph q hq).trans
    ((ClosureSpace.Homeomorph.refl (NormedSpace.geometricStructureClosure ℂ A)).subspace
      _ _ (fun J => Set.ext_iff.mp htarget J))
```

In the source	Mathematical meaning
Surrounding **Exact declaration** `A`, `H` and their instances	The nonzero complex C*-algebra A with compatible order, and the complete complex Hilbert space H. A unit is not required.
Exact declaration `pi`, `hpi`, `I`, `hI`	The nonzero representation π, the point $I_0 \in S(A)$, and the equality $\bigcap_{I \in C(I_0)} I = \ker \pi$. The representation need not be faithful or irreducible.
Exact declaration `NonUnitalStarAlgHom.range pi` and the local instances	The image $\pi(A)$, with its inherited C*-algebra structure, nontriviality, and compatible spectral order.
Exact declaration `q := NonUnitalStarAlgHom.rangeRestrict pi`	The same action $a \mapsto \pi(a)$, now with codomain $\pi(A)$.

In the source	Mathematical meaning
Exact declaration `hq : Function.Surjective q`	Every element of $\pi(A)$ equals $\pi(a)$ for some $a \in A$.
Exact declaration `hker`	The equality $\ker q = \ker \pi$ as subsets of A. Restricting the codomain does not change this kernel.
Exact declaration `geometricStructurePreimageHomeomorph q hq`	The homeomorphism $J \mapsto \pi^{-1}(J)$ from $S(\pi(A))$ onto $\{I \in S(A) : \ker \pi \subseteq I\}$, with inverse $I \mapsto \pi(I)$.
Exact declaration `htarget`	The set $\{I \in S(A) : \ker \pi \subseteq I\}$ equals $c_A(C(I_0))$, by the component-intersection hypothesis.
Exact declaration `Homeomorph.refl ... .subspace`	The identity on these geometric points, regarded in the target $c_A(C(I_0))$ with its inherited closure.
Final **Exact declaration** `.trans`	First take $J \mapsto \pi^{-1}(J)$, then identify its codomain with $c_A(C(I_0))$. The point map and its inverse remain those displayed above.

Exact surrounding source context (separate excerpts)

Source lines 32–32:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Source lines 33–33:

```
variable [PartialOrder A] [StarOrderedRing A]
```

Source lines 34–34:

```
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```

Source lines 35–35:

```
variable (pi : NonUnitalCStarRepresentation A H)
```

Source line 45 supplies the image C*-algebra instance used by the declaration; it is not an additional hypothesis:

```
attribute [local instance] HomKernelQuotient.imageCStarAlgebra
```

The continuous-map wrapper in the same source, lines 37–43, is copied separately; it retains the given action with the norm bound supplied by `NonUnitalStarAlgHom.norm_apply_le`:

```
private def imageSubspaceContinuousMap {B : Type*} [NonUnitalCStarAlgebra B]
    (q : A →⋆ₙₐ[ℂ] B) : A →L[ℂ] B :=
  LinearMap.mkContinuous
    { toFun := q, map_add' := map_add q, map_smul' := map_smul q }
    1 (fun a => by
      change ‖q a‖ ≤ 1 * ‖a‖
      simpa only [one_mul] using NonUnitalStarAlgHom.norm_apply_le q a)
```

The same reduced atomic representation and its exact diagonal range

`MathlibAnnex.Analysis.CStarAlgebra.PrimitiveSpectrum.reducedAtomicRepresentation_spec_of_isCompactlyRepresented`

theorem

Collects the simultaneous properties and exact diagonal range of the same reduced atomic representation.

Statement

Let A be a complex C*-algebra, let H be a complex Hilbert space, and let $\pi : A \to \mathcal{B}(H)$ be a faithful $*$-representation with $\pi(A) \subseteq \mathcal{K}(H)$. Here $\mathcal{K}(L)$ denotes the algebra of compact operators on a complex Hilbert space L. Write $\mathrm{Prim}(A)$ for the primitive ideal spectrum, whose points are the kernels of pure-state GNS representations. For each $P \in \mathrm{Prim}(A)$, fix the pure state φ_P selected by the primitive-point representative. Its GNS representation satisfies $\ker \pi_{\varphi_P} = P$ by the kernel-identification theorem. Write H_P for its GNS Hilbert space and $\rho_P = \pi_{\varphi_P}$. On $\mathcal{H} = \bigoplus^2_P H_P$, the reduced atomic representation $R : A \to \mathcal{B}(\mathcal{H})$ is specified by the coordinate-action theorem:

$$(R(a)\xi)_P = \rho_P(a)\xi_P \qquad (a \in A,\ \xi \in \mathcal{H}).$$

For a bounded family of compact operators $T = (T_P)$ with $T_P \in \mathcal{K}(H_P)$, let $\Delta_b(T)$ act diagonally on $\mathcal{H}$:

$$(\Delta_b(T)\xi)_P = T_P\xi_P, \qquad \sup_P \|T_P\| < \infty.$$

Write Δ_0 for the restriction to the c_0-sum: for every $\varepsilon > 0$, the set $\{P : \|T_P\| \geq \varepsilon\}$ is finite. These bounded diagonal and c0-diagonal action formulas refer to the same Hilbert sum. Then, simultaneously:

- R is injective and compact-valued;
- $\operatorname{ran} R = \operatorname{ran} \Delta_0$, where Δ_0 is the diagonal action of $\bigoplus^{c_0}_P \mathcal{K}(H_P)$;
- $\operatorname{ran} R = \operatorname{ran} \Delta_b \cap \mathcal{K}(\mathcal{H})$, where Δ_b is the diagonal action of the bounded product of compact blocks;
- R is an isometry and $\|R(a)\| = \|a\|$ for every $a \in A$;
- for the coordinate projection p_P and embedding j_P, $p_P R(a) j_P = \rho_P(a)$;
- $R(a)^* = R(a^*)$ for every $a \in A$.

Assumptions

The given π acts on the complete complex Hilbert space H. Only its injectivity and pointwise compactness are assumed. No irreducibility, nonzero algebra, nonempty index, countability, or separability is imposed. The Hilbert sum and its selected fibers are the already defined ones.

Conclusion

All outputs concern this one R, not a replacement action. Its range is block diagonal; the theorem does not assert that it is all of $\mathcal{K}(\mathcal{H})$ or that arbitrary off-diagonal compact operators are present. It also does not claim unitary equivalence between R and the original π.

Main citations

The simultaneous reduced-atomic specification concerns the already defined reduced-atomic map. Its range as a compact c0-diagonal action and range as the compact part of the bounded diagonal action are exact equalities for that same map, not the whole compact-operator algebra.

Lean source signature (exact)

Exact source and declaration. Only the literal signature is displayed here; the linked source retains the proof.

```
theorem reducedAtomicRepresentation_spec_of_isCompactlyRepresented
    (pi : NonUnitalCStarRepresentation A H) (hpi : pi.IsCompactlyRepresented) :
    (reducedAtomicRepresentation (A := A)).IsCompactlyRepresented ∧
    (NonUnitalStarAlgHom.range (reducedAtomicRepresentation (A := A))) =
      (NonUnitalStarAlgHom.range (CompactOperatorC0Sum.diagonalRepresentation
        (fun P : PrimitiveSpectrum A => P.pureStateRepresentative.GNS))) ∧
    (NonUnitalStarAlgHom.range (reducedAtomicRepresentation (A := A))) =
      (NonUnitalStarAlgHom.range (CompactOperatorBoundedProduct.diagonalRepresentation
        (fun P : PrimitiveSpectrum A => P.pureStateRepresentative.GNS))) ⊓
       ContinuousLinearMap.compactOperatorStarSubalgebra ℂ
         (ReducedAtomicHilbertSpace A) ∧
    Isometry (reducedAtomicRepresentation (A := A)) ∧
    (∀ a : A, ‖reducedAtomicRepresentation (A := A) a‖ = ‖a‖) ∧
    (∀ (a : A) (P : PrimitiveSpectrum A),
      (coordinateProjection P).comp
        ((reducedAtomicRepresentation (A := A) a).comp (coordinateEmbedding P)) =
          P.pureStateRepresentative.gnsRepresentation a) ∧
    (∀ a : A, ContinuousLinearMap.adjoint (reducedAtomicRepresentation (A := A) a) =
      reducedAtomicRepresentation (star a))
```

In the source	Mathematical meaning

Algebra, Hilbert space and positivity
`universe u v`
 	
`variable {A : Type u} [NonUnitalCStarAlgebra A]`
`variable {H : Type v} [NormedAddCommGroup H]`
`[InnerProductSpace ℂ H] [CompleteSpace H]`
	The complex C*-algebra A and complex Hilbert space H are independent inputs. Completeness is part of the Hilbert structure. The local spectral-order instances displayed separately provide the positivity used for pure states.

Given faithful compact action
`(pi : NonUnitalCStarRepresentation A H) (hpi :`
`pi.IsCompactlyRepresented) :`
	The input is the given $\pi : A \to \mathcal{B}(H)$, faithful with $\pi(A) \subseteq \mathcal{K}(H)$. This hypothesis places no irreducibility or nonzero requirement on A.

In the source	Mathematical meaning
Meaning of compact representability `def NonUnitalCStarRepresentation.IsCompactlyRepresented (pi : NonUnitalCStarRepresentation A H) : Prop := Function.Injective pi ∧ ∀ a : A, IsCompactOperator (pi a)`	Injectivity is faithfulness, and the universal compactness condition says that every $\pi(a)$ is compact. The same definition is applied to R in the first conclusion.
Fixed pure-state representatives `def pureStateRepresentative (P : PrimitiveSpectrum A) : NonUnitalPureState A := P.property.choose`	For each $P \in \mathrm{Prim}(A)$ this selects the fixed pure state φ_P. The separately cited kernel theorem identifies $\ker \rho_P = P$, where $\rho_P = \pi_{\varphi_P}$ on H_P. These are the same representatives in every conclusion.
Hilbert sum `abbrev ReducedAtomicHilbertSpace (A : Type u) [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A] := HilbertSum (fun P : PrimitiveSpectrum A => P.pureStateRepresentative.GNS)`	The representation space is $$\mathcal{H} = \bigoplus_{P \in \mathrm{Prim}(A)}^{2} H_P.$$ This is the Hilbert direct sum of the selected GNS spaces, not the algebra of bounded operator families.
Reduced atomic action `theorem reducedAtomicRepresentation_apply (a : A) (x : ReducedAtomicHilbertSpace A) (P : PrimitiveSpectrum A) : reducedAtomicRepresentation a x P = P.pureStateRepresentative.gnsRepresentation a (x P) := by`	The coordinate formula specifies the same $R : A \to \mathcal{B}(\mathcal{H})$: $$(R(a)\xi)_P = \rho_P(a)\xi_P.$$ It holds for every $a \in A$, $\xi \in \mathcal{H}$ and $P \in \mathrm{Prim}(A)$.
Faithfulness and compact values of R `(reducedAtomicRepresentation (A := A)).IsCompactlyRepresented ∧`	The first conclusion is that this R is injective and that $R(a)$ is compact for every $a \in A$.
Range as c0 diagonal operators `(NonUnitalStarAlgHom.range (reducedAtomicRepresentation (A := A))) = (NonUnitalStarAlgHom.range (CompactOperatorC0Sum.diagonalRepresentation (fun P : PrimitiveSpectrum A => P.pureStateRepresentative.GNS))) ∧`	The family argument assigns the same H_P to each primitive ideal. The equality is $$\mathrm{ran}\, R = \mathrm{ran}\, \Delta_0,$$ where Δ_0 acts on the c_0-family of compact blocks defined in the Statement, on the same $\mathcal{H}$.

Range as compact bounded-diagonal operators
`(NonUnitalStarAlgHom.range (reducedAtomicRepresentation (A := A))) = (NonUnitalStarAlgHom.range (CompactOperatorBoundedProduct.diagonalRepresentation (fun P : PrimitiveSpectrum A => P.pureStateRepresentative.GNS))) ⊓`
 	
`ContinuousLinearMap.compactOperatorStarSubalgebra ℂ (ReducedAtomicHilbertSpace A) ∧`
	The meet ⊓ is intersection of subalgebras of $\mathcal{B}(\mathcal{H})$. Thus
	$$\mathrm{ran}\, R = \mathrm{ran}\, \Delta_b \cap \mathcal{K}(\mathcal{H}).$$
	The bounded family and the compact-operator algebra use the same selected Hilbert sum; no off-diagonal operators are added.

In the source	Mathematical meaning
Isometry and norm equality `Isometry (reducedAtomicRepresentation (A := A)) ∧ (∀ a : A, ‖reducedAtomicRepresentation (A := A) a‖ = ‖a‖) ∧`	The same R preserves distances and satisfies $$\|R(a)\| = \|a\| \qquad (a \in A).$$ Both assertions occur explicitly as conjuncts.
Coordinate compression `(∀ (a : A) (P : PrimitiveSpectrum A), (coordinateProjection P).comp ((reducedAtomicRepresentation (A := A) a).comp (coordinateEmbedding P)) = P.pureStateRepresentative.gnsRepresentation a) ∧`	The projection $p_P : \mathcal{H} \to H_P$ and inclusion $j_P : H_P \to \mathcal{H}$ give $$p_P R(a) j_P = \rho_P(a) \qquad (a \in A,\ P \in \mathrm{Prim}(A)).$$ The order of the two `comp` operations is exactly this composition.
Adjoint identity `(∀ a : A, ContinuousLinearMap.adjoint (reducedAtomicRepresentation (A := A) a) = reducedAtomicRepresentation (star a)) := by`	The last conjunct is $$R(a)^* = R(a^*) \qquad (a \in A).$$ The left side is the Hilbert adjoint on $\mathcal{H}$ and the right side uses the involution of A. It concerns the same R as all preceding conjuncts.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
attribute [local instance] _root_.CStarAlgebra.spectralOrder
  _root_.CStarAlgebra.spectralOrderedRing
```

Exact surrounding source:

```
universe u v

variable {A : Type u} [NonUnitalCStarAlgebra A]
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```

Related exact source for the reading guide

Meaning of compact representability · Fixed source

```
def NonUnitalCStarRepresentation.IsCompactlyRepresented
    (pi : NonUnitalCStarRepresentation A H) : Prop :=
  Function.Injective pi ∧ ∀ a : A, IsCompactOperator (pi a)
```

Hilbert sum · Fixed source

```
abbrev ReducedAtomicHilbertSpace (A : Type u) [NonUnitalCStarAlgebra A]
    [PartialOrder A] [StarOrderedRing A] :=
  HilbertSum (fun P : PrimitiveSpectrum A => P.pureStateRepresentative.GNS)
```

Reduced atomic action · Fixed source

```
theorem reducedAtomicRepresentation_apply
    (a : A) (x : ReducedAtomicHilbertSpace A) (P : PrimitiveSpectrum A) :
    reducedAtomicRepresentation a x P =
      P.pureStateRepresentative.gnsRepresentation a (x P) := by
```

Adjoint identity · Fixed source

```
    (∀ a : A, ContinuousLinearMap.adjoint (reducedAtomicRepresentation (A := A) a) =
      reducedAtomicRepresentation (star a)) := by
```

Fixed pure-state representatives · Fixed source

```
def pureStateRepresentative (P : PrimitiveSpectrum A) : NonUnitalPureState A :=
  P.property.choose
```

A coefficient-kernel component equals the normal part

`MathlibAnnex.Analysis.CStarAlgebra.component_restrictedMatrixCoefficientKernel_eq_normalPart`

theorem

Identifies the normal part with one component of the ambient geometric structure space.

Statement

Let H be a complex Hilbert space, and let $A \subseteq \mathcal{B}(H)$ be a nonzero norm-closed complex $*$-subalgebra acting irreducibly on H. No identity is required. For $x, y \in H \setminus \{0\}$, define the bounded linear functional

$$f_{x,y} : A \to \mathbb{C}, \qquad f_{x,y}(a) = \langle y, a(x) \rangle,$$

where the inner product is conjugate-linear in the first variable. The geometric structure space $S(A)$ consists of the kernels of extreme points of the closed unit ball of the Banach dual of A, with convexity over $\mathbb{R}$. The coefficient-kernel membership theorem guarantees that every $\ker f_{x,y}$ above belongs to $S(A)$. For $D \subseteq S(A)$, the hull–kernel closure is

$$c_A(D) = \left\{ I \in S(A) : \bigcap_{J \in D} J \subseteq I \right\}.$$

Two points are linked if they are equal or their two-point set is not closed. For $I \in S(A)$, let $C(I)$ be its component under finite chains of links, including I itself. Then, for every $x, y \in H \setminus \{0\}$,

$$C(\ker f_{x,y}) = N(A) = \{\ker f_{z,w} : z, w \in H \setminus \{0\}\}. \quad (1)$$

The set $N(A)$ is called the normal part of A.

Assumptions

The algebra A has its inherited operator norm and adjoint. Here irreducibility means that the only closed A-invariant subspaces of H are $\{0\}$ and H; nonzero action is ensured by $A \neq \{0\}$. The vectors x, y need not be unit vectors.

Conclusion

Thus $N(A)$ is one component of the ambient space $S(A)$, independent of the nonzero pair x, y used to specify the starting kernel. This does not assert $N(A) = S(A)$.

Main citations

- The normal-part definition: the membership condition with a single nonzero pair.

- Whole-component theorem: `component_matrixCoefficientKernel_eq_family`, the result retained from the preceding component analysis.
- Exact concrete component theorem: the equality for the inclusion representation.
- arXiv:2504.05551v1, Lemma 6.3 and Definition 6.4: the coefficient component and the normal-part terminology.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem component_restrictedMatrixCoefficientKernel_eq_normalPart
    (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))
    (x y : H) (hx : x ≠ 0) (hy : y ≠ 0) :
    (NormedSpace.geometricStructureClosure ℂ S).component
      (restrictedMatrixCoefficientKernel S hS x y hx hy) = normalPart S
```

In the source, `S` denotes the algebra A used here.

In the source	Mathematical meaning
Hilbert space `variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`	The complete complex Hilbert space H on which A acts.
Closed operator algebra `variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) [IsClosed (S : Set (H →L[ℂ] H))]`	The source name S denotes the algebra $A \subseteq \mathcal{B}(H)$. The closedness assumption concerns this subset of bounded operators.

Inherited C*-algebra structure
`private abbrev normalPartCStarAlgebra : NonUnitalCStarAlgebra S := NonUnitalStarSubalgebra.nonUnitalCStarAlgebra S (h_closed := inferInstance)`
`attribute [local instance] normalPartCStarAlgebra`
	The closed subalgebra has the inherited C*-algebra structure. The local instance makes that structure available when forming $S(A)$ and its closure; it does not impose a new hypothesis.

Irreducibility and coefficient vectors `(hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S)) (x y : H) (hx : x ≠ 0) (hy : y ≠ 0) :`	The inclusion action of A is nonzero and irreducible. The two vectors are $x, y \in H \setminus \{0\}$, defining $\ker f_{x,y}$ in (1).
Ambient component equality `(NormedSpace.geometricStructureClosure ℂ S).component (restrictedMatrixCoefficientKernel S hS x y hx hy) = normalPart S := by`	The complete starting point is the coefficient kernel for the same A, x, y. The component is taken in the ambient space $S(A)$, and $$C(\ker f_{x,y}) = N(A).$$ It is not a component formed after restricting to $N(A)$, and it does not assert $N(A) = S(A)$.

In the source	Mathematical meaning
Coefficient-kernel point `def restrictedMatrixCoefficientKernel (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S)) (x y : H) (hx : x ≠ 0) (hy : y ≠ 0) : NormedSpace.GeometricStructureSpace ℂ S := ⟨{a : S | inner ℂ y ((a : H →L[ℂ] H) x) = 0}, setOf_inner_apply_eq_zero_mem_geometricStructureFamily S hS x y hx hy⟩`	The constructed point has underlying subset $\ker f_{x,y}$; the stated irreducibility and nonzero-vector conditions certify its membership in $S(A)$. The coefficient-kernel evaluation records that same subset.
Normal part `def normalPart (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) : Set (NormedSpace.GeometricStructureSpace ℂ S) := {I | ∃ x y : H, x ≠ 0 ∧ y ≠ 0 ∧ {a : S | inner ℂ y ((a : H →L[ℂ] H) x) = 0} = I.val}`	The set $N(A)$ consists of geometric points $I \in S(A)$ for which there exist $z, w \in H \setminus \{0\}$ with $I = \ker f_{z,w}$. Under the present hypotheses every such kernel is a geometric point, as used in (1).

The local instance below is a consequence of the closed-subalgebra hypothesis. The following Image section and its new variables are not part of this theorem.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```

Exact surrounding source:

```
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) [IsClosed (S : Set (H →L[ℂ] H))]
```

Exact surrounding source:

```
private abbrev normalPartCStarAlgebra : NonUnitalCStarAlgebra S :=
  NonUnitalStarSubalgebra.nonUnitalCStarAlgebra S (h_closed := inferInstance)

attribute [local instance] normalPartCStarAlgebra
```

Related exact source for the reading guide

Normal part · Fixed source

```
def normalPart (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) :
    Set (NormedSpace.GeometricStructureSpace ℂ S) :=
  {I | ∃ x y : H, x ≠ 0 ∧ y ≠ 0 ∧
    {a : S | inner ℂ y ((a : H →L[ℂ] H) x) = 0} = I.val}
```

Coefficient-kernel point · Fixed source

```
def restrictedMatrixCoefficientKernel
    (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))
    (x y : H) (hx : x ≠ 0) (hy : y ≠ 0) : NormedSpace.GeometricStructureSpace ℂ S :=
  ⟨{a : S | inner ℂ y ((a : H →L[ℂ] H) x) = 0},
    setOf_inner_apply_eq_zero_mem_geometricStructureFamily S hS x y hx hy⟩
```

Ambient component equality · Fixed source

```
    (NormedSpace.geometricStructureClosure ℂ S).component
      (restrictedMatrixCoefficientKernel S hS x y hx hy) = normalPart S := by
```

Every geometric-spectrum component contains a pure-state kernel

`MathlibAnnex.Analysis.CStarAlgebra.exists_pureState_kernel_mem_geometricSpectrum`

theorem

Places the kernel of a pure state in every geometric-spectrum component.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity. Its geometric structure space $S(A)$ consists of kernels $\ker \rho \subseteq A$ of bounded complex-linear functionals $\rho : A \to \mathbb{C}$ that are extreme points, for real convexity, of the entire closed dual unit ball B_{A^*}, as stated by the extreme-kernel identification, with closure

$$c_A(\mathcal{T}) = \left\{ I \in S(A) : \bigcap_{J \in \mathcal{T}} J \subseteq I \right\}.$$

Two points are linked if they are equal or their pair is not closed. Their finite-chain equivalence classes are the components. A point C of the geometric spectrum $\mathrm{GS}(A)$ is one such component subset of $S(A)$.

For every $C \in \mathrm{GS}(A)$ there is a pure state φ of A such that

$$\ker \varphi \in C. \qquad (1)$$

Here $\ker \varphi$ is a point of $S(A)$: the pure-state extremality theorem guarantees that φ is extreme in the entire closed dual unit ball.

Assumptions

The source fixes a nonzero complex C*-algebra and its compatible order. The component C is arbitrary; it need not be presented initially by a functional or representation. A pure state is a positive norm-one bounded complex-linear functional $A \to \mathbb{C}$, extreme among positive norm-one functionals for real convexity. The geometric spectrum consists of component **sets**, so (1) places a functional kernel, rather than a functional, in C.

Conclusion

The theorem produces one state together with its purity and the membership of that same state's kernel in the specified component. It does not identify $\ker \varphi$ with the intersection of all points of C; that intersection is determined in the component-kernel identity.

Main citations

- Exact component representative theorem: `exists_pureState_kernel_mem_geometricSpectrum`.
- Ryotaro Tanaka, arXiv:2504.05551v1, Corollary 5.9: the component representative assertion, read with the type-correct pure-state **kernel**.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem exists_pureState_kernel_mem_geometricSpectrum (C : NormedSpace.GeometricSpectrum ℂ A) :
    ∃ (phi : StrongDual ℂ A) (hpure : IsPureNonUnitalState A phi),
      extremeKernel ⟨phi, mem_extremePoints_dual_closedBall_of_isPureNonUnitalState phi hpure⟩
∈ C.val
```

In the source	Mathematical meaning

Algebra and order
`variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]`
`variable [PartialOrder A] [StarOrderedRing A]`
	The nonzero complex C*-algebra A and its compatible order support the pure-state and geometric-point constructions.

Specified component `theorem exists_pureState_kernel_mem_geometricSpectrum (C : NormedSpace.GeometricSpectrum ℂ A) :`	The input $C \in \mathrm{GS}(A)$ is a component subset of $S(A)$, not an individual functional or representation.
Pure state `∃ (phi : StrongDual ℂ A) (hpure : IsPureNonUnitalState A phi),`	There is a bounded complex-linear functional $\varphi : A \to \mathbb{C}$ which is a pure state: positive, of norm one, and extreme among states for real convexity.
Kernel point in that component `extremeKernel ⟨phi, mem_extremePoints_dual_closedBall_of_isPureNonUnitalState phi hpure⟩ ∈ C.val := by`	The extremality result applied to this same φ certifies $\ker\varphi$ as a point of $S(A)$. The full conclusion is $$\ker\varphi \in C.$$ The projection of C is its component subset, so the membership places the kernel point in the originally specified component.

Components as geometric-spectrum points
`def GeometricSpectrum : Type u :=`
`  {C : Set (GeometricStructureSpace 𝕜 E) // C ∈ geometricSpectrumFamily 𝕜 E}`
	A geometric-spectrum point is a component subset of $S(A)$ with its component membership. Its underlying set is the C used in (1).

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Exact surrounding source:

```
variable [PartialOrder A] [StarOrderedRing A]
```

Related exact source for the reading guide

Algebra and order · Fixed source

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
variable [PartialOrder A] [StarOrderedRing A]
```

Kernel point in that component · Fixed source

```
      extremeKernel ⟨phi, mem_extremePoints_dual_closedBall_of_isPureNonUnitalState phi hpure⟩
∈ C.val := by
```

Components as geometric-spectrum points · Fixed source

```
def GeometricSpectrum : Type u :=
  {C : Set (GeometricStructureSpace 𝕜 E) // C ∈ geometricSpectrumFamily 𝕜 E}
```

Finite-homeomorphisms detect finite dimension

`MathlibAnnex.Analysis.CStarAlgebra.finite_and_finrank_eq_of_normalPart_finiteHomeomorph`

theorem

Recovers finite dimensionality and equal dimensions from a finite-homeomorphism of normal parts.

Statement

Let H, K be complex Hilbert spaces. Let $A \subseteq \mathcal{B}(H)$ and $B \subseteq \mathcal{B}(K)$ be nonzero norm-closed complex $*$-subalgebras acting irreducibly on H and K, respectively. Neither algebra is assumed unital. For $x, y \in H \setminus \{0\}$ and $z, w \in K \setminus \{0\}$, define the coefficient functionals

$$\begin{aligned} f^A_{x,y} &: A \to \mathbb{C}, \qquad f^A_{x,y}(a) = \langle y, a(x)\rangle, \\ f^B_{z,w} &: B \to \mathbb{C}, \qquad f^B_{z,w}(b) = \langle w, b(z)\rangle. \end{aligned}$$

The inner products are conjugate-linear in the first variable. The normal parts are

$$\begin{aligned} N(A) &= \{\ker f^A_{x,y} : x, y \in H \setminus \{0\}\}, \\ N(B) &= \{\ker f^B_{z,w} : z, w \in K \setminus \{0\}\}. \end{aligned} \tag{1}$$

Give the two normal parts their inherited hull–kernel closures c_A, c_B. For $D \subseteq N(A)$, write $c_A(D) = \{J \in N(A) : \bigcap_{I \in D} I \subseteq J\}$, and define c_B analogously.

Suppose $e : N(A) \to N(B)$ is a finite-homeomorphism, that is, a bijection satisfying

$$e(c_A(D)) = c_B(e(D)) \qquad \text{for every finite } D \subseteq N(A).$$

If either H or K is finite dimensional, then both are finite dimensional and

$$\dim_{\mathbb{C}} H = \dim_{\mathbb{C}} K. \tag{2}$$

Assumptions

The given finite-homeomorphism is a bijection whose map and inverse preserve closure on finite subsets. Continuity on arbitrary subsets is not assumed. The nonzero norm-closed $*$-subalgebras A, B act irreducibly on H, K, respectively.

Finite dimensionality is required on at least one side. There is no dimension-at-least-three hypothesis: dimensions one and two are included. The nonzero actions exclude the zero Hilbert space.

Conclusion

Both spaces are finite dimensional, and their complex dimensions agree. The conclusion asserts finite dimensionality as well as equality of dimensions.

Main citations

The normal-part dimension theorem gives both finite-dimensionality assertions and equality of dimensions. Its algebraic input is the finite-incidence dimension theorem. The finite-homeomorphism definition specifies the two finite-continuity conditions. See also the retained paper's Theorem 6.18, pages 42–45.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

The parameter e is declared earlier by `variable`, rather than repeated after the theorem name. Immediately before this declaration, `include hS hT e in` tells Lean to add the two irreducibility hypotheses and e as parameters of this theorem. Thus the theorem takes the given finite-homeomorphism $e : N(A) \to N(B)$ as an input, although its conclusion mentions only the dimensions of H and K. The exact `variable` and `include` declarations are quoted in the source context below.

```
theorem finite_and_finrank_eq_of_normalPart_finiteHomeomorph
    (hfinite : Module.Finite ℂ H ∨ Module.Finite ℂ K) :
    Module.Finite ℂ H ∧ Module.Finite ℂ K ∧ Module.finrank ℂ H = Module.finrank ℂ K
```

In the source, `S` and `T` denote the algebras A and B used here, respectively.

In the source	Mathematical meaning

Hilbert spaces and irreducible algebras
`variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`
`variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]`
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))`
`variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]`
`variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
	The complex Hilbert spaces H, K carry the nonzero norm-closed irreducible algebras A, B, called `S`,`T` in the source. Completeness belongs to each Hilbert structure. No unit is required.

In the source	Mathematical meaning
Finite-homeomorphism input `variable (e : ClosureSpace.FiniteHomeomorph ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)) ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)))`	The parameter is a bijection $e : N(A) \to N(B)$, with the induced closures on these normal parts. For every finite $D \subseteq N(A)$, $$e(c_A(D)) = c_B(e(D)).$$ This is an input, not a map obtained in the conclusion.
Parameters retained by include `include hS hT e in`	The command retains the previously declared irreducibility hypotheses and the same e as parameters of this theorem, although the displayed conclusion concerns only dimensions.
One finite-dimensional space `(hfinite : Module.Finite ℂ H ∨ Module.Finite ℂ K) :`	The hypothesis is that H or K is finite-dimensional over $\mathbb{C}$. It is a disjunction, not a requirement that both are already known to be finite-dimensional.
Both dimensions and their equality `Module.Finite ℂ H ∧ Module.Finite ℂ K ∧ Module.finrank ℂ H = Module.finrank ℂ K :=`	The conclusion has three conjuncts: H is finite-dimensional, K is finite-dimensional, and $$\dim_{\mathbb{C}} H = \dim_{\mathbb{C}} K.$$ The numerical equality is additional to the two finite-dimensionality assertions; no lower dimension bound is imposed.

The displayed declaration prefix and the following ambient context are separate excerpts. The next `include` and comment in the source are not part of this declaration.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
universe u v
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))
variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
S))
variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
T))
variable (e : ClosureSpace.FiniteHomeomorph
  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))
  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)))
```

Exact surrounding source:

```
include hS hT e in
```

Related exact source for the reading guide

Both dimensions and their equality · Fixed source

```
    Module.Finite ℂ H ∧ Module.Finite ℂ K ∧ Module.finrank ℂ H = Module.finrank ℂ K :=
```

Maximal linked families are projective fibers

`MathlibAnnex.Analysis.CStarAlgebra.maximal_isLinkedSet_normalPart_iff`

theorem

Characterizes maximal linked subsets of the normal part as projective fibers.

Statement

Let H be a complex Hilbert space, and let $A \subseteq \mathcal{B}(H)$ be a nonzero norm-closed complex $*$-subalgebra acting irreducibly on H. No identity is required. For $x, y \in H \setminus \{0\}$, define $f_{x,y} : A \to \mathbb{C}$ by $f_{x,y}(a) = \langle y, a(x) \rangle$. The inner product is conjugate-linear in its first variable. The normal part is $N(A) = \{\ker f_{x,y} : x, y \in H \setminus \{0\}\}$.

Let $\mathbb{P}(H)$ be the set of one-dimensional complex subspaces of H, with $[x] = \mathbb{C}x$ for $x \neq 0$. The projective-coordinate equivalence is the bijection $e : \mathbb{P}(H) \times \mathbb{P}(H) \to N(A)$ with $e([x], [y]) = \ker f_{x,y}$, as given by its coefficient-kernel formula.

For $D \subseteq N(A)$, use the inherited closure $c(D) = \{I \in N(A) : \bigcap_{J \in D} J \subseteq I\}$. A set is pairwise linked if any two distinct members form a nonclosed two-point set. This is equivalent to requiring the linked relation, which also includes equality, for every pair of members. For $p, q \in \mathbb{P}(H)$ define the source and target fibers

$$\mathcal{R}_p = \{e(p,t) : t \in \mathbb{P}(H)\}, \qquad \mathcal{C}_q = \{e(t,q) : t \in \mathbb{P}(H)\}. \tag{1}$$

For any $L \subseteq N(A)$,

$$L \text{ is maximal among pairwise-linked sets} \iff \begin{array}{l}(\exists p \in \mathbb{P}(H),\ L = \mathcal{R}_p) \\ \text{or } (\exists q \in \mathbb{P}(H),\ L = \mathcal{C}_q).\end{array} \tag{2}$$

Assumptions

Maximality is with respect to inclusion, among pairwise-linked subsets of the same normal part. There is no hypothesis that L is closed, finite, or initially nonempty. The nonzero-action hypothesis excludes the zero Hilbert space. Dimension one is allowed; no lower bound of two is added.

Conclusion

Every maximal family fixes either the source projective line or the target projective line, and every fiber in (1) is maximal. The two descriptions need not be different: in dimension one both fibers are the whole singleton normal part. No uniqueness between the two alternatives of (2) is asserted.

Main citations

- The projective-product bijection: every point has a unique ordered pair of projective coordinates.
- The linked-pair criterion: two coefficient kernels are linked exactly when their source lines or their target lines agree.
- Exact maximal-family theorem: the equivalence (2) for all subsets of the normal part.
- arXiv:2504.05551v1, Definition 6.11 and Lemma 6.13: pairwise-linked sets and their maximal fibers; dimension one is included.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem maximal_isLinkedSet_normalPart_iff (L : Set (normalPart S)) :
    Maximal
      ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).IsLinkedSet L ↔
      (∃ p : ℙ ℂ H, L = normalPartSourceFiber S hS p) ∨
      (∃ q : ℙ ℂ H, L = normalPartTargetFiber S hS q)
```

In the source, `S` denotes the algebra A used here.

In the source	Mathematical meaning

Hilbert space and irreducible algebra
`variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable [IsClosed (S : Set (H →L[ℂ] H))]`
`variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
	The source algebra `S` is the nonzero norm-closed complex $*$-subalgebra $A \subseteq \mathcal{B}(H)$ acting irreducibly on the complex Hilbert space H.

Set to be classified
`theorem maximal_isLinkedSet_normalPart_iff (L : Set (normalPart S)) :`
`  Maximal`
`  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).IsLinkedSet L ↔`
	The input L is any subset of $N(A)$. It is maximal by inclusion among pairwise-linked subsets for the relative geometric closure on $N(A)$. Neither closedness of L nor a greatest cardinality is required.

Pairwise linkedness
`def IsLinkedSet (c : ClosureSpace α) (S : Set α) : Prop :=`
`  ∀ x ∈ S, ∀ y ∈ S, c.Linked x y`
	The condition on L is that every two of its points are linked for that closure. The `Maximal` in the preceding row also requires that no strictly larger subset has this property.

Source fiber
`def normalPartSourceFiber (p : ℙ ℂ H) : Set (normalPart S) :=`
`  {I | ((normalPartProjectivizationEquiv S hS).symm I).1 = p}`
	The equivalence is $e : \mathbb{P}(H) \times \mathbb{P}(H) \to N(A)$ from the Statement. The first coordinate of $e^{-1}(I)$ equals p exactly on
	$$\mathcal{R}_p = \{e(p,t) : t \in \mathbb{P}(H)\}.$$
	Here `symm` denotes the inverse of that same e.

In the source	Mathematical meaning

Target fiber
`def normalPartTargetFiber (q : ℙ ℂ H) : Set (normalPart S) :=`
`  {I | ((normalPartProjectivizationEquiv S hS).symm I).2 = q}`
	The second coordinate of $e^{-1}(I)$ equals q exactly on
	$$\mathcal{C}_q = \{e(t,q) : t \in \mathbb{P}(H)\}.$$
	Both fibers use the same projective space and equivalence.

The two kinds of maximal family
`(∃ p : ℙ ℂ H, L = normalPartSourceFiber S hS p) ∨`
`(∃ q : ℙ ℂ H, L = normalPartTargetFiber S hS q) := by`
	The conclusion is the disjunction
	$$\big(\exists p \in \mathbb{P}(H),\ L = \mathcal{R}_p\big) \quad \text{or} \quad \big(\exists q \in \mathbb{P}(H),\ L = \mathcal{C}_q\big).$$
	It classifies all maximal linked families. The two kinds may coincide in dimension one; no dimension-at-least-two assumption is made.

The effective context requires nonzero irreducibility. There is no dimension-at-least-two or nonempty-family assumption.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable [IsClosed (S : Set (H →L[ℂ] H))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
S))
```

Related exact source for the reading guide

Source fiber · Fixed source

```
def normalPartSourceFiber (p : ℙ ℂ H) : Set (normalPart S) :=
  {I | ((normalPartProjectivizationEquiv S hS).symm I).1 = p}
```

Target fiber · Fixed source

```
def normalPartTargetFiber (q : ℙ ℂ H) : Set (normalPart S) :=
  {I | ((normalPartProjectivizationEquiv S hS).symm I).2 = q}
```

The two kinds of maximal family · Fixed source

```
      (∃ p : ℙ ℂ H, L = normalPartSourceFiber S hS p) ∨
      (∃ q : ℙ ℂ H, L = normalPartTargetFiber S hS q) := by
```

Pairwise linkedness · Fixed source

```
def IsLinkedSet (c : ClosureSpace α) (S : Set α) : Prop :=
  ∀ x ∈ S, ∀ y ∈ S, c.Linked x y
```

Semilinear factors recover all coefficient kernels

`MathlibAnnex.Analysis.CStarAlgebra.normalPart_semilinear_factors_evaluation`

theorem

Evaluates a normal-part finite-homeomorphism and its inverse using the same semilinear factors.

Statement

Let H, K be complex Hilbert spaces. Let $A \subseteq \mathcal{B}(H)$ and $B \subseteq \mathcal{B}(K)$ be nonzero norm-closed complex $*$-subalgebras acting irreducibly on H and K, respectively. Units are not required. For $x, y \in H \setminus \{0\}$ and $z, w \in K \setminus \{0\}$, define the coefficient functionals

$$\begin{aligned} f^A_{x,y} &: A \to \mathbb{C}, \qquad f^A_{x,y}(a) = \langle y, a(x)\rangle, \\ f^B_{z,w} &: B \to \mathbb{C}, \qquad f^B_{z,w}(b) = \langle w, b(z)\rangle. \end{aligned}$$

The inner products are conjugate-linear in the first variable. The normal parts are

$$N(A) = \{\ker f^A_{x,y} : x, y \in H \setminus \{0\}\}, \qquad N(B) = \{\ker f^B_{z,w} : z, w \in K \setminus \{0\}\}.$$

For $D \subseteq N(A)$, let $c_A(D) = \{J \in N(A) : \bigcap_{I \in D} I \subseteq J\}$, and define c_B similarly. Let $e : N(A) \to N(B)$ be a finite-homeomorphism: it is a bijection and $e(c_A(D)) = c_B(e(D))$ for every finite $D \subseteq N(A)$. Suppose

$$3 \le \operatorname{rank}_{\mathbb{C}} H, \tag{1}$$

where rank is cardinal-valued algebraic dimension.

Let $f, g : \mathbb{P}(H) \to \mathbb{P}(K)$ be the projective factors associated with e by the projective factorization. Here $\mathbb{P}(H)$ is the space of one-dimensional complex subspaces of H. The semilinear realization theorem provides lifts of both factors. Use the lifts fixed by the first and second definitions:

$$\begin{aligned} \sigma_1, \sigma_2 &: \mathbb{C} \longrightarrow \mathbb{C} && \text{field automorphisms,} \\ U &: H \longrightarrow K && \text{a } \sigma_1\text{-semilinear equivalence,} \\ V &: H \longrightarrow K && \text{a } \sigma_2\text{-semilinear equivalence.} \end{aligned} \tag{2}$$

Thus $f([x]) = [Ux]$ and $g([y]) = [Vy]$ for every nonzero representative. The same maps are used for all points. For these fixed U, V, one of the following two alternatives holds globally:

$$\begin{aligned} e(\ker f^A_{x,y}) &= \ker f^B_{Ux,Vy} && (x, y \neq 0), \\ e^{-1}(\ker f^B_{z,w}) &= \ker f^A_{U^{-1}z,V^{-1}w} && (z, w \neq 0); \end{aligned} \tag{3}$$

or

$$\begin{aligned} e(\ker f^A_{x,y}) &= \ker f^B_{Vy,Ux} && (x, y \neq 0), \\ e^{-1}(\ker f^B_{z,w}) &= \ker f^A_{U^{-1}w,V^{-1}z} && (z, w \neq 0). \end{aligned} \tag{4}$$

Each alternative includes both of its universally quantified equations. Here e^{-1} is the inverse of the given bijection.

Assumptions

The normal-part bijection and its inverse satisfy finite continuity. The spaces are complex Hilbert spaces, and the two norm-closed $*$-subalgebras act nontrivially and irreducibly. The algebraic dimension of H is at least three; it may be infinite.

The two projective factors come from the same factorization of e. Their semilinear lifts may have different scalar automorphisms. Neither those automorphisms nor the lifts are assumed continuous or bounded.

Conclusion

One of the alternatives (3) and (4) holds for every nonzero pair of vectors, in both directions. These are equalities of points of the normal parts and hence of their coefficient kernels. The orientation is fixed globally, and the formulas use the same U, V and their inverses throughout. Continuity on arbitrary subsets and boundedness of the lifts are not conclusions of this theorem.

Main citations

The projective factorization and its evaluation formulas specify the two factors of the given map. The semilinear realization theorem gives lifts on every projective line. The evaluation theorem states (3) or (4) for the lifts fixed by the definitions of the first lift and second lift. The retained paper's Theorem 6.18 gives the broader context.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

Here e is a parameter declared earlier by `variable`, not a map constructed by this theorem. Lean adds the section parameters used in the statement as theorem parameters. The first two `let` expressions name the semilinear maps chosen from this same e; `e.toEquiv` is the given bijection e, and `e.toEquiv.symm` is e^{-1}. The four equations are identified individually in the table below.

```
theorem normalPart_semilinear_factors_evaluation :
    let U := normalPartFirstSemilinearEquiv S T hS hT e hdim
    let V := normalPartSecondSemilinearEquiv S T hS hT e hdim
    ((∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),
      e.toEquiv (normalPartPoint S hS x y hx hy) =
        normalPartPoint T hT (U x) (V y)
          (by simpa only [map_zero] using U.injective.ne hx)
          (by simpa only [map_zero] using V.injective.ne hy)) ∧
     (∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),
      e.toEquiv.symm (normalPartPoint T hT z w hz hw) =
        normalPartPoint S hS (U.symm z) (V.symm w)
          (by simpa only [map_zero] using U.symm.injective.ne hz)
          (by simpa only [map_zero] using V.symm.injective.ne hw))) ∨
    ((∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),
      e.toEquiv (normalPartPoint S hS x y hx hy) =
        normalPartPoint T hT (V y) (U x)
          (by simpa only [map_zero] using V.injective.ne hy)
          (by simpa only [map_zero] using U.injective.ne hx)) ∧
     (∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),
      e.toEquiv.symm (normalPartPoint T hT z w hz hw) =
        normalPartPoint S hS (U.symm w) (V.symm z)
          (by simpa only [map_zero] using U.symm.injective.ne hw)
          (by simpa only [map_zero] using V.symm.injective.ne hz)))
```

In the four equation rows below, $x, y \in H \setminus \{0\}$ and $z, w \in K \setminus \{0\}$ are arbitrary.

In the source, `S` and `T` denote the algebras A and B used here, respectively.

In the source	Mathematical meaning

Ambient spaces and algebras
`variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`
`variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]`
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))`
`variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]`
`variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
	The complex Hilbert spaces H, K and the nonzero norm-closed irreducible complex $*$-subalgebras A, B, denoted S, T in the source.

Given finite homeomorphism
`variable (e : ClosureSpace.FiniteHomeomorph`
`  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)))`
	The given bijection $e : N(A) \to N(B)$ and its actual inverse, with finite continuity for the relative closures c_A, c_B in both directions.

Finite homeomorphism
`structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where`
`  finitelyContinuous_toFun : c.FinitelyContinuous d toFun`
`  finitelyContinuous_invFun : d.FinitelyContinuous c invFun`
	One equivalence together with finite continuity of its forward map and its inverse. Here its underlying equivalence is e, and its inverse is e^{-1}; these are not newly chosen maps.

Dimension input `variable (hdim : 3 ≤ Module.rank ℂ H)`	The bound $3 \leq \operatorname{rank}_{\mathbb{C}} H$ in (1), allowing infinite dimension.

The selected lifts
`let U := normalPartFirstSemilinearEquiv S T hS hT e hdim`
`let V := normalPartSecondSemilinearEquiv S T hS hT e hdim`
	The maps U, V in (2) are chosen from this same e, the two algebras, their irreducibility, and the dimension bound. Their scalar automorphisms are σ_1, σ_2; no equality of these automorphisms is asserted.

Coefficient-kernel point
`def normalPartPoint (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`  (x y : H) (hx : x ≠ 0) (hy : y ≠ 0) : normalPart S :=`
`  ⟨restrictedMatrixCoefficientKernel S hS x y hx hy, ⟨x, y, hx, hy, rfl⟩⟩`
	The point $\ker f^A_{x,y} \in N(A)$ for $x, y \neq 0$; using B gives $\ker f^B_{z,w} \in N(B)$. The displayed nonzero conditions ensure that these are normal-part points.

In the source	Mathematical meaning

Direct forward equation
`((∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),`
`  e.toEquiv (normalPartPoint S hS x y hx hy) =`
`    normalPartPoint T hT (U x) (V y)`
`      (by simpa only [map_zero] using`
`U.injective.ne hx)`
`      (by simpa only [map_zero] using`
`V.injective.ne hy)) ∧`
	For every $x, y \in H \setminus \{0\}$, the first equation of (3):
	$$e(\ker f^A_{x,y}) = \ker f^B_{Ux,Vy}.$$

Direct inverse equation
`(∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),`
` e.toEquiv.symm (normalPartPoint T hT z w hz hw) =`
`   normalPartPoint S hS (U.symm z) (V.symm w)`
`     (by simpa only [map_zero] using`
`U.symm.injective.ne hz)`
`     (by simpa only [map_zero] using`
`V.symm.injective.ne hw))) ∨`
	For every $z, w \in K \setminus \{0\}$, the second equation of (3), for the inverse of the same e:
	$$e^{-1}(\ker f^B_{z,w}) = \ker f^A_{U^{-1}z,V^{-1}w}.$$
	The disjunction following this pair separates the two global orientations.

Swapped forward equation
`((∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),`
`  e.toEquiv (normalPartPoint S hS x y hx hy) =`
`    normalPartPoint T hT (V y) (U x)`
`      (by simpa only [map_zero] using`
`V.injective.ne hy)`
`      (by simpa only [map_zero] using`
`U.injective.ne hx)) ∧`
	For every $x, y \in H \setminus \{0\}$, the first equation of (4):
	$$e(\ker f^A_{x,y}) = \ker f^B_{Vy,Ux}.$$

Swapped inverse equation
`(∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),`
` e.toEquiv.symm (normalPartPoint T hT z w hz hw) =`
`   normalPartPoint S hS (U.symm w) (V.symm z)`
`     (by simpa only [map_zero] using`
`U.symm.injective.ne hw)`
`     (by simpa only [map_zero] using`
`V.symm.injective.ne hz))) := by`
	For every $z, w \in K \setminus \{0\}$, the second equation of (4):
	$$e^{-1}(\ker f^B_{z,w}) = \ker f^A_{U^{-1}w,V^{-1}z}.$$
	Each branch joins its two universally quantified equations; one orientation holds for the entire map.

Inverse scalar maps
`local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair`
`σ.toRingHom σ.symm.toRingHom :=`
`  RingHomInvPair.of_ringEquiv σ`
`local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair`
`σ.symm.toRingHom σ.toRingHom :=`
`  RingHomInvPair.of_ringEquiv_symm σ`
	The scalar maps and their actual inverses compose to the identity in either order. This does not assert $\sigma_i^2 = \mathrm{id}_{\mathbb{C}}$ or $\sigma_1 = \sigma_2$.

The short proof terms attached to transported vectors only certify that those vectors are nonzero, so that their coefficient kernels define the displayed normal-part points.

The declaration prefix includes both internal `let` assignments and all four universally quantified equations. It ends before the proof assignment. The surrounding context below is quoted separately.

Exact surrounding source context (separate excerpts)

Exact ambient excerpt:

```
local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair σ.toRingHom σ.symm.toRingHom :=
  RingHomInvPair.of_ringEquiv σ
local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair σ.symm.toRingHom σ.toRingHom :=
  RingHomInvPair.of_ringEquiv_symm σ
```

Exact ambient excerpt:

```
universe u v
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))
variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
S))
variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
T))
variable (e : ClosureSpace.FiniteHomeomorph
  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))
  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)))
variable (hdim : 3 ≤ Module.rank ℂ H)
```

Related exact source for the reading guide

Coefficient-kernel point · Fixed source

```
def normalPartPoint (hS : NonUnitalCStarRepresentation.IsIrreducible
(NonUnitalStarSubalgebraClass.subtype S))
    (x y : H) (hx : x ≠ 0) (hy : y ≠ 0) : normalPart S :=
  ⟨restrictedMatrixCoefficientKernel S hS x y hx hy, ⟨x, y, hx, hy, rfl⟩⟩
```

Swapped inverse equation · Fixed source

```
     (∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),
      e.toEquiv.symm (normalPartPoint T hT z w hz hw) =
        normalPartPoint S hS (U.symm w) (V.symm z)
          (by simpa only [map_zero] using U.symm.injective.ne hw)
          (by simpa only [map_zero] using V.symm.injective.ne hz))) := by
```

Finite homeomorphism · Fixed source

```
structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  finitelyContinuous_toFun : c.FinitelyContinuous d toFun
  finitelyContinuous_invFun : d.FinitelyContinuous c invFun
```

From components to their common kernel

`NormedSpace.geometricSpectrumToPrimitive`

def

From components to their common kernel

Statement

The natural map from the geometric spectrum to the primitive geometric spectrum sends a component to the intersection of all its geometric points. The map is surjective. With the separately defined spectrum closures, it preserves the closure of each selected set by an exact image formula.

Definition

Let E be a normed space over a real-or-complex scalar field $\mathbb{K}$, with closed unit ball $B_E = \{x \in E : \|x\| \le 1\}$ and continuous $\mathbb{K}$-linear dual E^*. The geometric point set $S_{\mathbb{K}}(E)$ consists of the subsets

$$I_{\mathbb{K}}(M) = \bigcup_{f \in \Sigma_{\mathbb{K}}(M)} \ker f,$$
$$\Sigma_{\mathbb{K}}(M) = \{f \in E^* : \|f\| \le 1,\ f|_M = 1\},$$

where M ranges over nonempty maximal proper real convex faces of B_E. A face is a convex subset containing both endpoints of every segment in the ball whose interior meets it; maximality is among proper convex faces. Equal resulting subsets are the same geometric point. Here $f|_M = 1$ means $f(x) = 1$ for every $x \in M$.

For $\mathscr{A} \subseteq S_{\mathbb{K}}(E)$, the geometric structure closure is

$$c_E(\mathscr{A}) = \left\{ I \in S_{\mathbb{K}}(E) : \bigcap_{J \in \mathscr{A}} J \subseteq I \right\}.$$

Two points I, J are linked if $I = J$ or $c_E(\{I, J\}) \neq \{I, J\}$. Let $C_E(I)$ be all points reachable from I by a finite chain of such linked pairs, including I itself. Define

$$\mathrm{GS}_{\mathbb{K}}(E) = \{C_E(I) : I \in S_{\mathbb{K}}(E)\},$$
$$J_C = \bigcap_{I \in C} I,$$
$$\mathrm{PS}_{\mathbb{K}}(E) = \{J_C : C \in \mathrm{GS}_{\mathbb{K}}(E)\}.$$

The **component-intersection map** is

$$\gamma_E : \mathrm{GS}_{\mathbb{K}}(E) \longrightarrow \mathrm{PS}_{\mathbb{K}}(E), \qquad \gamma_E(C) = J_C.$$

It takes a set of geometric points to a subset of E. It uses the component set itself, so changing its representative I does not change the output.

The cited surjectivity theorem has a direct witness argument. Given $J \in \mathrm{PS}_{\mathbb{K}}(E)$, choose a geometric point I_0 witnessing $J = J_{C_E(I_0)}$. Then $C = C_E(I_0)$ satisfies

$$\gamma_E(C) = J_{C_E(I_0)} = J.$$

This choice proves existence of a preimage. It makes no uniqueness assertion about the witnessing point or component.

Assumptions

The space E is normed over $\mathbb{K}$, without completeness or nontriviality hypotheses. All component kernels are intersections of subsets of E. The map has no additional input, and injectivity is not an assumption or a conclusion of this declaration.

Conclusion

The output is the specified map $\gamma_E(C) = J_C$; the separately cited theorem proves that it is surjective. Its role in the closure constructions is expressed by the following formulas. For selections $\mathscr{T} \subseteq \mathrm{PS}_{\mathbb{K}}(E)$ and $\mathscr{U} \subseteq \mathrm{GS}_{\mathbb{K}}(E)$, put

$$\begin{aligned} d_E(\mathscr{T}) &= \left\{ J \in \mathrm{PS}_{\mathbb{K}}(E) : \bigcap_{J' \in \mathscr{T}} J' \subseteq J \right\}, \\ g_E(\mathscr{U}) &= \gamma_E^{-1}\big(d_E(\gamma_E(\mathscr{U}))\big). \end{aligned}$$

Empty intersections are E. These separately defined operations are the primitive and geometric spectrum closures. The pullback formula makes γ_E closure-continuous, and surjectivity gives the stronger identity

$$\gamma_E(g_E(\mathscr{U})) = d_E(\gamma_E(\mathscr{U})).$$

Indeed, surjectivity gives $\gamma_E(\gamma_E^{-1}(\mathscr{T})) = \mathscr{T}$ for every subset of the codomain; apply this to $\mathscr{T} = d_E(\gamma_E(\mathscr{U}))$. These closure statements are supplied by the cited companion declarations. They do not turn γ_E into a homeomorphism without an additional injectivity result. For the zero space, both carriers are empty and the same definition applies.

Main citations

- The face-defined geometric points and closure: `NormedSpace.geometricStructureFamily`, `NormedSpace.GeometricStructureSpace` and `NormedSpace.geometricStructureClosure`, lines 37–65; the face condition, `NormedSpace.IsMaximalFace`, lines 32–39.
- Linked pairs and components: `ClosureSpace.Linked`, `ClosureSpace.Related` and `ClosureSpace.component`, lines 30–71.
- The two spectrum carriers and component kernel: `NormedSpace.GeometricSpectrum`, `NormedSpace.geometricSpectrumKernel` and `NormedSpace.PrimitiveGeometricSpectrum`, lines 36–54; the primitive range characterization, lines 93–101.
- Exact map, its value and surjectivity: `NormedSpace.geometricSpectrumToPrimitive`, lines 68–74; `NormedSpace.geometricSpectrumToPrimitive_surjective`, lines 88–90. Surjectivity is supplied by this separate theorem.
- The witness used by the generic construction: `ClosureSpace.HullKernel.componentPointToComponentKernelPoint` and its surjectivity theorem, lines 80–95.
- The closures and their exact image formula: `NormedSpace.primitiveGeometricSpectrumClosure`, `NormedSpace.geometricSpectrumClosure`, `NormedSpace.continuous_geometricSpectrumToPrimitive` and `NormedSpace.image_geometricSpectrumToPrimitive_closure`, lines 109–157; the generic image-of-pullback identity is here, `ClosureSpace.image_induced_closure`, lines 74–80.
- Ryotaro Tanaka, arXiv:2504.05551v1, Definition 4.6 and Theorem 4.7: the map γ and the two closure formulas. The formal source allows the weaker normed-space assumptions stated above.

Lean source signature (exact)

The scalar and normed-space declarations used in this construction are reproduced separately below.

```
def geometricSpectrumToPrimitive : GeometricSpectrum 𝕜 E → PrimitiveGeometricSpectrum 𝕜 E :=
  ClosureSpace.HullKernel.componentPointToComponentKernelPoint
    (geometricStructureFamily 𝕜 E) (univ_notMem_geometricStructureFamily 𝕜 E)
```

In the source	Mathematical meaning
Exact declaration `𝕜, [RCLike 𝕜]`	The real or complex scalar field $\mathbb{K}$.
Exact declaration `E, [NormedAddCommGroup E], [NormedSpace 𝕜 E]`	A normed vector space E over $\mathbb{K}$, possibly zero and not necessarily complete.
Exact declaration `GeometricSpectrum 𝕜 E → PrimitiveGeometricSpectrum 𝕜 E`	The entire output type is a map $\gamma_E : \mathrm{GS}_{\mathbb{K}}(E) \to \mathrm{PS}_{\mathbb{K}}(E)$, from component subsets to their intersection subsets.

In the source	Mathematical meaning
Exact declaration `ClosureSpace.HullKernel.componentPointToComponentKernelPoint`	For every component C, the defining value is $\gamma_E(C) = J_C = \bigcap_{I \in C} I$. The component membership supplies a witness that this intersection lies in the target family.
Exact declaration `geometricStructureFamily 𝕜 E`	The family $\mathcal{F}_{\mathbb{K}}(E)$ of geometric points on which the generic component-intersection construction is applied.
Exact declaration `univ_notMem_geometricStructureFamily 𝕜 E`	The already proved properness $E \notin \mathcal{F}_{\mathbb{K}}(E)$ is the constructor input; it is not an extra assumption on E.
The complete RHS **Exact declaration**	These arguments specify the map $C \mapsto J_C$ on all components. Its cited surjectivity theorem gives $\forall J \in \mathrm{PS}_{\mathbb{K}}(E),\ \exists C,\ \gamma_E(C) = J$; neither a unique preimage nor closure continuity is a field of this function definition.

Source context (separate exact excerpt)

The surrounding declarations below are from lines 32–33; their mathematical meaning is included in the guide above.

```
variable (𝕜 : Type v) [RCLike 𝕜]
variable (E : Type u) [NormedAddCommGroup E] [NormedSpace 𝕜 E]
```

Exact declaration for the source guide

```
def geometricSpectrumToPrimitive : GeometricSpectrum 𝕜 E → PrimitiveGeometricSpectrum 𝕜 E :=
  ClosureSpace.HullKernel.componentPointToComponentKernelPoint
    (geometricStructureFamily 𝕜 E) (univ_notMem_geometricStructureFamily 𝕜 E)

@[simp]
```

Faithful normal-part transport through a specified GSS homeomorphism

`MathlibAnnex.Analysis.CStarAlgebra.exists_faithful_normalPart_transport`

theorem

Transports the normal part through one faithful pure-GNS representation and the same preimage map.

Statement

Let H be a complex Hilbert space, and let $A \subseteq \mathcal{B}(H)$ be a nonzero norm-closed complex $*$-subalgebra acting irreducibly on H. Let B be a complex C*-algebra with its compatible order. Write $S(D)$ for the geometric structure space of an algebra D, with hull–kernel closure c_D. For each maximal proper face of the closed unit ball of D, take the union of the kernels of all bounded complex-linear functionals of norm one that equal 1 on that face. The distinct subsets so obtained are the points of $S(D)$, by the geometric structure definition and the norm-one support characterization.

For a concrete algebra $D \subseteq \mathcal{B}(L)$, define

$$f^D_{x,y} : D \to \mathbb{C}, \qquad f^D_{x,y}(a) = \langle y, a(x) \rangle, \qquad x, y \in L \setminus \{0\},$$

and

$$N(D) = \{I \in S(D) : I = \ker f^D_{x,y} \text{ for some } x, y \in L \setminus \{0\}\}.$$

The inner product is linear in the second variable. Suppose that $e : S(A) \to S(B)$ is a homeomorphism of closure spaces, so that

$$e(c_A(E)) = c_B(e(E)) \qquad (E \subseteq S(A)).$$

Then there are a pure state φ of B and a homeomorphism

$$r : S(\pi_\varphi(B)) \longrightarrow S(B),$$

where $\pi_\varphi : B \to \mathcal{B}(H_\varphi)$ is the GNS representation of this same state, with the following properties:

- π_φ is faithful.
- For its range restriction $\pi_{\varphi,r} : B \to \pi_\varphi(B)$,

$$r(J) = \pi^{-1}_{\varphi,r}(J) \qquad (J \in S(\pi_\varphi(B))). \tag{1}$$

- The inverse of that same homeomorphism satisfies

$$\pi_{\varphi,r}^{-1}(r^{-1}(I)) = I \qquad (I \in S(B)). \qquad (2)$$

- The normal parts satisfy

$$(r^{-1} \circ e)(N(A)) = N(\pi_\varphi(B)). \qquad (3)$$

Assumptions

The Hilbert spaces are complete. A pure state is a positive bounded complex-linear functional of norm one that is extreme in the state space for real convexity. Closedness and irreducibility concern the given action of A on H. The homeomorphism e is specified, with continuity for all subsets in both directions. Nonzero H and B follow from these hypotheses; no extra nontriviality assumption on B is imposed.

Conclusion

One faithful pure-GNS representation and one inverse-image homeomorphism satisfy all three formulas. In (3), the same map e is followed by r^{-1}. The conclusion identifies the normal parts under this composite; it does not identify a normal part with its entire geometric structure space.

Main citations

Faithful normal-part transport gives all the outputs for the same pure state and the same homeomorphism. The component-density criterion characterizes faithfulness of the pure-GNS representation. The forward preimage formula and inverse preimage formula concern the same range restriction; the normal-part image identity specifies which subset is transported.

Lean source signature (exact)

Exact source and declaration. Only the literal signature is displayed here; the linked source retains the proof.

```
theorem exists_faithful_normalPart_transport
    (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) [IsClosed (S : Set (H →L[ℂ] H))]
    (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))
    (e : ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ S)
      (NormedSpace.geometricStructureClosure ℂ B)) :
    ∃ phi : NonUnitalPureState B, Function.Injective phi.gnsRepresentation ∧
      ∃ r : ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ
(NonUnitalStarAlgHom.range phi.gnsRepresentation))
          (NormedSpace.geometricStructureClosure ℂ B),
        (∀ J, (r.toEquiv J).val = (NonUnitalStarAlgHom.rangeRestrict phi.gnsRepresentation) ⁻¹'
J.val) ∧
        (∀ I, (NonUnitalStarAlgHom.rangeRestrict phi.gnsRepresentation) ⁻¹' (r.toEquiv.symm
I).val = I.val) ∧
        (e.trans r.symm).toEquiv '' normalPart S = normalPart (NonUnitalStarAlgHom.range
phi.gnsRepresentation)
```

In the source, `S` denotes the algebra A used here.

In the source	Mathematical meaning
Original Hilbert space `variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`	The original representation space H is a complete complex Hilbert space. It is independent of the GNS space of the state found in the conclusion.
Target algebra `variable {B : Type w} [NonUnitalCStarAlgebra B] variable [PartialOrder B] [StarOrderedRing B]`	The complex C*-algebra B has its compatible order. No separate nontriviality assumption on B appears here.
Concrete irreducible action `(S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) [IsClosed (S : Set (H →L[ℂ] H))] (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`	The source name S is the closed complex $*$-subalgebra $A \subseteq \mathcal{B}(H)$. Its inclusion action is nonzero and irreducible.
Given homeomorphism `(e : ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ S) (NormedSpace.geometricStructureClosure ℂ B)) :`	The input is the specified closure-space homeomorphism $e : S(A) \to S(B)$. Its domain and codomain use the ambient geometric closures, not closures restricted to normal parts.
One faithful pure-state representation `∃ phi : NonUnitalPureState B, Function.Injective phi.gnsRepresentation ∧`	There exists one pure state φ of B whose GNS representation π_φ is injective, hence faithful. This same state and representation occur in every subsequent condition.
Homeomorphism from the representation image `∃ r : ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ (NonUnitalStarAlgHom.range phi.gnsRepresentation)) (NormedSpace.geometricStructureClosure ℂ B),`	There exists one homeomorphism $r : S(\pi_\varphi(B)) \to S(B)$. The range is the actual image algebra of this same representation, not all bounded operators on its GNS space.
Forward preimage formula `(∀ J, (r.toEquiv J).val = (NonUnitalStarAlgHom.rangeRestrict phi.gnsRepresentation) ⁻¹' J.val) ∧`	The underlying bijection of r sends every geometric point J of the image algebra to $$r(J) = \pi_{\varphi,r}^{-1}(J).$$ Here $\pi_{\varphi,r} : B \to \pi_\varphi(B)$ is the range restriction. The projections expose the underlying subsets.
Inverse preimage formula `(∀ I, (NonUnitalStarAlgHom.rangeRestrict phi.gnsRepresentation) ⁻¹' (r.toEquiv.symm I).val = I.val) ∧`	The inverse of that same bijection satisfies $$\pi_{\varphi,r}^{-1}(r^{-1}(I)) = I \qquad (I \in S(B)).$$ The inverse is not chosen independently.
Transport of normal parts `(e.trans r.symm).toEquiv '' normalPart S = normalPart (NonUnitalStarAlgHom.range phi.gnsRepresentation) := by`	Transitivity composes e first and r^{-1} second. Taking its underlying bijection and its set image gives $$(r^{-1} \circ e)(N(A)) = N(\pi_\varphi(B)).$$ This identifies the normal parts, not the whole geometric structure spaces.

In the source	Mathematical meaning

Same bijection in both continuity fields
`structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where`
`  continuous_toFun : c.Continuous d toFun`
`  continuous_invFun : d.Continuous c invFun`
	For a homeomorphism $q : (X, c_X) \to (Y, c_Y)$, the underlying equivalence supplies q and its inverse. The fields assert
	$$q(c_X(E)) \subseteq c_Y(q(E)), \qquad E \subseteq X,$$
	$$q^{-1}(c_Y(F)) \subseteq c_X(q^{-1}(F)), \qquad F \subseteq Y.$$
	This applies to both e and r with their displayed domains.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
universe u v w
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
variable [PartialOrder A] [StarOrderedRing A]
variable {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]

attribute [local instance] HomKernelQuotient.imageCStarAlgebra HomKernelQuotient.isClosed_image
```

Exact surrounding source:

```
variable {B : Type w} [NonUnitalCStarAlgebra B]
variable [PartialOrder B] [StarOrderedRing B]
```

Related exact source for the reading guide

Transport of normal parts · Fixed source

```
        (e.trans r.symm).toEquiv '' normalPart S = normalPart (NonUnitalStarAlgHom.range
phi.gnsRepresentation) := by
```

Same bijection in both continuity fields · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```

The intersection of a pure-state component is its GNS representation kernel

`MathlibAnnex.Analysis.CStarAlgebra.geometricSpectrumToPrimitive_toGeometricSpectrum_pureStateKernel_val`

theorem

Identifies a pure-state component intersection with the kernel of its GNS representation.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity. A state on A is a positive bounded complex-linear functional of norm one. It is pure if it is an extreme point of the state space, regarded as a real convex set. Let φ be a pure state. Let $(H_\varphi, \widetilde{\pi}_\varphi, \xi_\varphi)$ be the GNS model of its canonical state extension $\widetilde{\varphi}(\lambda, a) = \lambda + \varphi(a)$ to the forced unitization $\mathbb{C} \oplus A$, and let $\pi_\varphi = \widetilde{\pi}_\varphi|_A$.

By the extreme-kernel theorem, $S(A)$ is the family of kernels $\ker \rho \subseteq A$ of bounded complex-linear functionals $\rho : A \to \mathbb{C}$ that are extreme points, for real convexity, of the entire closed dual unit ball B_{A^*}, with closure $c_A(\mathcal{T}) = \{I \in S(A) : \bigcap_{J \in \mathcal{T}} J \subseteq I\}$. The pure-state kernel $\ker \varphi$ is one such point, by the pure-state extremality theorem. Let $C(\ker \varphi)$ denote its component for the finite-chain relation generated by equal points and nonclosed pairs. Write $\mathrm{GS}(A)$ for the set of these components. For a component C, put $J_C = \bigcap_{I \in C} I$, and write $\mathrm{PS}(A)$ for the family of all such intersections. The canonical map is

$$\gamma : \mathrm{GS}(A) \longrightarrow \mathrm{PS}(A), \qquad \gamma(C) = J_C.$$

Then, as subsets of the original algebra,

$$\gamma(C(\ker \varphi)) = J_{C(\ker \varphi)} = \ker \pi_\varphi. \qquad (1)$$

On the left we display the underlying subset of the primitive geometric point.

Assumptions

The given functional $\varphi \in A^*$ is positive, has norm one, and is extreme in the state space for real convexity; A carries its compatible C*-order. The GNS action is formed from this very state's canonical extension. The representation is fixed by φ; faithfulness is not assumed.

Conclusion

The whole component's intersection is the representation kernel $\ker \pi_\varphi$, using this fixed GNS representation. An individual functional kernel $\ker \varphi$ need not equal that intersection. Formula (1) is an equality of subsets; it gives neither injectivity of γ nor a comparison of topologies by itself.

Main citations

- Exact pure-state gamma identity: `geometricSpectrumToPrimitive_toGeometricSpectrum_pureStateKernel_val`.
- Component intersections, their family and the component-intersection map: the definitions of J_C, $\mathrm{PS}(A)$ and γ.
- Ryotaro Tanaka, arXiv:2504.05551v1, Lemma 5.10: the component-family and representation-kernel comparison.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem geometricSpectrumToPrimitive_toGeometricSpectrum_pureStateKernel_val
    (phi : StrongDual ℂ A) (hpure : IsPureNonUnitalState A phi) :
    (NormedSpace.geometricSpectrumToPrimitive ℂ A
      (NormedSpace.toGeometricSpectrum ℂ A
        (extremeKernel ⟨phi, mem_extremePoints_dual_closedBall_of_isPureNonUnitalState phi
hpure⟩))).val =
      ((absoluteStateContinuousMap (NonUnitalStateGNS.representation phi hpure.1)).ker : Set A)
```

In the source	Mathematical meaning

Algebra and order
`variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]`
`variable [PartialOrder A] [StarOrderedRing A]`
	The nonzero complex C*-algebra A and its compatible order are fixed.

Given pure state `(phi : StrongDual ℂ A) (hpure : IsPureNonUnitalState A phi) :`	The given φ is a positive bounded complex-linear functional of norm one, extreme in the state space for real convexity. This same state determines the GNS model.

Kernel, component and its intersection
`(NormedSpace.geometricSpectrumToPrimitive ℂ A`
`  (NormedSpace.toGeometricSpectrum ℂ A`
`    (extremeKernel ⟨phi, mem_extremePoints_dual_closedBall_of_isPureNonUnitalState phi hpure⟩))).val =`
	The innermost input is the geometric point $\ker\varphi$, with its extremality certificate. Sending it to its component gives $C(\ker\varphi) \in \mathrm{GS}(A)$. Applying γ and taking the underlying subset gives
	$$\gamma(C(\ker\varphi)) = J_{C(\ker\varphi)} \subseteq A.$$
	The nested applications act on this same point in that order.

The component map
`def toGeometricSpectrum (I : GeometricStructureSpace 𝕜 E) : GeometricSpectrum 𝕜 E :=`
`  (geometricStructureClosure 𝕜 E).toComponentPoint I`
	This map sends a point $I \in S(A)$ to its entire finite-chain component. In the preceding application, $I = \ker\varphi$.

In the source	Mathematical meaning

The intersection map
`def geometricSpectrumToPrimitive : GeometricSpectrum 𝕜 E → PrimitiveGeometricSpectrum 𝕜 E :=`
 	
`ClosureSpace.HullKernel.componentPointToComponentKernelPoint (geometricStructureFamily 𝕜 E) (univ_notMem_geometricStructureFamily 𝕜 E)`
	This is $\gamma : \mathrm{GS}(A) \to \mathrm{PS}(A)$, sending a component C to $J_C = \bigcap_{I \in C} I$. Its value is a subset of the algebra, rather than a subset of $S(A)$.

Kernel of the same GNS representation `((absoluteStateContinuousMap (NonUnitalStateGNS.representation phi hpure.1)).ker : Set A) := by`	The right side is $\ker \pi_\varphi \subseteq A$. The GNS action is formed from the canonical extension of this same state, and the continuous-linear-map wrapper changes no values. Thus (1) identifies the whole component intersection with the representation kernel, not with the individual functional kernel.
Operator-valued linear map `private def absoluteStateContinuousMap {H : Type v} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H] (pi : NonUnitalCStarRepresentation A H) : A →L[ℂ] (H →L[ℂ] H) := LinearMap.mkContinuous { toFun := pi, map_add' := map_add pi, map_smul' := map_smul pi }`	The wrapper has domain A and codomain $\mathcal{B}(H_\varphi)$. Its function is the given representation π_φ, with the same addition and complex-scalar action. It supplies the operator-valued map whose kernel appears on the right of (1).

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
```

Exact surrounding source:

```
variable [PartialOrder A] [StarOrderedRing A]
```

The continuous-map wrapper in the same source is copied separately; its forward function is the given action; the continuous-map constructor receives the existing contractive bound:

```
private def absoluteStateContinuousMap {H : Type v}
    [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
    (pi : NonUnitalCStarRepresentation A H) : A →L[ℂ] (H →L[ℂ] H) :=
  LinearMap.mkContinuous
    { toFun := pi, map_add' := map_add pi, map_smul' := map_smul pi }
    1 (fun a => by
      change ‖pi a‖ ≤ 1 * ‖a‖
      simpa only [one_mul] using NonUnitalStarAlgHom.norm_apply_le pi a)
```

Related exact source for the reading guide

Algebra and order · Fixed source

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
variable [PartialOrder A] [StarOrderedRing A]
```

Kernel of the same GNS representation · Fixed source

```
      ((absoluteStateContinuousMap (NonUnitalStateGNS.representation phi hpure.1)).ker : Set A)
:= by
```

The component map · Fixed source

```
def toGeometricSpectrum (I : GeometricStructureSpace 𝕜 E) : GeometricSpectrum 𝕜 E :=
  (geometricStructureClosure 𝕜 E).toComponentPoint I
```

The intersection map · Fixed source

```
def geometricSpectrumToPrimitive : GeometricSpectrum 𝕜 E → PrimitiveGeometricSpectrum 𝕜 E :=
  ClosureSpace.HullKernel.componentPointToComponentKernelPoint
    (geometricStructureFamily 𝕜 E) (univ_notMem_geometricStructureFamily 𝕜 E)
```

Forward continuity of normal parts preserves Hilbert-basis cardinality

`MathlibAnnex.Analysis.CStarAlgebra.hilbertBasis_lift_mk_eq_of_normalPart_continuous`

theorem

Preserves Hilbert-basis cardinality under forward full continuity of a finite normal-part homeomorphism.

Statement

Let H, K be complex Hilbert spaces. Let $A_1 \subseteq \mathcal{B}(H)$ and $A_2 \subseteq \mathcal{B}(K)$ be nonzero norm-closed complex $*$-subalgebras acting irreducibly on H and K, respectively. No unit is required. The inner product is linear in its second variable. For $x, y \in H \setminus \{0\}$ and $z, w \in K \setminus \{0\}$, define the bounded linear functionals $f^{A_1}_{x,y} : A_1 \to \mathbb{C}$ and $f^{A_2}_{z,w} : A_2 \to \mathbb{C}$ by $f^{A_1}_{x,y}(a) = \langle y, a(x)\rangle$ and $f^{A_2}_{z,w}(b) = \langle w, b(z)\rangle$. Their kernels form the normal parts

$$N(A_1) = \{\ker f^{A_1}_{x,y} : x, y \in H \setminus \{0\}\}, \qquad N(A_2) = \{\ker f^{A_2}_{z,w} : z, w \in K \setminus \{0\}\}. \tag{1}$$

Write c_{A_1} and c_{A_2} for the relative geometric closures on these normal parts. Thus, for $\mathcal{D} \subseteq N(A_1)$, $c_{A_1}(\mathcal{D}) = \{I \in N(A_1) : \bigcap_{J \in \mathcal{D}} J \subseteq I\}$, and similarly for A_2. Let $e : N(A_1) \to N(A_2)$ be a bijection satisfying $e(c_{A_1}(\mathcal{D})) = c_{A_2}(e(\mathcal{D}))$ for every finite $\mathcal{D} \subseteq N(A_1)$. This is the finite-homeomorphism condition. Assume in addition that its forward map is fully continuous:

$$e(c_{A_1}(\mathcal{D})) \subseteq c_{A_2}(e(\mathcal{D})) \qquad (\mathcal{D} \subseteq N(A_1)). \tag{2}$$

For any Hilbert bases $b = (b_i)_{i \in \iota}$ of H and $c = (c_j)_{j \in \kappa}$ of K, their index cardinalities satisfy

$$|\iota| = |\kappa|. \tag{3}$$

The invariant is Hilbert-basis index cardinality, not the cardinality of the vector spaces as sets and not their Hamel dimension.

Assumptions

The bases are arbitrary complete orthonormal families. They need not be countable, and neither space is assumed separable. No infinite-dimensionality, same-type condition on two projective factors, or full continuity of e^{-1} is assumed. Dimensions one and two are included through the finite-dimensional branch. The inclusion representations are nonzero and irreducible, and the algebras are norm closed; unitality is not added.

Conclusion

Forward full continuity preserves Hilbert-basis index cardinality, without assuming full continuity of the inverse. The conclusion neither chooses a basis-to-basis map nor asserts an isometry.

Main citations

The normal-part definition and coefficient-kernel membership theorem identify (1) under the stated closedness and irreducibility assumptions.

The Hilbert-dimension theorem gives (3) from forward full continuity. The related results are the finite-dimensional comparison, the continuous first lift and its inverse, and Hilbert-basis comparison.

Lean source signature (exact)

The preceding `variable` declarations introduce the two Hilbert spaces, the closed algebras, their irreducible actions, and `e`. The `include hS hT in` at the effective inclusion retains the irreducibility assumptions for this theorem. The map `e` is also a parameter because it occurs in `hcont`; `e.toEquiv` is the same bijection with its finite-continuity fields omitted. The following contiguous prefix ends before the proof assignment.

```
theorem hilbertBasis_lift_mk_eq_of_normalPart_continuous
    (hcont : ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Continuous
      ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)) e.toEquiv)
    (b : HilbertBasis ι ℂ H) (c : HilbertBasis κ ℂ K) :
    Cardinal.lift.{uJ} (#ι) = Cardinal.lift.{uI} (#κ)
```

In the source, `S` and `T` denote the algebras A_1 and A_2 used here, respectively.

In the source	Mathematical meaning

Two Hilbert spaces and basis indices
`universe u v uI uJ`
`variable {H : Type u} [NormedAddCommGroup H]`
`[InnerProductSpace ℂ H] [CompleteSpace H]`
`variable {K : Type v} [NormedAddCommGroup K]`
`[InnerProductSpace ℂ K] [CompleteSpace K]`
`variable {ι : Type uI} {κ : Type uJ}`
	The spaces H, K are complex Hilbert spaces, including completeness. The sets ι, κ index their Hilbert bases.

In the source	Mathematical meaning

Closed irreducible operator algebras
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))`
`variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]`
`variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
	The algebras A_1, A_2 are norm-closed complex $*$-subalgebras of $\mathcal{B}(H), \mathcal{B}(K)$. Their inclusion representations are nonzero and irreducible. No unit is required.

The same finite comparison of normal parts
`variable (e : ClosureSpace.FiniteHomeomorph`
`  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)))`
	The relative closures c_{A_1}, c_{A_2} are those on $N(A_1), N(A_2)$, rather than on the whole geometric structure spaces. One bijection $e : N(A_1) \to N(A_2)$ satisfies, for every finite $\mathcal{D} \subseteq N(A_1)$,
	$$e(c_{A_1}(\mathcal{D})) = c_{A_2}(e(\mathcal{D})).$$

Effective irreducibility assumptions `include hS hT in`	The preceding inclusion retains both irreducibility assumptions as theorem parameters. The already declared e is also a parameter because it occurs in the following continuity assumption.

Forward continuity for all subsets
`(hcont : ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Continuous`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)) e.toEquiv)`
	The same bijection e is fully continuous in the forward direction: for every $\mathcal{D} \subseteq N(A_1)$,
	$$e(c_{A_1}(\mathcal{D})) \subseteq c_{A_2}(e(\mathcal{D})).$$
	This is (2); full continuity of e^{-1} is not an input.

Arbitrary Hilbert bases `(b : HilbertBasis ι ℂ H) (c : HilbertBasis κ ℂ K) :`	The bases are $b = (b_i)_{i \in \iota}$ of H and $c = (c_j)_{j \in \kappa}$ of K: arbitrary complete orthonormal families.
Equality of basis-index cardinalities `Cardinal.lift.{uJ} (#ι) = Cardinal.lift.{uI} (#κ) := by`	The two basis-index sets have the same cardinality: $$\lvert\iota\rvert = \lvert\kappa\rvert,$$ as in (3). The lifts allow Lean to compare cardinalities belonging to different universes. This is Hilbert-basis cardinality, not Hamel dimension or the cardinality of the vector spaces as sets.

In the source	Mathematical meaning

Finite continuity of the map and its inverse
`structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where`
`  finitelyContinuous_toFun : c.FinitelyContinuous d toFun`
`  finitelyContinuous_invFun : d.FinitelyContinuous c invFun`
	The inherited equivalence gives e and e^{-1}. Its fields require finite continuity in both directions: for finite $\mathcal{D} \subseteq N(A_1)$ and finite $\mathcal{E} \subseteq N(A_2)$, respectively,
	$$e(c_{A_1}(\mathcal{D})) \subseteq c_{A_2}(e(\mathcal{D})),$$
	$$e^{-1}(c_{A_2}(\mathcal{E})) \subseteq c_{A_1}(e^{-1}(\mathcal{E})).$$
	This is the finite-homeomorphism condition used in the Statement.

Exact surrounding source context (separate excerpts)

NormalPartHilbertDimension.lean:

```
universe u v uI uJ
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
variable {ι : Type uI} {κ : Type uJ}
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))
variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
S))
variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
T))
variable (e : ClosureSpace.FiniteHomeomorph
  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))
  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)))
```

NormalPartHilbertDimension.lean:

```
include hS hT in
```

Related exact source for the reading guide

Equality of basis-index cardinalities · Fixed source

```
    Cardinal.lift.{uJ} (#ι) = Cardinal.lift.{uI} (#κ) := by
```

Finite continuity of the map and its inverse · Fixed source

```
structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  finitelyContinuous_toFun : c.FinitelyContinuous d toFun
  finitelyContinuous_invFun : d.FinitelyContinuous c invFun
```

The semilinear factors and their inverses are continuous

`MathlibAnnex.Analysis.CStarAlgebra.normalPart_semilinear_factors_continuous`

theorem

Establishes continuity of the fixed semilinear factors and their inverses.

Statement

Let H, K be complex Hilbert spaces, and let $A \subseteq \mathcal{B}(H)$ and $B \subseteq \mathcal{B}(K)$ be nonzero norm-closed complex $*$-subalgebras acting irreducibly on H and K, respectively. Neither algebra is assumed unital. For $x, y \in H \setminus \{0\}$ and $z, w \in K \setminus \{0\}$, define the bounded complex-linear functionals

$$\begin{aligned} f^A_{x,y} &: A \to \mathbb{C}, \qquad f^A_{x,y}(a) = \langle y, a(x) \rangle, \\ f^B_{z,w} &: B \to \mathbb{C}, \qquad f^B_{z,w}(b) = \langle w, b(z) \rangle. \end{aligned}$$

Their normal parts are

$$\begin{aligned} N(A) &= \{\ker f^A_{x,y} : x, y \in H \setminus \{0\}\}, \\ N(B) &= \{\ker f^B_{z,w} : z, w \in K \setminus \{0\}\}. \end{aligned}$$

For $\mathcal{D} \subseteq N(A)$, give the normal part the relative geometric closure

$$c_A(\mathcal{D}) = \{J \in N(A) : \bigcap_{I \in \mathcal{D}} I \subseteq J\}, \tag{1}$$

and similarly c_B. Let $e : N(A) \to N(B)$ be a specified finite-homeomorphism, that is, a bijection satisfying $e(c_A(\mathcal{D})) = c_B(e(\mathcal{D}))$ for every finite $\mathcal{D} \subseteq N(A)$. Assume that H is infinite-dimensional and that

$$e\big(c_A(\mathcal{D})\big) \subseteq c_B\big(e(\mathcal{D})\big) \qquad \text{for every } \mathcal{D} \subseteq N(A). \tag{2}$$

Let $U, V : H \to K$ be the semilinear equivalences fixed in the algebraic realization of e. Then all four functions

$$U : H \to K, \quad U^{-1} : K \to H, \quad V : H \to K, \quad V^{-1} : K \to H \tag{3}$$

are norm continuous. The inverses in (3) are those of the same maps U, V.

Assumptions

The two nonzero concrete $*$-algebras are norm closed and act irreducibly on their respective Hilbert spaces. They need not be unital. The inner product is conjugate-linear in its first variable. The closure on each normal part is the intersection closure displayed above.

For their scalar field automorphisms σ_1, σ_2 of $\mathbb{C}$, the semilinear maps satisfy $U(cx) = \sigma_1(c)U(x)$ and $V(cx) = \sigma_2(c)V(x)$. They are the lifts fixed in the algebraic realization of the given e, with one global direct or swapped orientation. The rank theorem supplies the bound $3 \leq \dim_{\mathbb{C}}^{\mathrm{alg}} H$ required for this realization.

Full continuity is required only in the forward direction for e. Neither full continuity of e^{-1} nor initial continuity of the two semilinear maps is assumed.

Conclusion

The two fixed semilinear equivalences are homeomorphisms for the norm topologies. The conclusion states continuity of U, U^{-1}, V, V^{-1} in this order. It neither replaces the chosen factors nor asserts full continuity of e^{-1}.

Main citations

The four-continuity theorem concerns the factors fixed in the algebraic realization. The first lift and second lift specify the maps. The scalar-type theorem gives the separate scalar alternatives.

Lean source signature (exact)

The preceding `variable` declarations introduce the spaces, algebras, irreducibility hypotheses, the finite-homeomorphism `e`, and `hdim`. They occur in the conclusion through the selected factors. The earlier `include hS hT e` remains in scope, and `include hH hcont` additionally retains infinite-dimensionality and forward full continuity as theorem parameters. Thus the absence of an argument list on the next line does not make this an unconditional theorem. These declarations and commands are quoted separately below.

The following contiguous declaration prefix ends before its proof assignment.

```
theorem normalPart_semilinear_factors_continuous :
    Continuous (normalPartFirstSemilinearEquiv S T hS hT e hdim) ∧
    Continuous (normalPartFirstSemilinearEquiv S T hS hT e hdim).symm ∧
    Continuous (normalPartSecondSemilinearEquiv S T hS hT e hdim) ∧
    Continuous (normalPartSecondSemilinearEquiv S T hS hT e hdim).symm
```

In the source, `S` and `T` denote the algebras A and B used here, respectively.

In the source	Mathematical meaning

Hilbert spaces and irreducible algebras
`variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`
`variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]`
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))`
`variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]`
`variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
	The complex Hilbert spaces H, K carry the nonzero norm-closed irreducible algebras $A \subseteq \mathcal{B}(H)$ and $B \subseteq \mathcal{B}(K)$, called `S`,`T`. The inclusion representations in the last two lines express these irreducible actions.

Finite closure comparison
`variable (e : ClosureSpace.FiniteHomeomorph`
`  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)))`
	The same bijection $e : N(A) \to N(B)$ compares the relative closures. For every finite $\mathcal{D} \subseteq N(A)$,
	$$e(c_A(\mathcal{D})) = c_B(e(\mathcal{D})).$$
	Its underlying equivalence is obtained by `toEquiv`; no new map is selected by that projection.

Included geometric inputs `include hS hT e`	The two irreducibility hypotheses and e are retained as parameters of the later theorem. They also occur as arguments of the selected scalar maps and semilinear maps.

Dimension and forward continuity
`variable (hdim : 3 ≤ Module.rank ℂ H)`
`variable (hH : ¬ FiniteDimensional ℂ H)`
`variable (hcont : ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Continuous`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)) e.toEquiv)`
 	
`include hH hcont`
	The bound $3 \leq \dim_{\mathbb{C}}^{\mathrm{alg}} H$ specifies the selected lifts. Under the present hypotheses it is guaranteed by the rank theorem cited above. The additional hypotheses are infinite-dimensionality of H and, for every $\mathcal{D} \subseteq N(A)$,
	$$e(c_A(\mathcal{D})) \subseteq c_B(e(\mathcal{D})).$$
	The final `include` retains these hypotheses. Full continuity of e^{-1} is not assumed.

Inverse scalar structures
`local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair σ.toRingHom σ.symm.toRingHom :=`
`  RingHomInvPair.of_ringEquiv σ`
`local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair σ.symm.toRingHom σ.toRingHom :=`
`  RingHomInvPair.of_ringEquiv_symm σ`
	For a scalar automorphism σ, these instances pair σ with its inverse σ^{-1} in both orders. They make the inverse of a σ-semilinear equivalence a σ^{-1}-semilinear map; they do not impose $\sigma^2 = \mathrm{id}$.

In the source	Mathematical meaning
First selected semilinear equivalence `def normalPartFirstSemilinearEquiv : LinearEquiv (normalPartFirstScalarEquiv S T hS hT e hdim).toRingHom (σ' := (normalPartFirstScalarEquiv S T hS hT e hdim).symm.toRingHom) H K := Classical.choose (Classical.choose_spec (exists_normalPart_first_semilinearEquiv S T hS hT e hdim))`	This is the fixed additive bijection $U : H \to K$ with $U(cx) = \sigma_1(c)U(x)$. Its type specifies the inverse scalar map as σ_1^{-1}, so `symm` below is the inverse $U^{-1} : K \to H$.
Second selected semilinear equivalence `def normalPartSecondSemilinearEquiv : LinearEquiv (normalPartSecondScalarEquiv S T hS hT e hdim).toRingHom (σ' := (normalPartSecondScalarEquiv S T hS hT e hdim).symm.toRingHom) H K := Classical.choose (Classical.choose_spec (exists_normalPart_second_semilinearEquiv S T hS hT e hdim))`	This is the fixed additive bijection $V : H \to K$ with $V(cx) = \sigma_2(c)V(x)$, from the same geometric inputs. Its inverse is $V^{-1} : K \to H$, semilinear for σ_2^{-1}.
Continuity of U `Continuous (normalPartFirstSemilinearEquiv S T hS hT e hdim) ∧`	$U : H \to K$ is continuous for the norm topologies, the first assertion in (3).
Continuity of the inverse of U `Continuous (normalPartFirstSemilinearEquiv S T hS hT e hdim).symm ∧`	$U^{-1} : K \to H$ is norm continuous, the second assertion in (3). The suffix `symm` takes the inverse of the same selected U.
Continuity of V `Continuous (normalPartSecondSemilinearEquiv S T hS hT e hdim) ∧`	$V : H \to K$ is norm continuous, the third assertion in (3).
Continuity of the inverse of V `Continuous (normalPartSecondSemilinearEquiv S T hS hT e hdim).symm :=`	$V^{-1} : K \to H$ is norm continuous, the fourth assertion in (3). All four assertions are joined by conjunctions; this is not an assertion of full closure continuity of e^{-1}.

Exact surrounding source context (separate excerpts)

These separate excerpts record inverse scalar instances, the ambient spaces and algebras, and the included hypotheses. They are not spliced into the declaration prefix.

`NormalPartSemilinearRegularity.lean` — separate exact context:

```
local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair σ.toRingHom σ.symm.toRingHom :=
  RingHomInvPair.of_ringEquiv σ
local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair σ.symm.toRingHom σ.toRingHom :=
  RingHomInvPair.of_ringEquiv_symm σ
```

`NormalPartSemilinearRegularity.lean` — separate exact context:

```
universe u v
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))
variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
S))
variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
T))
variable (e : ClosureSpace.FiniteHomeomorph
  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))
  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)))
```

`NormalPartSemilinearRegularity.lean` — separate exact context:

```
include hS hT e
```

`NormalPartSemilinearRegularity.lean` — separate exact context:

```
variable (hdim : 3 ≤ Module.rank ℂ H)
variable (hH : ¬ FiniteDimensional ℂ H)
variable (hcont : ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart
S)).Continuous
  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)) e.toEquiv)

include hH hcont
```

Related exact source for the reading guide

First selected semilinear equivalence · Fixed source

```
def normalPartFirstSemilinearEquiv :
    LinearEquiv (normalPartFirstScalarEquiv S T hS hT e hdim).toRingHom (σ' :=
(normalPartFirstScalarEquiv S T hS hT e hdim).symm.toRingHom) H K :=
  Classical.choose (Classical.choose_spec
    (exists_normalPart_first_semilinearEquiv S T hS hT e hdim))
```

Second selected semilinear equivalence · Fixed source

```
def normalPartSecondSemilinearEquiv :
    LinearEquiv (normalPartSecondScalarEquiv S T hS hT e hdim).toRingHom (σ' :=
(normalPartSecondScalarEquiv S T hS hT e hdim).symm.toRingHom) H K :=
  Classical.choose (Classical.choose_spec
    (exists_normalPart_second_semilinearEquiv S T hS hT e hdim))
```

Continuity of the inverse of V · Fixed source

```
    Continuous (normalPartSecondSemilinearEquiv S T hS hT e hdim).symm :=
```

Each selected scalar automorphism is identity or conjugation

`MathlibAnnex.Analysis.CStarAlgebra.normalPart_semilinear_factors_scalar_types`

theorem

Identifies each scalar automorphism of the fixed semilinear factors as identity or conjugation.

Statement

Let H, K be complex Hilbert spaces, and let $A \subseteq \mathcal{B}(H)$ and $B \subseteq \mathcal{B}(K)$ be nonzero norm-closed complex $*$-subalgebras acting irreducibly on H and K, respectively. Neither algebra is assumed unital. For $x, y \in H \setminus \{0\}$ and $z, w \in K \setminus \{0\}$, define the bounded complex-linear functionals

$$\begin{aligned} f^A_{x,y} &: A \to \mathbb{C}, \qquad f^A_{x,y}(a) = \langle y, a(x) \rangle, \\ f^B_{z,w} &: B \to \mathbb{C}, \qquad f^B_{z,w}(b) = \langle w, b(z) \rangle. \end{aligned}$$

Their normal parts are

$$\begin{aligned} N(A) &= \{\ker f^A_{x,y} : x, y \in H \setminus \{0\}\}, \\ N(B) &= \{\ker f^B_{z,w} : z, w \in K \setminus \{0\}\}. \end{aligned} \tag{1}$$

For $\mathcal{D} \subseteq N(A)$, the relative geometric closure is

$$c_A(\mathcal{D}) = \left\{ J \in N(A) : \bigcap_{I \in \mathcal{D}} I \subseteq J \right\}. \tag{2}$$

Define c_B analogously. Let $e : N(A) \to N(B)$ be a specified finite-homeomorphism: a bijection satisfying $e(c_A(\mathcal{D})) = c_B(e(\mathcal{D}))$ for every finite $\mathcal{D} \subseteq N(A)$. Assume that H is infinite-dimensional and that e is also **forward continuous for arbitrary families**:

$$e\big(c_A(\mathcal{D})\big) \subseteq c_B\big(e(\mathcal{D})\big) \qquad (\mathcal{D} \subseteq N(A)). \tag{3}$$

Keep the two semilinear equivalences $U, V : H \to K$ and their scalar automorphisms σ_1, σ_2 fixed by the algebraic realization of this same e. Then

$$(\sigma_1 = \mathrm{id} \text{ or } \sigma_1 = \mathrm{conj}) \quad \text{and} \quad (\sigma_2 = \mathrm{id} \text{ or } \sigma_2 = \mathrm{conj}). \tag{4}$$

Here $\mathrm{conj}(z) = \overline{z}$. The conclusion concerns the scalar automorphisms of these same two maps.

Assumptions

The two nonzero concrete $*$-algebras are norm closed and act irreducibly on their respective Hilbert spaces. They need not be unital. The inner product is conjugate-linear in its first variable. The closure on each normal part is the intersection closure displayed above.

For their scalar field automorphisms σ_1, σ_2 of $\mathbb{C}$, the semilinear maps satisfy $U(cx) = \sigma_1(c)U(x)$ and $V(cx) = \sigma_2(c)V(x)$. They are the lifts fixed in the algebraic realization of the given e, with one global direct or swapped orientation. The rank theorem supplies the bound $3 \leq \dim_{\mathbb{C}}^{\mathrm{alg}} H$ required for this realization.

Full continuity is required only in the forward direction for e. Neither full continuity of e^{-1} nor initial continuity of the two semilinear maps is assumed.

Conclusion

Each of σ_1, σ_2 is identity or complex conjugation. The conclusion is a conjunction of two alternatives. It does not assert that the two scalar automorphisms are equal.

Main citations

The scalar-type theorem states the two alternatives. The algebraic realization identifies the factors of the given map; the source fixes their scalar automorphisms in the first and second definitions.

Lean source signature (exact)

The preceding `variable` declarations introduce the spaces, algebras, irreducibility hypotheses, the finite-homeomorphism `e`, and `hdim`. They occur in the conclusion through the selected factors. The earlier `include hS hT e` remains in scope, and `include hH hcont` additionally retains infinite-dimensionality and forward full continuity as theorem parameters. Thus the absence of an argument list on the next line does not make this an unconditional theorem. These declarations and commands are quoted separately below.

The following contiguous declaration prefix ends before its proof assignment.

```
theorem normalPart_semilinear_factors_scalar_types :
    (normalPartFirstScalarEquiv S T hS hT e hdim = RingEquiv.refl ℂ ∨
      normalPartFirstScalarEquiv S T hS hT e hdim = starRingAut (R := ℂ)) ∧
    (normalPartSecondScalarEquiv S T hS hT e hdim = RingEquiv.refl ℂ ∨
      normalPartSecondScalarEquiv S T hS hT e hdim = starRingAut (R := ℂ))
```

In the source, `S` and `T` denote the algebras A and B used here, respectively.

In the source	Mathematical meaning

Hilbert spaces and irreducible algebras
`variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`
`variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]`
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))`
`variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]`
`variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
	The complex Hilbert spaces H, K carry the nonzero norm-closed irreducible algebras $A \subseteq \mathcal{B}(H)$ and $B \subseteq \mathcal{B}(K)$, called `S`,`T`. The inclusion representations in the last two lines express these irreducible actions.

Finite closure comparison
`variable (e : ClosureSpace.FiniteHomeomorph`
`  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)))`
	The same bijection $e : N(A) \to N(B)$ compares the relative closures. For every finite $\mathcal{D} \subseteq N(A)$,
	$$e(c_A(\mathcal{D})) = c_B(e(\mathcal{D})).$$
	Its underlying equivalence is obtained by `toEquiv`; no new map is selected by that projection.

Included geometric inputs `include hS hT e`	The two irreducibility hypotheses and e are retained as parameters of the later theorem. They also occur as arguments of the selected scalar maps and semilinear maps.

Dimension and forward continuity
`variable (hdim : 3 ≤ Module.rank ℂ H)`
`variable (hH : ¬ FiniteDimensional ℂ H)`
`variable (hcont : ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Continuous`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)) e.toEquiv)`
 	
`include hH hcont`
	The bound $3 \leq \dim_{\mathbb{C}}^{\mathrm{alg}} H$ specifies the selected lifts. Under the present hypotheses it is guaranteed by the rank theorem cited above. The additional hypotheses are infinite-dimensionality of H and, for every $\mathcal{D} \subseteq N(A)$,
	$$e(c_A(\mathcal{D})) \subseteq c_B(e(\mathcal{D})).$$
	The final `include` retains these hypotheses. Full continuity of e^{-1} is not assumed.

Inverse scalar structures
`local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair σ.toRingHom σ.symm.toRingHom :=`
`  RingHomInvPair.of_ringEquiv σ`
`local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair σ.symm.toRingHom σ.toRingHom :=`
`  RingHomInvPair.of_ringEquiv_symm σ`
	For a scalar automorphism σ, these instances pair σ with its inverse σ^{-1} in both orders. They make the inverse of a σ-semilinear equivalence a σ^{-1}-semilinear map; they do not impose $\sigma^2 = \mathrm{id}$.

First selected scalar automorphism
`def normalPartFirstScalarEquiv : ℂ ≃+* ℂ :=`
`  Classical.choose (exists_normalPart_first_semilinearEquiv S T hS hT e hdim)`
	The source defines the fixed automorphism σ_1 from the same A, B, e and dimension bound. For the same selected map U it is characterized by $U(cx) = \sigma_1(c)U(x)$. It is not selected separately for each vector.

In the source	Mathematical meaning

Second selected scalar automorphism
`def normalPartSecondScalarEquiv : ℂ ≃+* ℂ :=`
`  Classical.choose`
`(exists_normalPart_second_semilinearEquiv S T hS hT e hdim)`
	This is the fixed scalar automorphism σ_2 of the same selected map V, characterized by $V(cx) = \sigma_2(c)V(x)$, from the same inputs. The first and second scalar automorphisms are not identified with each other.

First scalar alternative
`(normalPartFirstScalarEquiv S T hS hT e hdim =`
`RingEquiv.refl ℂ ∨`
`  normalPartFirstScalarEquiv S T hS hT e hdim =`
`starRingAut (R := ℂ)) ∧`
	The first factor obeys
	$$\sigma_1 = \mathrm{id} \quad \text{or} \quad \sigma_1 = \mathrm{conj},$$
	where $\mathrm{conj}(c) = \overline{c}$ for every complex scalar. The trailing conjunction joins this whole alternative to the next one.

Second scalar alternative
`(normalPartSecondScalarEquiv S T hS hT e hdim =`
`RingEquiv.refl ℂ ∨`
`  normalPartSecondScalarEquiv S T hS hT e hdim =`
`starRingAut (R := ℂ)) :=`
	The second factor obeys
	$$\sigma_2 = \mathrm{id} \quad \text{or} \quad \sigma_2 = \mathrm{conj}.$$
	Both alternatives in (4) hold for the fixed selected maps. This theorem alone does not assert $\sigma_1 = \sigma_2$.

Exact surrounding source context (separate excerpts)

These separate excerpts record inverse scalar instances, the ambient spaces and algebras, and the included hypotheses. They are not spliced into the declaration prefix.

`NormalPartSemilinearRegularity.lean` — separate exact context:

```
local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair σ.toRingHom σ.symm.toRingHom :=
  RingHomInvPair.of_ringEquiv σ
local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair σ.symm.toRingHom σ.toRingHom :=
  RingHomInvPair.of_ringEquiv_symm σ
```

`NormalPartSemilinearRegularity.lean` — separate exact context:

```
universe u v
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))
variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
S))
variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
T))
variable (e : ClosureSpace.FiniteHomeomorph
  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))
  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)))
```

`NormalPartSemilinearRegularity.lean` — separate exact context:

```
include hS hT e
```

`NormalPartSemilinearRegularity.lean` — separate exact context:

```
variable (hdim : 3 ≤ Module.rank ℂ H)
variable (hH : ¬ FiniteDimensional ℂ H)
variable (hcont : ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart
S)).Continuous
  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)) e.toEquiv)

include hH hcont
```

Related exact source for the reading guide

First selected scalar automorphism · Fixed source

```
def normalPartFirstScalarEquiv : ℂ ≃+* ℂ :=
  Classical.choose (exists_normalPart_first_semilinearEquiv S T hS hT e hdim)
```

Second selected scalar automorphism · Fixed source

```
def normalPartSecondScalarEquiv : ℂ ≃+* ℂ :=
  Classical.choose (exists_normalPart_second_semilinearEquiv S T hS hT e hdim)
```

Second scalar alternative · Fixed source

```
    (normalPartSecondScalarEquiv S T hS hT e hdim = RingEquiv.refl ℂ v
      normalPartSecondScalarEquiv S T hS hT e hdim = starRingAut (R := ℂ)) :=
```

Classifying the specified normal-part equivalence with the selected bounded lifts

`MathlibAnnex.Analysis.CStarAlgebra.exists_normalPart_selected_classification_eq`

theorem

Classifies the given normal-part homeomorphism using its fixed factors and an operator image equality.

Statement

Let H, K be complex Hilbert spaces, and let $A \subseteq \mathcal{B}(H)$ and $B \subseteq \mathcal{B}(K)$ be nonzero norm-closed complex $*$-subalgebras acting irreducibly on H and K, respectively. Neither algebra is assumed unital. For $x, y \in H \setminus \{0\}$ and $z, w \in K \setminus \{0\}$, define the bounded complex-linear functionals

$$\begin{aligned} f^A_{x,y} &: A \to \mathbb{C}, \qquad f^A_{x,y}(a) = \langle y, a(x)\rangle, \\ f^B_{z,w} &: B \to \mathbb{C}, \qquad f^B_{z,w}(b) = \langle w, b(z)\rangle. \end{aligned}$$

Their normal parts are

$$\begin{aligned} N(A) &= \{\ker f^A_{x,y} : x, y \in H \setminus \{0\}\}, \\ N(B) &= \{\ker f^B_{z,w} : z, w \in K \setminus \{0\}\}. \end{aligned} \tag{1}$$

The inner product is conjugate-linear in its first variable. For $\mathcal{D} \subseteq N(A)$, the relative geometric closure is $c_A(\mathcal{D}) = \{J \in N(A) : \bigcap_{I \in \mathcal{D}} I \subseteq J\}$; define c_B analogously. Let $e : N(A) \to N(B)$ be a finite-homeomorphism: a bijection satisfying $e(c_A(\mathcal{D})) = c_B(e(\mathcal{D}))$ for every finite $\mathcal{D} \subseteq N(A)$. Assume H is infinite-dimensional and, additionally, both e and e^{-1} are fully continuous:

$$\begin{aligned} e(c_A(\mathcal{D})) &\subseteq c_B(e(\mathcal{D})) \qquad (\mathcal{D} \subseteq N(A)), \\ e^{-1}(c_B(\mathcal{E})) &\subseteq c_A(e^{-1}(\mathcal{E})) \qquad (\mathcal{E} \subseteq N(B)). \end{aligned} \tag{2}$$

Let $U, V : H \to K$ and σ_1, σ_2 be the semilinear lifts and scalar automorphisms selected in the algebraic realization of this e, using the dimension bound guaranteed by infinite-dimensionality. Thus $U(cx) = \sigma_1(c)U(x)$ and $V(cx) = \sigma_2(c)V(x)$ for $c \in \mathbb{C}$ and $x \in H$. Then $\sigma_1 = \sigma_2$. There exist either two continuous complex-linear equivalences $U_c, V_c : H \to K$, or two continuous conjugate-linear equivalences $U_c, V_c : H \to K$, such that

$$\begin{aligned} U_c(x) &= U(x), & V_c(x) &= V(x) & (x \in H), \\ U_c^{-1}(z) &= U^{-1}(z), & V_c^{-1}(z) &= V^{-1}(z) & (z \in K). \end{aligned} \tag{3}$$

For either common scalar type, one of the following **global** orientation branches holds. Write F_{U_c,V_c} for the normal-part equivalence given by $F_{U_c,V_c}(\ker f^A_{x,y}) = \ker f^B_{U_c x, V_c y}$. Let s_B be the closure-space homeomorphism induced by the involution, so that $s_B(\ker f^B_{z,w}) = \ker f^B_{w,z}$.

Direct branch. The specified equivalence is $e = F_{U_c,V_c}$. For every $x, y \in H \setminus \{0\}$ and $z, w \in K \setminus \{0\}$,

$$\begin{aligned} e(\ker f^A_{x,y}) &= \ker f^B_{U_c x, V_c y}, \\ e^{-1}(\ker f^B_{z,w}) &= \ker f^A_{U_c^{-1} z, V_c^{-1} w}. \end{aligned} \tag{4}$$

Swapped branch. The specified equivalence is $e = s_B \circ F_{U_c,V_c}$. For every $x, y \in H \setminus \{0\}$ and $z, w \in K \setminus \{0\}$,

$$\begin{aligned} e(\ker f^A_{x,y}) &= \ker f^B_{V_c y, U_c x}, \\ e^{-1}(\ker f^B_{z,w}) &= \ker f^A_{U_c^{-1} w, V_c^{-1} z}. \end{aligned} \tag{5}$$

The kernels on each side are subsets of the corresponding algebra. The forward and inverse formulas use the same branch and the same maps in (3). In either orientation, the common scalar type determines the operator-set equality

$$\begin{cases} V_c^* B U_c = A, & U_c, V_c \text{ complex-linear}, \\ V_c^\dagger B U_c = A, & U_c, V_c \text{ conjugate-linear}. \end{cases} \tag{6}$$

Here $V_c^* B U_c = \{V_c^* b U_c : b \in B\}$, and similarly for $V_c^\dagger B U_c$. In the second case, the conjugate adjoint $V_c^\dagger : K \to H$ is characterized by $\langle V_c^\dagger p, q\rangle = \langle V_c q, p\rangle$ for $p \in K$ and $q \in H$. For each $b \in B$, the corresponding product $V_c^* b U_c$ or $V_c^\dagger b U_c$ is a bounded complex-linear operator on H.

Assumptions

The two nonzero concrete $*$-algebras are norm closed and act irreducibly on their respective Hilbert spaces. Neither algebra is assumed unital. The source space H is infinite dimensional.

Both full-continuity conditions (2) are required, in addition to finite continuity in both directions. Continuity of the semilinear factors, equality of their scalar types, and the four identities (3) are conclusions.

Conclusion

The given e has one common scalar type and one global direct or swapped orientation. The same maps U_c, V_c and their inverses satisfy all point and kernel formulas. Their underlying functions are precisely those fixed by the algebraic realization of e.

In either scalar case the corresponding operator-set equality (6) holds. This is equality of concrete operator sets; it does not assert that $U_c = V_c$, that either map is unitary, or that the transformation is a multiplicative $*$-isomorphism.

Main citations

The selected classification theorem gives the common scalar type, the four function identities, both orientations, all coefficient-kernel formulas and the operator image equality. The algebraic realization fixes the underlying maps. The linear and conjugate-linear operator criteria specify the

two transformations.

Lean source signature (exact)

The preceding `variable` declarations introduce H, K, the algebras `S`, `T`, the irreducibility hypotheses and the finite-homeomorphism e. Their occurrence in the arguments and conclusion makes them theorem parameters. The three arguments displayed below are infinite-dimensionality and the two directions of full continuity. In the conclusion type, the three `let` bindings name the rank bound and the selected maps U, V; they are not proof steps or a second choice of maps.

The following contiguous literal signature ends immediately before the final proof assignment. The full source retains the proof, and the surrounding declarations are quoted separately below.

```
theorem exists_normalPart_selected_classification_eq
    (hH : ¬ FiniteDimensional ℂ H)
    (hcont : ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Continuous
      ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)) e.toEquiv)
    (hcontInv : ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart
T)).Continuous
      ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)) e.toEquiv.symm) :
    let hd := (three_le_rank_of_normalPart_finiteHomeomorph S T hS hT e hH).1
    let U := normalPartFirstSemilinearEquiv S T hS hT e hd
    let V := normalPartSecondSemilinearEquiv S T hS hT e hd
    normalPartFirstScalarEquiv S T hS hT e hd =
      normalPartSecondScalarEquiv S T hS hT e hd ∧
    (
    (∃ Uc Vc : H ≃L[ℂ] K,
      (∀ x, Uc x = U x) ∧ (∀ x, Vc x = V x) ∧
      (∀ z, Uc.symm z = U.symm z) ∧ (∀ z, Vc.symm z = V.symm z) ∧
      (((e.toEquiv = normalPartSemilinearEquiv S T hS hT (fun _ => rfl)
            Uc.toLinearEquiv Vc.toLinearEquiv) ∧
        (∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),
          e.toEquiv (normalPartPoint S hS x y hx hy) =
            normalPartPoint T hT (Uc x) (Vc y)
              (by simpa only [map_zero] using Uc.injective.ne hx)
              (by simpa only [map_zero] using Vc.injective.ne hy)) ∧
        (∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),
          e.toEquiv.symm (normalPartPoint T hT z w hz hw) =
            normalPartPoint S hS (Uc.symm z) (Vc.symm w)
              (by simpa only [map_zero] using Uc.symm.injective.ne hz)
              (by simpa only [map_zero] using Vc.symm.injective.ne hw)) ∧
        (∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),
          (e.toEquiv (normalPartPoint S hS x y hx hy)).val.val =
            {b : T | inner ℂ (Vc y) ((b : K →L[ℂ] K) (Uc x)) = 0}) ∧
        (∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),
          (e.toEquiv.symm (normalPartPoint T hT z w hz hw)).val.val =
            {a : S | inner ℂ (Vc.symm w) ((a : H →L[ℂ] H) (Uc.symm z)) = 0})) ∨
       ((e.toEquiv = (normalPartSemilinearEquiv S T hS hT (fun _ => rfl)
            Uc.toLinearEquiv Vc.toLinearEquiv).trans (normalPartStarHomeomorph T hT).toEquiv) ∧
        (∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),
          e.toEquiv (normalPartPoint S hS x y hx hy) =
            normalPartPoint T hT (Vc y) (Uc x)
              (by simpa only [map_zero] using Vc.injective.ne hy)
              (by simpa only [map_zero] using Uc.injective.ne hx)) ∧
        (∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),
```

```
          e.toEquiv.symm (normalPartPoint T hT z w hz hw) =
            normalPartPoint S hS (Uc.symm w) (Vc.symm z)
              (by simpa only [map_zero] using Uc.symm.injective.ne hw)
              (by simpa only [map_zero] using Vc.symm.injective.ne hz)) ∧
        (∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),
          (e.toEquiv (normalPartPoint S hS x y hx hy)).val.val =
            {b : T | inner ℂ (Uc x) ((b : K →L[ℂ] K) (Vc y)) = 0}) ∧
        (∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),
          (e.toEquiv.symm (normalPartPoint T hT z w hz hw)).val.val =
            {a : S | inner ℂ (Vc.symm z) ((a : H →L[ℂ] H) (Uc.symm w)) = 0}))) ∧
      InnerProductSpace.adjointSandwich Uc.toContinuousLinearMap Vc.toContinuousLinearMap ''
        (T : Set (K →L[ℂ] K)) = (S : Set (H →L[ℂ] H))) ∨
    (∃ Uc Vc : H ≃SL[starRingEnd ℂ] K,
      (∀ x, Uc x = U x) ∧ (∀ x, Vc x = V x) ∧
      (∀ z, Uc.symm z = U.symm z) ∧ (∀ z, Vc.symm z = V.symm z) ∧
      (((e.toEquiv = normalPartSemilinearEquiv S T hS hT (fun _ => rfl)
            Uc.toLinearEquiv Vc.toLinearEquiv) ∧
        (∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),
          e.toEquiv (normalPartPoint S hS x y hx hy) =
            normalPartPoint T hT (Uc x) (Vc y)
              (by simpa only [map_zero] using Uc.injective.ne hx)
              (by simpa only [map_zero] using Vc.injective.ne hy)) ∧
        (∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),
          e.toEquiv.symm (normalPartPoint T hT z w hz hw) =
            normalPartPoint S hS (Uc.symm z) (Vc.symm w)
              (by simpa only [map_zero] using Uc.symm.injective.ne hz)
              (by simpa only [map_zero] using Vc.symm.injective.ne hw)) ∧
        (∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),
          (e.toEquiv (normalPartPoint S hS x y hx hy)).val.val =
            {b : T | inner ℂ (Vc y) ((b : K →L[ℂ] K) (Uc x)) = 0}) ∧
        (∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),
          (e.toEquiv.symm (normalPartPoint T hT z w hz hw)).val.val =
            {a : S | inner ℂ (Vc.symm w) ((a : H →L[ℂ] H) (Uc.symm z)) = 0})) ∨
       ((e.toEquiv = (normalPartSemilinearEquiv S T hS hT (fun _ => rfl)
            Uc.toLinearEquiv Vc.toLinearEquiv).trans (normalPartStarHomeomorph T hT).toEquiv) ∧
        (∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),
          e.toEquiv (normalPartPoint S hS x y hx hy) =
            normalPartPoint T hT (Vc y) (Uc x)
              (by simpa only [map_zero] using Vc.injective.ne hy)
              (by simpa only [map_zero] using Uc.injective.ne hx)) ∧
        (∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),
          e.toEquiv.symm (normalPartPoint T hT z w hz hw) =
            normalPartPoint S hS (Uc.symm w) (Vc.symm z)
              (by simpa only [map_zero] using Uc.symm.injective.ne hw)
              (by simpa only [map_zero] using Vc.symm.injective.ne hz)) ∧
        (∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),
          (e.toEquiv (normalPartPoint S hS x y hx hy)).val.val =
            {b : T | inner ℂ (Uc x) ((b : K →L[ℂ] K) (Vc y)) = 0}) ∧
        (∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),
          (e.toEquiv.symm (normalPartPoint T hT z w hz hw)).val.val =
            {a : S | inner ℂ (Vc.symm z) ((a : H →L[ℂ] H) (Uc.symm w)) = 0}))) ∧
      InnerProductSpace.conjugateAdjointSandwich Uc.toContinuousLinearMap
Vc.toContinuousLinearMap ''
        (T : Set (K →L[ℂ] K)) = (S : Set (H →L[ℂ] H))))
```

In the source, `S` and `T` denote the algebras A and B used here, respectively.

In the source	Mathematical meaning

Ambient spaces and algebras
`variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]`
`variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]`
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))`
`variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]`
`variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
	The Hilbert spaces H, K and the nonzero norm-closed irreducible complex $*$-subalgebras A, B, denoted S, T in the source.

Given finite homeomorphism
`variable (e : ClosureSpace.FiniteHomeomorph`
`  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)))`
	The same bijection $e : N(A) \to N(B)$ and its actual inverse, with finite continuity for the relative closures c_A, c_B in both directions.

Full continuity hypotheses
`(hH : ¬ FiniteDimensional ℂ H)`
`(hcont : ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)).Continuous`
`  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)) e.toEquiv)`
`(hcontInv : ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)).Continuous`
`  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S)) e.toEquiv.symm) :`
	The infinite-dimensionality of H, and both full-continuity inclusions in (2), for arbitrary subsets $\mathcal{D} \subseteq N(A)$ and $\mathcal{E} \subseteq N(B)$. The functions supplied are the given e and its inverse.

Derived rank and selected maps
`let hd := (three_le_rank_of_normalPart_finiteHomeomorph S T hS hT e hH).1`
`let U := normalPartFirstSemilinearEquiv S T hS hT e hd`
`let V := normalPartSecondSemilinearEquiv S T hS hT e hd`
	The rank theorem supplies $3 \leq \dim_{\mathbb{C}}^{\mathrm{alg}} H$ under the stated hypotheses. It is not an additional assumption. The same e and this bound select the same $U, V : H \to K$ used in the Statement.

Common scalar type
`normalPartFirstScalarEquiv S T hS hT e hd =`
`  normalPartSecondScalarEquiv S T hS hT e hd ∧`
	The two scalar automorphisms of the selected lifts agree:
	$$\sigma_1 = \sigma_2.$$
	The alternatives that follow classify their common type; they do not allow a mixed linear/conjugate-linear pair.

In the source	Mathematical meaning

Complex-linear realization
`(∃ Uc Vc : H ≃L[ℂ] K,`
`  (∀ x, Uc x = U x) ∧ (∀ x, Vc x = V x) ∧`
`  (∀ z, Uc.symm z = U.symm z) ∧ (∀ z, Vc.symm z = V.symm z) ∧`
	Continuous complex-linear isomorphisms $U_c, V_c : H \to K$ have the same functions and the same inverses as the selected lifts:
	$$U_c(x) = U(x), \qquad V_c(x) = V(x) \quad (x \in H),$$
	$$U_c^{-1}(z) = U^{-1}(z), \qquad V_c^{-1}(z) = V^{-1}(z) \quad (z \in K).$$

Complex-linear direct map
`(((e.toEquiv = normalPartSemilinearEquiv S T hS hT (fun _ => rfl)`
`      Uc.toLinearEquiv Vc.toLinearEquiv) ∧`
	The direct branch identifies the given map with the transport by these same U_c, V_c:
	$$e = F_{U_c,V_c}.$$
	Taking underlying linear or semilinear maps changes no function.

Complex-linear direct forward point
`(∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),`
`  e.toEquiv (normalPartPoint S hS x y hx hy) =`
`    normalPartPoint T hT (Uc x) (Vc y)`
`      (by simpa only [map_zero] using Uc.injective.ne hx)`
`      (by simpa only [map_zero] using Vc.injective.ne hy)) ∧`
	For every $x, y \in H \setminus \{0\}$, the first equation of (4):
	$$e(\ker f^A_{x,y}) = \ker f^B_{U_c x, V_c y}.$$

Complex-linear direct inverse point
`(∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),`
`  e.toEquiv.symm (normalPartPoint T hT z w hz hw) =`
`    normalPartPoint S hS (Uc.symm z) (Vc.symm w)`
`      (by simpa only [map_zero] using Uc.symm.injective.ne hz)`
`      (by simpa only [map_zero] using Vc.symm.injective.ne hw)) ∧`
	For every $z, w \in K \setminus \{0\}$, the second equation of (4):
	$$e^{-1}(\ker f^B_{z,w}) = \ker f^A_{U_c^{-1} z, V_c^{-1} w}.$$

Complex-linear direct underlying sets
`(∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),`
`  (e.toEquiv (normalPartPoint S hS x y hx hy)).val.val =`
`    {b : T | inner ℂ (Vc y) ((b : K →L[ℂ] K) (Uc x)) = 0}) ∧`
`(∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),`
`  (e.toEquiv.symm (normalPartPoint T hT z w hz hw)).val.val =`
`    {a : S | inner ℂ (Vc.symm w) ((a : H →L[ℂ] H) (Uc.symm z)) = 0})) v`
	The same two equalities are also equalities of subsets of B and A, respectively, with the same nonzero-vector quantifiers:
	$$e(\ker f^A_{x,y}) = \ker f^B_{U_c x, V_c y},$$
	$$e^{-1}(\ker f^B_{z,w}) = \ker f^A_{U_c^{-1} z, V_c^{-1} w}.$$
	The projections expose the underlying kernel sets, not different points.

Complex-linear swapped map
`((e.toEquiv = (normalPartSemilinearEquiv S T hS hT (fun _ => rfl)`
`     Uc.toLinearEquiv Vc.toLinearEquiv).trans (normalPartStarHomeomorph T hT).toEquiv) ∧`
	The other orientation composes with the target star map:
	$$e = s_B \circ F_{U_c,V_c}.$$
	Here $s_B(\ker f^B_{z,w}) = \ker f^B_{w,z}$. The disjunction selects one orientation for all vectors.

In the source	Mathematical meaning

Complex-linear swapped forward point
`(∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),`
`  e.toEquiv (normalPartPoint S hS x y hx hy) =`
`  normalPartPoint T hT (Vc y) (Uc x)`
`  (by simpa only [map_zero] using Vc.injective.ne hy)`
`  (by simpa only [map_zero] using Uc.injective.ne hx)) ∧`
	For every $x, y \in H \setminus \{0\}$, the first equation of (5):
	$$e(\ker f^A_{x,y}) = \ker f^B_{V_c y, U_c x}.$$

Complex-linear swapped inverse point
`(∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),`
`  e.toEquiv.symm (normalPartPoint T hT z w hz hw) =`
`  normalPartPoint S hS (Uc.symm w) (Vc.symm z)`
`  (by simpa only [map_zero] using Uc.symm.injective.ne hw)`
`  (by simpa only [map_zero] using Vc.symm.injective.ne hz)) ∧`
	For every $z, w \in K \setminus \{0\}$, the second equation of (5):
	$$e^{-1}(\ker f^B_{z,w}) = \ker f^A_{U_c^{-1} w, V_c^{-1} z}.$$

Complex-linear swapped underlying sets
`(∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),`
`  (e.toEquiv (normalPartPoint S hS x y hx hy)).val.val =`
`  {b : T | inner ℂ (Uc x) ((b : K →L[ℂ] K) (Vc y)) = 0}) ∧`
`(∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),`
`  (e.toEquiv.symm (normalPartPoint T hT z w hz hw)).val.val =`
`  {a : S | inner ℂ (Vc.symm z) ((a : H →L[ℂ] H) (Uc.symm w)) = 0})) ∧`
	The underlying subsets satisfy the same two equations in (5), for all the same nonzero pairs:
	$$e(\ker f^A_{x,y}) = \ker f^B_{V_c y, U_c x},$$
	$$e^{-1}(\ker f^B_{z,w}) = \ker f^A_{U_c^{-1} w, V_c^{-1} z}.$$
	The orientation and actual inverse are unchanged.

Complex-linear operator image
`InnerProductSpace.adjointSandwich Uc.toContinuousLinearMap Vc.toContinuousLinearMap ''`
`  (T : Set (K →L[ℂ] K)) = (S : Set (H →L[ℂ] H))) v`
	In either orientation of the complex-linear case,
	$$V_c^* B U_c = \{V_c^* b U_c : b \in B\} = A \subseteq \mathcal{B}(H).$$
	This is equality, not just inclusion, and uses the bounded Hilbert-space adjoint.

Conjugate-linear realization
`(∃ Uc Vc : H ≃SL[starRingEnd ℂ] K,`
`  (∀ x, Uc x = U x) ∧ (∀ x, Vc x = V x) ∧`
`  (∀ z, Uc.symm z = U.symm z) ∧ (∀ z, Vc.symm z = V.symm z) ∧`
	Continuous conjugate-linear isomorphisms $U_c, V_c : H \to K$ instead realize the selected lifts, again with the same inverse functions:
	$$U_c(x) = U(x), \qquad V_c(x) = V(x) \quad (x \in H),$$
	$$U_c^{-1}(z) = U^{-1}(z), \qquad V_c^{-1}(z) = V^{-1}(z) \quad (z \in K).$$
	This is the other scalar-type alternative, not a second choice within the complex-linear case.

Conjugate-linear direct map
`(((e.toEquiv = normalPartSemilinearEquiv S T hS hT (fun _ => rfl)`
`  Uc.toLinearEquiv Vc.toLinearEquiv) ∧`
	The direct branch identifies the given map with the transport by these same U_c, V_c:
	$$e = F_{U_c, V_c}.$$
	Taking underlying linear or semilinear maps changes no function.

In the source	Mathematical meaning
Conjugate-linear direct forward point `(∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0), e.toEquiv (normalPartPoint S hS x y hx hy) = normalPartPoint T hT (Uc x) (Vc y) (by simpa only [map_zero] using Uc.injective.ne hx) (by simpa only [map_zero] using Vc.injective.ne hy)) ∧`	For every $x, y \in H \setminus \{0\}$, the first equation of (4): $$e(\ker f^A_{x,y}) = \ker f^B_{U_c x, V_c y}.$$
Conjugate-linear direct inverse point `(∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0), e.toEquiv.symm (normalPartPoint T hT z w hz hw) = normalPartPoint S hS (Uc.symm z) (Vc.symm w) (by simpa only [map_zero] using Uc.symm.injective.ne hz) (by simpa only [map_zero] using Vc.symm.injective.ne hw)) ∧`	For every $z, w \in K \setminus \{0\}$, the second equation of (4): $$e^{-1}(\ker f^B_{z,w}) = \ker f^A_{U_c^{-1} z, V_c^{-1} w}.$$
Conjugate-linear direct underlying sets `(∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0), (e.toEquiv (normalPartPoint S hS x y hx hy)).val.val = {b : T | inner ℂ (Vc y) ((b : K →L[ℂ] K) (Uc x)) = 0}) ∧ (∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0), (e.toEquiv.symm (normalPartPoint T hT z w hz hw)).val.val = {a : S | inner ℂ (Vc.symm w) ((a : H →L[ℂ] H) (Uc.symm z)) = 0})) ∨`	The same two equalities are also equalities of subsets of B and A, respectively, with the same nonzero-vector quantifiers: $$e(\ker f^A_{x,y}) = \ker f^B_{U_c x, V_c y},$$ $$e^{-1}(\ker f^B_{z,w}) = \ker f^A_{U_c^{-1} z, V_c^{-1} w}.$$ The projections expose the underlying kernel sets, not different points.
Conjugate-linear swapped map `((e.toEquiv = (normalPartSemilinearEquiv S T hS hT (fun _ => rfl) Uc.toLinearEquiv Vc.toLinearEquiv).trans (normalPartStarHomeomorph T hT).toEquiv) ∧`	The other orientation composes with the target star map: $$e = s_B \circ F_{U_c, V_c}.$$ Here $s_B(\ker f^B_{z,w}) = \ker f^B_{w,z}$. The disjunction selects one orientation for all vectors.
Conjugate-linear swapped forward point `(∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0), e.toEquiv (normalPartPoint S hS x y hx hy) = normalPartPoint T hT (Vc y) (Uc x) (by simpa only [map_zero] using Vc.injective.ne hy) (by simpa only [map_zero] using Uc.injective.ne hx)) ∧`	For every $x, y \in H \setminus \{0\}$, the first equation of (5): $$e(\ker f^A_{x,y}) = \ker f^B_{V_c y, U_c x}.$$
Conjugate-linear swapped inverse point `(∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0), e.toEquiv.symm (normalPartPoint T hT z w hz hw) = normalPartPoint S hS (Uc.symm w) (Vc.symm z) (by simpa only [map_zero] using Uc.symm.injective.ne hw) (by simpa only [map_zero] using Vc.symm.injective.ne hz)) ∧`	For every $z, w \in K \setminus \{0\}$, the second equation of (5): $$e^{-1}(\ker f^B_{z,w}) = \ker f^A_{U_c^{-1} w, V_c^{-1} z}.$$

In the source	Mathematical meaning

Conjugate-linear swapped underlying sets
`(∀ (x y : H) (hx : x ≠ 0) (hy : y ≠ 0),`
`  (e.toEquiv (normalPartPoint S hS x y hx hy)).val.val =`
`  {b : T | inner ℂ (Uc x) ((b : K →L[ℂ] K) (Vc y)) = 0}) ∧`
`(∀ (z w : K) (hz : z ≠ 0) (hw : w ≠ 0),`
`  (e.toEquiv.symm (normalPartPoint T hT z w hz hw)).val.val =`
`  {a : S | inner ℂ (Vc.symm z) ((a : H →L[ℂ] H) (Uc.symm w)) = 0})) ∧`
	The underlying subsets satisfy the same two equations in (5), for all the same nonzero pairs:
	$$e(\ker f^A_{x,y}) = \ker f^B_{V_c y, U_c x},$$
	$$e^{-1}(\ker f^B_{z,w}) = \ker f^A_{U_c^{-1} w, V_c^{-1} z}.$$
	The orientation and actual inverse are unchanged.

Conjugate-linear operator image
`InnerProductSpace.conjugateAdjointSandwich Uc.toContinuousLinearMap Vc.toContinuousLinearMap ''`
`  (T : Set (K →L[ℂ] K)) = (S : Set (H →L[ℂ] H))) := by`
	In either orientation of the conjugate-linear case,
	$$V_c^\dagger B U_c = \{V_c^\dagger b U_c : b \in B\} = A \subseteq \mathcal{B}(H).$$
	The dagger denotes the conjugate adjoint. Each displayed composite is a bounded complex-linear operator.

Adjoint composite
`def adjointSandwich (U V : H →L[𝕜] K) (b : K →L[𝕜] K) : H →L[𝕜] H :=`
`  V.adjoint.comp (b.comp U)`
	The composite $V_c^* \circ b \circ U_c$ in the complex-linear case.

Conjugate-adjoint composite
`def conjugateAdjointSandwich (U V : H →L⋆[𝕜] K) (b : K →L[𝕜] K) : H →L[𝕜] H :=`
`  (conjugateAdjoint V).comp (b.comp U)`
	The composite $V_c^\dagger \circ b \circ U_c$ in the conjugate-linear case, with the same order of factors.

Exact surrounding source context (separate excerpts)

The following are separate exact ambient excerpts, not lines spliced into the signature. The local inverse-pair instances express genuine inverse scalar maps, not an assumption that every field automorphism is involutive.

```
local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair σ.toRingHom σ.symm.toRingHom :=
  RingHomInvPair.of_ringEquiv σ
local instance (σ : ℂ ≃+* ℂ) : RingHomInvPair σ.symm.toRingHom σ.toRingHom :=
  RingHomInvPair.of_ringEquiv_symm σ
```

The ambient declarations are active here. The local scalar variables of the earlier `Evaluation` and `SameFunctions` sections have gone out of scope before the selected theorem; they are not additional inputs.

```
universe u v
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))
variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
S))
variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
T))
variable (e : ClosureSpace.FiniteHomeomorph
  ((NormedSpace.geometricStructureClosure ℂ S).subspace (normalPart S))
  ((NormedSpace.geometricStructureClosure ℂ T).subspace (normalPart T)))
```

Related exact source for the reading guide

Conjugate-linear operator image · Fixed source

```
      InnerProductSpace.conjugateAdjointSandwich Uc.toContinuousLinearMap
Vc.toContinuousLinearMap ''
        (T : Set (K →L[ℂ] K)) = (S : Set (H →L[ℂ] H))) := by
```

Conjugate-adjoint composite · Fixed source

```
def conjugateAdjointSandwich (U V : H →L⋆[𝕜] K) (b : K →L[𝕜] K) : H →L[𝕜] H :=
  (conjugateAdjoint V).comp (b.comp U)
```

Adjoint composite · Fixed source

```
def adjointSandwich (U V : H →L[𝕜] K) (b : K →L[𝕜] K) : H →L[𝕜] H :=
  V.adjoint.comp (b.comp U)
```

Primitive representation kernels are exactly geometric component intersections

`MathlibAnnex.Analysis.CStarAlgebra.primitiveKernelFamily_eq_primitiveGeometricSpectrumFamily`

theorem

Identifies primitive representation kernels with intersections of geometric components.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity. A pure state is a positive norm-one bounded complex-linear functional on A that is extreme in the state space for real convexity. For a pure state φ, let π_φ be the representation of A obtained by restricting the GNS representation of its canonical state extension to the forced unitization. The independently defined primitive-kernel family is

$$\mathrm{Prim}(A) = \{\ker \pi_\varphi : \varphi \text{ is a pure state of } A\}.$$

Let $S(A)$ be the geometric structure space, whose points are kernels $\ker \rho \subseteq A$ of bounded complex-linear functionals $\rho : A \to \mathbb{C}$ that are extreme points, for real convexity, of the entire closed dual unit ball B_{A^*} by the extreme-kernel theorem, with the hull–kernel closure $c_A(\mathcal{T}) = \{I \in S(A) : \bigcap_{J \in \mathcal{T}} J \subseteq I\}$. Let $\mathrm{GS}(A)$ be its finite-chain components (chains of equal points or nonclosed pairs), and put $J_C = \bigcap_{I \in C} I$. The primitive geometric family is $\mathrm{PS}(A) = \{J_C : C \in \mathrm{GS}(A)\}$. Then

$$\mathrm{Prim}(A) = \mathrm{PS}(A). \tag{1}$$

Here both $\mathrm{Prim}(A)$ and $\mathrm{PS}(A)$ are treated as families of subsets of A.

Assumptions

The source uses a nonzero complex C*-algebra with its compatible order. The family $\mathrm{Prim}(A)$ is constructed from the pure-state GNS representations independently of geometric components. The compared objects are families of subsets of this same algebra; no additional representation or component is an input.

Conclusion

The equality identifies precisely which subsets occur. It does not identify an individual component with its intersection. No injectivity of the representation-kernel map or of $C \mapsto J_C$ is asserted. The primitive-spectrum homeomorphism is a separate construction using this family equality.

Main citations

- Exact family equality: `primitiveKernelFamily_eq_primitiveGeometricSpectrumFamily`.

- Pure-state component representative and pure-state component intersection: the related representative and same-state intersection statements.
- Independent primitive family and geometric component-intersection family: the definitions compared here.

Lean source signature (exact)

The declaration prefix below is a contiguous source excerpt, ending before its proof assignment.

```
theorem primitiveKernelFamily_eq_primitiveGeometricSpectrumFamily :
    primitiveKernelFamily A = NormedSpace.primitiveGeometricSpectrumFamily ℂ A
```

In the source	Mathematical meaning
Exact declaration `(A : Type u)` and its nonunital C*-algebra, order and nontriviality instances	The same nonzero complex C*-algebra underlies both independently defined families. No extra representation is an input to this declaration.
Exact declaration `primitiveKernelFamily A`	Kernels $\ker \pi_\varphi$ for pure states φ, forming the family $\mathrm{Prim}(A)$ of subsets of A.
Exact declaration `NormedSpace.primitiveGeometricSpectrumFamily ℂ A`	The family $\mathrm{PS}(A)$ of intersections of all GSS points in a component. It is not the family of the component subsets themselves.
The complete equality **Exact declaration**	Every member of either family is a member of the other, as the same subset of A. This compares representation kernels with whole-component intersections, not components with individual functional kernels.

Exact surrounding source context (separate excerpts)

Source lines 27–27:

```
variable (A : Type u) [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A]
```

Source lines 28–28:

```
variable [Nontrivial A]
```

Exact declaration for the source guide

```
theorem primitiveKernelFamily_eq_primitiveGeometricSpectrumFamily :
    primitiveKernelFamily A = NormedSpace.primitiveGeometricSpectrumFamily ℂ A := by
```

The irreducible spectrum and the geometric spectrum

`MathlibAnnex.Analysis.CStarAlgebra.irreducibleSpectrumHomeomorph`

def

Identifies irreducible representation classes with components of the geometric structure space.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity. There is a specified homeomorphism

$$\Phi_A : \widehat{A} \longrightarrow \mathrm{GS}(A)$$

from its independent irreducible representation spectrum to its geometric spectrum. For a pure state φ and its GNS representation π_φ, its point map is

$$\Phi_A([\pi_\varphi]) = C(\ker \varphi). \tag{1}$$

Here $S(A)$ consists of kernels of extreme points of the closed unit ball of the Banach dual of A, with convexity over $\mathbb{R}$. The extreme-functional kernel characterization identifies this family with the original geometric structure space, with closure $c_A(\mathcal{U}) = \{I : \bigcap_{J \in \mathcal{U}} J \subseteq I\}$. The component is taken for the finite-chain relation generated by equal points and nonclosed pairs.

Definition

The source $\widehat{A}$ is the irreducible spectrum: pure norm-one positive bounded complex-linear functionals on A, modulo unitary equivalence of their GNS representations restricted to A. Equality of representation kernels is not the equivalence relation. Its topology is induced by the kernel map $\kappa([\pi]) = \ker \pi$ into the primitive spectrum with hull-kernel topology.

The target $\mathrm{GS}(A)$ consists of components of $S(A)$. For a component C, put $J_C = \bigcap_{I \in C} I$. Its closure is

$$\overline{\mathcal{D}} = \left\{ C : \bigcap_{D \in \mathcal{D}} J_D \subseteq J_C \right\},$$

with empty intersection A. The geometric-spectrum topology has this closure.

The spectral bijection sends $[\pi_\varphi]$ to $C(\ker \varphi)$. Its inverse assigns the representation class of a pure state whose functional kernel lies in the given component, independently of the chosen pure state. The closure-preservation theorem applies to every subset. The resulting closure-space homeomorphism, for the two specified ordinary topologies, is Φ_A.

Assumptions

The algebra is nonzero and has the compatible C*-order required by the pure-state and primitive-kernel constructions. The ordinary topology on $\widehat{A}$ is the kernel-induced topology just described; no topology from a quotient of the pure-state space is substituted. No injectivity of κ is imposed.

Conclusion

The output is the specified ordinary homeomorphism with action (1), for the kernel-induced topology on $\widehat{A}$ and the stated topology on $\mathrm{GS}(A)$. Its inverse is the inverse of this same bijection. No injectivity of the kernel map into the primitive spectrum is asserted.

Main citations

- Exact full definition: `irreducibleSpectrumHomeomorph`.
- Independent spectrum and topology, with its point construction in Spectrum/Basic.lean: the source of the displayed homeomorphism.
- SpectralComparison.lean, `toGeometricSpectrum_ofRepresentation`: the formula for any irreducible action and any nonzero coefficient pair.
- Ryotaro Tanaka, arXiv:2504.05551v1, Theorem 5.11 and Corollary 5.12: spectral reconstruction and the ordinary topological conclusion.
- image_irreducibleSpectrumClosure: full closure preservation.
- irreducibleSpectrumClosureHomeomorph: the bundled map and inverse.

Lean source signature (exact)

The full definition below is copied from the pinned source.

```
def irreducibleSpectrumHomeomorph :
    @_root_.Homeomorph (IrreducibleSpectrum A) (NormedSpace.GeometricSpectrum ℂ A)
      inferInstance (geometricSpectrumTopology A) :=
  (irreducibleSpectrumClosureHomeomorph A).toTopologicalHomeomorph
    (isTopologizable_irreducibleSpectrumClosure A) (isTopologizable_geometricSpectrumClosure A)
```

In the source	Mathematical meaning
Exact declaration `A` and the surrounding instances	The nonzero nonunital complex C*-algebra with compatible order.
Exact declaration `IrreducibleSpectrum A`	The pure-GNS representation classes, with the topology induced by their representation kernels; not the quotient topology of pure states.

In the source	Mathematical meaning
Exact declaration `NormedSpace.GeometricSpectrum ℂ A`	Components of the geometric structure space, with the ordinary topology having the stated component-intersection closure.
Exact declaration `irreducibleSpectrumClosureHomeomorph A`	The specific bijection $[\pi_\varphi] \mapsto C(\ker \varphi)$ with its inverse and full closure-continuity data.
Exact declaration `.toTopologicalHomeomorph`	Uses the two named topologizability guarantees, without replacing either closure by a new one.

Exact surrounding source context (separate excerpts)

Source lines 28–28:

```
variable (A : Type u) [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A]
```

Source lines 29–29:

```
variable [Nontrivial A]
```

The primitive spectrum and the primitive geometric spectrum

`MathlibAnnex.Analysis.CStarAlgebra.primitiveSpectrumHomeomorph`

def

Identifies the primitive ideal spectrum with the primitive geometric spectrum.

Statement

Let $A \neq \{0\}$ be a complex C*-algebra, possibly without an identity. There is a specified ordinary homeomorphism

$$\Psi_A : \mathrm{Prim}(A) \longrightarrow \mathrm{PS}(A),$$

from the independently defined primitive ideal spectrum to the primitive geometric spectrum. Both spaces have points that are subsets of A, and this homeomorphism retains each such subset. The two spectra have the hull-kernel topologies described below.

Definition

Write $\mathrm{Prim}(A)$ for the primitive spectrum: its points are subsets $P = \ker \pi_\varphi \subseteq A$, where φ is a pure norm-one positive bounded complex-linear functional and π_φ is its GNS representation of the original algebra. This is a representation kernel, not the zero set of φ.

For the geometric side, let $S(A)$ be the family of kernels of extreme points of the closed unit ball of the Banach dual of A, with convexity over $\mathbb{R}$. The identification of this family with the face-and-support geometric structure is the conclusion of the extreme-functional kernel characterization. Its closure is $c_A(\mathcal{U}) = \{I \in S(A) : \bigcap_{J \in \mathcal{U}} J \subseteq I\}$. Components use the equivalence relation generated by equality and nonclosed two-point sets. For a component C, put $J_C = \bigcap_{I \in C} I$ and $\mathrm{PS}(A) = \{J_C : C \text{ a component}\}$, with repeated subsets identified.

The source topology on $\mathrm{Prim}(A)$ and target topology on $\mathrm{PS}(A)$ have the hull-kernel closures

$$\overline{\mathcal{U}} = \left\{ P : \bigcap_{Q \in \mathcal{U}} Q \subseteq P \right\}, \qquad \overline{\mathcal{V}} = \left\{ J : \bigcap_{K \in \mathcal{V}} K \subseteq J \right\}.$$

The ambient space in each formula is the indicated spectrum; an empty intersection is A.

The map Ψ_A sends a primitive kernel to the identical subset regarded as a primitive geometric point; its inverse also retains the subset. The subset-preserving closure homeomorphism supplies this bijection with closure continuity in both directions. The primitive-spectrum topologizability

theorem and geometric-spectrum topologizability theorem supply ordinary topologies with exactly these closures, making the same bijection an ordinary homeomorphism.

Assumptions

The input is a nonzero complex C*-algebra, not necessarily unital, with the compatible partial order and star order in the displayed scope. Both spectra carry the hull-kernel topologies stated above. The correspondence of the two families and the topologizability results are provided by the cited source declarations, not additional assumptions on A.

Conclusion

The output is the ordinary homeomorphism Ψ_A with the same underlying subset in both directions. It includes continuity of that map and its inverse. It does not identify a functional kernel with a representation kernel.

Main citations

- Exact full definition: `primitiveSpectrumHomeomorph`.
- Primitive family equality and SpectralComparison.lean, `primitiveSpectrumEquivGeometric`: the forward and inverse carrier maps.
- Spectrum/Topology.lean: the independent hull–kernel topology and its finite-union proof.
- SpectralHomeomorph.lean, `primitiveSpectrumClosureHomeomorph`, `image_primitiveSpectrumClosure`, `image_symm_primitiveGeometricSpectrumClosure`: the two closure directions and their ordinary topological conversion.
- Ryotaro Tanaka, arXiv:2504.05551v1, Theorem 5.11 and Corollary 5.12: the primitive-spectrum comparison and topologizability.
- primitiveSpectrumClosureHomeomorph: the subset-preserving closure homeomorphism.
- isTopologizable_primitiveGeometricSpectrumClosure: the target topology guarantee.

Lean source signature (exact)

The full definition below is copied from the pinned source.

```
def primitiveSpectrumHomeomorph :
    @_root_.Homeomorph (PrimitiveSpectrum A) (NormedSpace.PrimitiveGeometricSpectrum ℂ A)
      inferInstance (primitiveGeometricSpectrumTopology A) :=
  (primitiveSpectrumClosureHomeomorph A).toTopologicalHomeomorph
    (isTopologizable_primitiveSpectrumClosure A)
    (isTopologizable_primitiveGeometricSpectrumClosure A)
```

In the source	Mathematical meaning
Exact declaration `A` and the surrounding instances	The same nonzero complex C*-algebra with compatible order; a unit is not required.
Exact declaration `PrimitiveSpectrum A` and `primitiveGeometricSpectrumTopology A`	The representation-kernel spectrum and the stated hull-kernel topology on the component-intersection family.
Exact declaration `primitiveSpectrumClosureHomeomorph A`	The same subset-preserving bijection with full closure continuity in both directions.
Exact declaration `.toTopologicalHomeomorph` and the two `isTopologizable` inputs	The same bijection is a homeomorphism for the two ordinary topologies with the specified closures.

Exact surrounding source context (separate excerpts)

Source lines 28–28:

```
variable (A : Type u) [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A]
```

Source lines 29–29:

```
variable [Nontrivial A]
```

A specified GSS homeomorphism preserves compact-operator equality

`MathlibAnnex.Analysis.CStarAlgebra.eq_compactOperatorStarSubalgebra_iff_of_geometricStructure_homeomorph`

theorem

Preserves equality with the compact operators under the specified full geometric homeomorphism.

Statement

Let H, K be complex Hilbert spaces, and let $A_1 \subseteq \mathcal{B}(H)$ and $A_2 \subseteq \mathcal{B}(K)$ be nonzero norm-closed complex $*$-subalgebras acting irreducibly on H and K, respectively. Write $S(D)$ for the geometric structure space of D, with hull–kernel closure c_D, and write $\mathcal{K}(L)$ for the algebra of compact operators on L.

For each maximal proper face of the closed unit ball of D, take the union of the kernels of all bounded complex-linear functionals of norm one that equal 1 on that face. The distinct subsets so obtained are the points of $S(D)$, by the geometric structure definition and the norm-one support characterization.

Suppose that $e : S(A_1) \to S(A_2)$ is a homeomorphism of closure spaces. Equivalently, e is a bijection preserving closure for all subsets:

$$e(c_{A_1}(E)) = c_{A_2}(e(E)) \qquad (E \subseteq S(A_1)). \qquad (1)$$

Then

$$A_1 = \mathcal{K}(H) \iff A_2 = \mathcal{K}(K). \qquad (2)$$

Assumptions

The Hilbert spaces are complete. Irreducibility supplies nonzero Hilbert spaces. No separability or finite-dimensionality is assumed. The homeomorphism uses the same e and its inverse. In terms of continuity, (1) requires

$$e(c_{A_1}(E)) \subseteq c_{A_2}(e(E)) \qquad (E \subseteq S(A_1)),$$

and

$$e^{-1}(c_{A_2}(F)) \subseteq c_{A_1}(e^{-1}(F)) \qquad (F \subseteq S(A_2)).$$

No hypothesis says that e directly maps $N(A_1)$ onto $N(A_2)$, the normal parts consisting of their coefficient kernels.

Conclusion

The conclusion is equality with the compact-operator algebra on each original Hilbert space, not merely containment or abstract $*$-isomorphism. The specified geometric homeomorphism is not asserted to be induced by a $*$-isomorphism. Finite continuity alone is not the hypothesis.

Main citations

Compact-equality transport is the equality branch for the specified full homeomorphism. It uses faithful normal-part transport, equality transport between normal parts, and equality reflection through a faithful irreducible image. The distinct compact-inclusion, proper-inclusion, and zero-intersection results are not substituted for equality.

Lean source signature (exact)

Exact source and declaration. Only the literal signature is displayed here; the linked source retains the proof.

```
theorem eq_compactOperatorStarSubalgebra_iff_of_geometricStructure_homeomorph :
    S = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H ↔
      T = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ K
```

In the source, `S` and `T` denote the algebras A_1 and A_2 used here, respectively.

In the source	Mathematical meaning

Hilbert spaces
`universe u v w`
`variable {H : Type u} [NormedAddCommGroup H]`
`[InnerProductSpace ℂ H] [CompleteSpace H]`
`variable {K : Type v} [NormedAddCommGroup K]`
`[InnerProductSpace ℂ K] [CompleteSpace K]`
	The complete complex Hilbert spaces H, K may have independent universe levels.

Closed operator algebras
`variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))`
`variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))`
`variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]`
	The source names S, T denote $A_1 \subseteq \mathcal{B}(H)$ and $A_2 \subseteq \mathcal{B}(K)$. Each is a norm-closed complex $*$-subalgebra.

Irreducible actions
`variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype S))`
`variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype T))`
	The inclusion representations of A_1 on H and A_2 on K are nonzero and irreducible. No finite-dimensionality or separability is assumed.

In the source	Mathematical meaning

Specified geometric homeomorphism
`variable (e : ClosureSpace.Homeomorph`
`  (NormedSpace.geometricStructureClosure ℂ S)`
`  (NormedSpace.geometricStructureClosure ℂ T))`
	The same given $e : S(A_1) \to S(A_2)$ is a homeomorphism of the ambient geometric closure spaces. It is not assumed to send one normal part directly onto the other.

Inherited algebra and order structures
`local instance : NonUnitalCStarAlgebra T :=`
`  NonUnitalStarSubalgebra.nonUnitalCStarAlgebra T`
`  (h_closed := inferInstance)`
`local instance : PartialOrder T :=`
`_root_.CStarAlgebra.spectralOrder T`
`local instance : StarOrderedRing T :=`
`_root_.CStarAlgebra.spectralOrderedRing T`
	The closed operator algebra A_2 carries its inherited C*-algebra structure and compatible order. These local instances are structures on the same algebra; they add no hypothesis to the displayed theorem.

Effective theorem inputs `include hS hT e in`	The two irreducibility assumptions and the specified e are included in the theorem even though its displayed conclusion mentions only the algebras and compact operators.

Equality with compact operators
`theorem`
`eq_compactOperatorStarSubalgebra_iff_of_geometricSt`
`ructure_homeomorph :`
`  S =`
`ContinuousLinearMap.compactOperatorStarSubalgebra ℂ`
`H ↔`
`  T =`
`ContinuousLinearMap.compactOperatorStarSubalgebra ℂ`
`K := by`
	The conclusion is exactly
	$$A_1 = \mathcal{K}(H) \quad \Longleftrightarrow \quad A_2 = \mathcal{K}(K).$$
	It concerns equality on the original Hilbert spaces, not merely compact containment or an abstract isomorphism.

Homeomorphism fields
`structure Homeomorph (c : ClosureSpace α) (d :`
`ClosureSpace β) extends α ≃ β where`
`  continuous_toFun : c.Continuous d toFun`
`  continuous_invFun : d.Continuous c invFun`
	The underlying equivalence is the same e with inverse e^{-1}. The forward continuity field states, for every $E \subseteq S(A_1)$,
	$$e(c_{A_1}(E)) \subseteq c_{A_2}(e(E)).$$
	The inverse continuity field states, for every $F \subseteq S(A_2)$,
	$$e^{-1}(c_{A_2}(F)) \subseteq c_{A_1}(e^{-1}(F)).$$
	The inverse is not an independently selected map.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
universe u v w
variable {H : Type u} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable {K : Type v} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]
```

Exact surrounding source:

```
variable (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H))
variable (T : NonUnitalStarSubalgebra ℂ (K →L[ℂ] K))
variable [IsClosed (S : Set (H →L[ℂ] H))] [IsClosed (T : Set (K →L[ℂ] K))]
variable (hS : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
S))
variable (hT : NonUnitalCStarRepresentation.IsIrreducible (NonUnitalStarSubalgebraClass.subtype
T))
variable (e : ClosureSpace.Homeomorph
  (NormedSpace.geometricStructureClosure ℂ S)
  (NormedSpace.geometricStructureClosure ℂ T))

local instance : NonUnitalCStarAlgebra T :=
  NonUnitalStarSubalgebra.nonUnitalCStarAlgebra T
    (h_closed := inferInstance)
local instance : PartialOrder T := _root_.CStarAlgebra.spectralOrder T
local instance : StarOrderedRing T := _root_.CStarAlgebra.spectralOrderedRing T
```

Exact surrounding source:

```
include hS hT e in
```

Related exact source for the reading guide

Equality with compact operators · Fixed source

```
theorem eq_compactOperatorStarSubalgebra_iff_of_geometricStructure_homeomorph :
    S = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H ↔
      T = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ K := by
```

Homeomorphism fields · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```

A singleton irreducible spectrum is witnessed by a faithful, fully normal representation

`MathlibAnnex.Analysis.CStarAlgebra.exists_forall_eq_irreducibleSpectrum_iff_exists_faithful_representation`

theorem

Characterizes a singleton irreducible spectrum by one faithful representation with a full normal part.

Statement

Let A be a nonzero complex C^*-algebra, not necessarily unital, with the compatible order structure used in its pure-state construction. Write $\widehat{A}$ for its irreducible spectrum: the space of unitary-equivalence classes of irreducible representations, realized here through pure-state GNS representations. It is not the space of primitive ideals obtained by identifying representations with the same kernel.

For a nonzero irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$ on a complex Hilbert space H, write $S(\pi(A))$ for the geometric structure of its image $\pi(A)$, whose points are subsets obtained as unions of supporting kernels for maximal proper real faces of the closed unit ball. The established C^*-algebra identification realizes these points as kernels of extreme norm-one functionals. Use the inner product linear in its second variable. For $x, y \in H \setminus \{0\}$, define the bounded linear functional $f_{x,y} : \pi(A) \to \mathbb{C}$ by $f_{x,y}(t) = \langle y, t(x) \rangle$. Its normal part consists of these coefficient kernels

$$N(\pi(A)) = \{\ker f_{x,y} : x, y \in H \setminus \{0\}\}. \qquad (1)$$

The following are equivalent:

(i) The irreducible spectrum $\widehat{A}$ has exactly one point.

(ii) There are a complex Hilbert space H and a nonzero irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$ such that

$$\ker \pi = \{0\}, \qquad N(\pi(A)) = S(\pi(A)).$$

The same Hilbert space and representation satisfy all the conditions in (ii).

Assumptions

The algebra is nontrivial. There is no separability, unitality, finite-dimensionality, or prescribed Hilbert space in the input. Irreducibility on the right includes a nonzero action; it is not a convention that permits the zero representation. The order and star-order structures in the ambient source are

retained. No full-normal-part or faithfulness assumption is hidden in a newly defined representation object.

Here “fully normal” means exactly the equality $N(\pi(A)) = S(\pi(A))$ of sets of geometric points in (ii). It is not a claim about normal functionals on a von Neumann algebra, nor about the ultraweak continuity of π.

Conclusion

A singleton unitary-equivalence spectrum is equivalent to the simultaneous existence of faithfulness and a full normal part for one nonzero irreducible representation. Neither condition alone replaces their conjunction. No elementary-algebra classification is asserted here.

Main citations

The singleton-spectrum representation theorem gives the equivalence of (i) and (ii). The pure-state witness theorem supplies one faithful GNS representation with full normal part; the converse for a given representation applies to the same representation in (ii).

Lean source signature (exact)

The preceding `variable` declarations supply the nonzero C*-algebra and its order assumptions. On the right of `↔`, the nested existentials choose one complex Hilbert space and one representation on it. The norm, inner product and completeness all belong to that same space; the local `letI` declarations make these structures available together. The three final conjuncts concern that same representation. The following contiguous prefix ends before the proof assignment.

```
theorem exists_forall_eq_irreducibleSpectrum_iff_exists_faithful_representation :
    (∃ q : IrreducibleSpectrum A, ∀ r : IrreducibleSpectrum A, r = q) ↔
    (∃ (H' : Type u) (instN : NormedAddCommGroup H'),
      letI := instN
      ∃ instI : InnerProductSpace ℂ H',
        letI := instI
        ∃ instC : CompleteSpace H',
          letI := instC
          ∃ pi : NonUnitalCStarRepresentation A H',
            pi.IsIrreducible ∧ Function.Injective pi ∧ normalPart (NonUnitalStarAlgHom.range
pi) = Set.univ)
```

In the source	Mathematical meaning

Nonzero algebra and compatible order
`variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]`
`variable [PartialOrder A] [StarOrderedRing A]`
	The input is the nonzero complex C*-algebra A. The partial order and star-order structure are those used for positivity and pure states. Unitality and separability are not required.

In the source	Mathematical meaning
A singleton irreducible spectrum `(∃ q : IrreducibleSpectrum A, ∀ r : IrreducibleSpectrum A, r = q) ↔`	The left side says that $\widehat{A}$ has exactly one point, including existence of that point. Its points are unitary-equivalence classes of irreducible representations; this is not a statement that their kernels alone coincide.

One Hilbert space and one representation
`(∃ (H' : Type u) (instN : NormedAddCommGroup H'),`
`  letI := instN`
`  ∃ instI : InnerProductSpace ℂ H',`
`  letI := instI`
`  ∃ instC : CompleteSpace H',`
`  letI := instC`
`  ∃ pi : NonUnitalCStarRepresentation A H',`
	One complex Hilbert space H is chosen, with its norm, compatible inner product and completeness. One $*$-representation $\pi : A \to \mathcal{B}(H)$ is then chosen on this same space. All conditions in the next row concern this same π.

Irreducibility, faithfulness, and the full normal part `pi.IsIrreducible ∧ Function.Injective pi ∧ normalPart (NonUnitalStarAlgHom.range pi) = Set.univ) := by`	The representation π is nonzero and irreducible, is faithful, and has full normal part: $$\ker \pi = \{0\},$$ $$N(\pi(A)) = S(\pi(A)).$$ The latter equality is with the whole geometric structure space of its image $\pi(A)$, not with the operator algebra $\mathcal{B}(H)$.

Coefficient kernels in the image algebra
`def normalPart (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) :`
`  Set (NormedSpace.GeometricStructureSpace ℂ S) :=`
`  {I | ∃ x y : H, x ≠ 0 ∧ y ≠ 0 ∧`
`  {a : S | inner ℂ y ((a : H →L[ℂ] H) x) = 0} = I.val}`
	Apply the definition to the image algebra $\pi(A)$. Its normal part consists of the coefficient kernels
	$$N(\pi(A)) = \{\ker f_{x,y} : x, y \in H \setminus \{0\}\},$$
	where $f_{x,y} : \pi(A) \to \mathbb{C}$ is given by $f_{x,y}(t) = \langle y, t(x)\rangle$, as in (1). The kernel is identified with the underlying subset of the geometric point I.

Exact surrounding source context (separate excerpts)

SpectrumSingleton.lean:

```
universe u v
variable {A : Type u} [NonUnitalCStarAlgebra A] [Nontrivial A]
variable [PartialOrder A] [StarOrderedRing A]
```

The existential Hilbert space has source level `u`, the same fixed level as A. The separate ambient level `v` does not enlarge this theorem's model quantifier.

Related exact source for the reading guide

Coefficient kernels in the image algebra · Fixed source

```
def normalPart (S : NonUnitalStarSubalgebra ℂ (H →L[ℂ] H)) :
    Set (NormedSpace.GeometricStructureSpace ℂ S) :=
  {I | ∃ x y : H, x ≠ 0 ∧ y ≠ 0 ∧
    {a : S | inner ℂ y ((a : H →L[ℂ] H) x) = 0} = I.val}
```

Irreducibility, faithfulness, and the full normal part · Fixed source

```
            pi.IsIrreducible ∧ Function.Injective pi ∧ normalPart (NonUnitalStarAlgHom.range
pi) = Set.univ) := by
```

Elementary C*-algebras from singleton spectrum and finite conjugation

`MathlibAnnex.Analysis.CStarAlgebra.exists_pureState_compactOperatorModel_iff_singleton_finiteConjugation`

theorem

Characterizes a pure-GNS compact model by singleton spectrum and continuous finite conjugation.

Statement

Let A be a nonzero complex C*-algebra with its compatible order. Write $\widehat{A}$ for its irreducible spectrum, the pure states modulo unitary equivalence of their GNS representations, and $S(A)$ for its geometric structure closure space. For $I \in S(A)$, let $C_A(I)$ be its finite-chain component, with the closure induced from $S(A)$. For a complex Hilbert space H, let $\mathcal{K}(H)$ denote the algebra of compact operators on H.

The following conditions are equivalent.

1. There is a pure state φ of A whose GNS representation $\pi_\varphi : A \to \mathcal{B}(H_\varphi)$ is faithful and satisfies

$$\pi_\varphi(A) = \mathcal{K}(H_\varphi). \tag{1}$$

2. The irreducible spectrum $\widehat{A}$ is a singleton, and there is a point $I \in S(A)$ whose component $C_A(I)$ satisfies the conjugation condition below.

For a closure space Y, write c_Y for its closure operator. For the geometric structure spaces $S(A)$ and $S(B)$, abbreviate these operators to c_A and c_B, respectively. Write $r(E)$ for the image of a subset E under a map r. The condition concerns every closure space X that is homeomorphic to $S(B)$ for some nonzero complex C*-algebra B. Here $S(B)$ is the same geometric structure construction used for $S(A)$. A homeomorphism $r : S(B) \to X$ is a bijection preserving closure on every subset:

$$r\big(c_B(E)\big) = c_X\big(r(E)\big) \qquad (E \subseteq S(B)).$$

The algebra B may vary; no unit, separability assumption or chosen representation of B is part of this condition.

For every such X, consider each of its finite-chain components D, with the induced closure $c_D(E) = D \cap c_X(E)$ for $E \subseteq D$, and all bijections

$$f : D \longrightarrow C_A(I), \qquad g : D \longrightarrow D.$$

A finite-chain component consists of points joined by a finite chain of linked pairs. Two points are linked if they are equal or their two-point set is not closed. These components are taken in the specified closure space, not as connected components of a topology.

A bijection $r : Y \to Z$ of closure spaces is a finite homeomorphism if it preserves closure on every finite subset:

$$r\big(c_Y(F)\big) = c_Z\big(r(F)\big) \qquad (F \subseteq Y \text{ finite}). \qquad (2)$$

A map $r : Y \to Z$ is continuous if

$$r\big(c_Y(E)\big) \subseteq c_Z\big(r(E)\big) \qquad (E \subseteq Y). \qquad (3)$$

The required condition is that whenever f and g are finite homeomorphisms and are also continuous in the sense of (3), transporting g to $C_A(I)$ by f gives a homeomorphism. More explicitly, both

$$\begin{aligned} h &= f \circ g \circ f^{-1}, \\ h^{-1} &= f \circ g^{-1} \circ f^{-1} \end{aligned} \qquad : C_A(I) \longrightarrow C_A(I) \qquad (4)$$

must be continuous on arbitrary subsets. This is required for every X, every component D and every pair f, g described above. Continuity of f^{-1} or g^{-1} on arbitrary subsets is not assumed. The inverses in (4) are those of the same bijections f and g.

Assumptions

A is a nonzero complex C*-algebra with its compatible order. It need not be unital or separable, and no finite-dimensionality assumption is imposed. The geometric condition requires both a singleton irreducible spectrum and the existence of $I \in S(A)$; neither condition is interpreted vacuously. The source guide records the fixed size bounds for the spaces in the conjugation condition.

Conclusion

The two geometric conditions characterize the existence of a pure state whose faithful GNS representation identifies A, as a C*-algebra, with $\mathcal{K}(H_\varphi)$. The Hilbert space and representation in this identification are those of the same state φ in (1).

Main citations

The compact-model characterization states this equivalence in every Hilbert dimension. Its two directions use the existence of a compact-operator model and the conjugation property of compact-operator models. The conjugation condition specifies the inputs and the two continuous maps in (4); the realization of closure spaces by C*-algebras specifies the spaces X. The conjugate and its inverse act by the formulas in (4).

Lean source signature (exact)

The preceding `variable` declarations supply `A` and its algebra and order structures; `[Nontrivial A]` appears explicitly in the theorem. The outer ↔ compares the GNS-model condition with a conjunction ∧: a singleton irreducible spectrum and the existence of a geometric point `I` with the component condition. The following is the contiguous prefix of the selected declaration, ending before the proof assignment.

```
theorem exists_pureState_compactOperatorModel_iff_singleton_finiteConjugation
    [Nontrivial A] :
    (∃ phi : NonUnitalPureState A, IsCompactOperatorModel phi.gnsRepresentation) ↔
      (∃ q : IrreducibleSpectrum A, ∀ r : IrreducibleSpectrum A, r = q) ∧
      ∃ I : NormedSpace.GeometricStructureSpace ℂ A,
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component
I)).HasContinuousFiniteConjugation
            (ClosureCat.IsCStarRealizable.{u, max u w})
```

In the source	Mathematical meaning

Ambient algebra
`universe u v w uA uL`
`variable {A : Type u} [NonUnitalCStarAlgebra A]`
`variable [PartialOrder A] [StarOrderedRing A]`
	The complex C*-algebra A with its compatible order; the size parameters are fixed for the selected theorem. No identity is required.

Nonzero algebra and one GNS model
`[Nontrivial A] :`
`(∃ phi : NonUnitalPureState A,`
`IsCompactOperatorModel phi.gnsRepresentation) ↔`
	The nonzero algebra A and one pure state φ whose same GNS representation $\pi_\varphi : A \to \mathcal{B}(H_\varphi)$ satisfies (1).

Compact-operator model
`def IsCompactOperatorModel [NonUnitalCStarAlgebra A]`
`(e : A →⋆ₙₐ[ℂ] (H →L[ℂ] H)) : Prop :=`
`Function.Injective e ∧`
`(∀ a : A, IsCompactOperator (e a)) ∧`
`∀ T : H →L[ℂ] H, IsCompactOperator T → ∃ a : A, e a = T`
	Applied to π_φ, this means faithfulness and the two inclusions giving
	$$\pi_\varphi(A) = \mathcal{K}(H_\varphi).$$
	Every image operator is compact, and every compact operator is an image of this same representation.

Singleton spectrum
`(∃ q : IrreducibleSpectrum A, ∀ r :`
`IrreducibleSpectrum A, r = q) ∧`
	The irreducible spectrum $\widehat{A}$ has exactly one point, including its existence. This is conjoined with the component condition that follows.

Point and applied component condition
`∃ I : NormedSpace.GeometricStructureSpace ℂ A,`
`((NormedSpace.geometricStructureClosure ℂ A).subspace`
`((NormedSpace.geometricStructureClosure ℂ A).component I)).HasContinuousFiniteConjugation`
`(ClosureCat.IsCStarRealizable.{u, max u w})`
`:= by`
	A point $I \in S(A)$ selects its finite-chain component $C_A(I)$. The ambient closure c_A is restricted to this component, and the resulting closure is the input of the conjugation condition, with precisely the stated class of test spaces.

In the source	Mathematical meaning
Geometric points `def GeometricStructureSpace : Type u := {I : Set E // I ∈ geometricStructureFamily 𝕜 E}`	A point of $S(A)$ is a subset of A belonging to the geometric structure family. Thus the quantified I is a geometric point, not an arbitrary subset or a normal-part point.
Ambient geometric closure `def geometricStructureClosure : ClosureSpace (GeometricStructureSpace 𝕜 E) := ClosureSpace.HullKernel.space (geometricStructureFamily 𝕜 E) (univ_notMem_geometricStructureFamily 𝕜 E)`	The hull–kernel closure c_A on the whole space $S(A)$, rather than the relative closure on a normal part.
Finite-chain component `def component (c : ClosureSpace α) (x : α) : Set α := {y | c.Related x y}`	The points related to I by finite chains form $C_A(I)$. This is not a claim about ordinary topological connectedness.
Induced component closure `def subspace (c : ClosureSpace α) (A : Set α) : ClosureSpace A := c.induced (Subtype.val : A → α)`	Restricting the ambient closure along inclusion gives the closure on the component. For $E \subseteq C_A(I)$, $$c_{C_A(I)}(E) = C_A(I) \cap c_A(E).$$
Class of test spaces `def IsCStarRealizable.{uA, uX} (X : ClosureCat.{uX}) : Prop := ∃ (A : Type uA) (instA : NonUnitalCStarAlgebra A), letI : NonUnitalCStarAlgebra A := instA Nontrivial A ∧ Nonempty (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A) X.str)`	A test space X is homeomorphic, for full closure, to $S(B)$ for some nonzero complex C*-algebra B. The bound algebra in this definition is the realizing B, not necessarily the original A.
All tests, components and first map `def HasContinuousFiniteConjugation (c : ClosureSpace α) (D : ClosureCat.{s} → Prop) : Prop := ∀ (L : ClosureCat.{s}), D L → ∀ t : L, ∀ f : FiniteHomeomorph (L.str.subspace (L.str.component t)) c, (L.str.subspace (L.str.component t)).Continuous c f.toEquiv →`	Apply the closure parameter to $c_{C_A(I)}$ and the test class to the preceding realizable spaces. For every such X, every chosen point selects a component $D \subseteq X$ with induced closure; the map $f : D \to C_A(I)$ is a finite homeomorphism and is fully continuous in the forward direction.
Second map and the two conclusions `∀ g : FiniteHomeomorph (L.str.subspace (L.str.component t)) (L.str.subspace (L.str.component t)), (L.str.subspace (L.str.component t)).Continuous (L.str.subspace (L.str.component t)) g.toEquiv → c.Continuous c (f.conjugate g).toEquiv ∧ c.Continuous c (f.conjugate g).toEquiv.symm`	For every finite homeomorphism $g : D \to D$ with forward full continuity, both the conjugate and its actual inverse are fully continuous on $C_A(I)$: $$h = f \circ g \circ f^{-1},$$ $$h^{-1} = f \circ g^{-1} \circ f^{-1}.$$ These are the same maps in (4); no full continuity of f^{-1} or g^{-1} is an input.

In the source	Mathematical meaning

Finite homeomorphism
`structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where`
`  finitelyContinuous_toFun : c.FinitelyContinuous d toFun`
`  finitelyContinuous_invFun : d.FinitelyContinuous c invFun`
	One bijection with finite continuity for it and its inverse. For f and g this gives the finite-set equality (2) in their respective induced closure spaces.

Finite continuity
`def FinitelyContinuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=`
`  ∀ s : Set α, s.Finite → f '' c.closure s ⊆ d.closure (f '' s)`
	Closure preservation by inclusion for every finite subset, including the empty subset. Applying this in both directions to the same bijection gives (2).

Full continuity
`def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=`
`  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`
	The inclusion (3) for every subset, without a finiteness restriction. In the conclusion the source and target closure are both $c_{C_A(I)}$.

The same conjugation
`def FiniteHomeomorph.conjugate (f : FiniteHomeomorph c d)`
`  (g : FiniteHomeomorph c c) : FiniteHomeomorph d d :=`
`  f.symm.trans (g.trans f)`
	The equivalence obtained by composing the same f^{-1}, then g, then f. Its forward map is $h = f \circ g \circ f^{-1}$ and its inverse is $h^{-1} = f \circ g^{-1} \circ f^{-1}$, not an independently chosen inverse.

Size convention: in `.{u, max u w}`, the theorem's algebra A and each realizing algebra B have level u; the underlying set of X has level $\max(u, w)$. The conjugation test ranges over all the stated spaces at these fixed levels, not over all universes at once.

Exact surrounding source context (separate excerpts)

Exact ambient source:

```
universe u v w uA uL
variable {A : Type u} [NonUnitalCStarAlgebra A]
variable [PartialOrder A] [StarOrderedRing A]
```

Exact ambient source installs the C*-algebra structure and closedness of representation images locally; this is separate from the selected declaration:

```
attribute [local instance] HomKernelQuotient.imageCStarAlgebra HomKernelQuotient.isClosed_image
```

Related exact source for the reading guide

Compact-operator model · Fixed source

```
def IsCompactOperatorModel [NonUnitalCStarAlgebra A]
    (e : A →⋆ₙₐ[ℂ] (H →L[ℂ] H)) : Prop :=
  Function.Injective e ∧
    (∀ a : A, IsCompactOperator (e a)) ∧
    ∀ T : H →L[ℂ] H, IsCompactOperator T → ∃ a : A, e a = T
```

Class of test spaces · Fixed source

```
def IsCStarRealizable.{uA, uX} (X : ClosureCat.{uX}) : Prop :=
  ∃ (A : Type uA) (instA : NonUnitalCStarAlgebra A),
    letI : NonUnitalCStarAlgebra A := instA
    Nontrivial A ∧ Nonempty
      (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A) X.str)
```

Point and applied component condition · Fixed source

```
      ∃ I : NormedSpace.GeometricStructureSpace ℂ A,
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component
I)).HasContinuousFiniteConjugation
            (ClosureCat.IsCStarRealizable.{u, max u w}) := by
```

Geometric points · Fixed source

```
def GeometricStructureSpace : Type u := {I : Set E // I ∈ geometricStructureFamily 𝕜 E}
```

Ambient geometric closure · Fixed source

```
def geometricStructureClosure : ClosureSpace (GeometricStructureSpace 𝕜 E) :=
  ClosureSpace.HullKernel.space (geometricStructureFamily 𝕜 E)
    (univ_notMem_geometricStructureFamily 𝕜 E)
```

Finite-chain component · Fixed source

```
def component (c : ClosureSpace α) (x : α) : Set α := {y | c.Related x y}
```

All tests, components and first map · Fixed source

```
def HasContinuousFiniteConjugation (c : ClosureSpace α) (D : ClosureCat.{s} → Prop) : Prop :=
  ∀ (L : ClosureCat.{s}), D L → ∀ t : L,
    ∀ f : FiniteHomeomorph (L.str.subspace (L.str.component t)) c,
      (L.str.subspace (L.str.component t)).Continuous c f.toEquiv →
      ∀ g : FiniteHomeomorph (L.str.subspace (L.str.component t))
          (L.str.subspace (L.str.component t)),
        (L.str.subspace (L.str.component t)).Continuous
          (L.str.subspace (L.str.component t)) g.toEquiv →
        c.Continuous c (f.conjugate g).toEquiv ∧
          c.Continuous c (f.conjugate g).toEquiv.symm
```

The same conjugation · Fixed source

```
def FiniteHomeomorph.conjugate (f : FiniteHomeomorph c d)
    (g : FiniteHomeomorph c c) : FiniteHomeomorph d d :=
  f.symm.trans (g.trans f)
```

Induced component closure · Fixed source

```
def subspace (c : ClosureSpace α) (A : Set α) : ClosureSpace A :=
  c.induced (Subtype.val : A → α)
```

Full continuity · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

Finite continuity · Fixed source

```
def FinitelyContinuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, s.Finite → f '' c.closure s ⊆ d.closure (f '' s)
```

Finite homeomorphism · Fixed source

```
structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  finitelyContinuous_toFun : c.FinitelyContinuous d toFun
  finitelyContinuous_invFun : d.FinitelyContinuous c invFun
```

Compact inclusion in every irreducible image

`MathlibAnnex.Analysis.CStarAlgebra.forall_irreducible_compact_le_iff_component_compact_comparison`

theorem

Characterizes compact containment in every irreducible image by forward-continuous component comparisons.

Statement

Let A be a nonzero complex C*-algebra.

For a complex C*-algebra B, write $S(B)$ for its geometric structure space. Its points are the unions of kernels of norm-one bounded complex-linear functionals supporting a maximal proper face of the closed unit ball of B; faces use real convexity. Write c_B for the hull-kernel closure on $S(B)$:

$$c_B(\mathcal{U}) = \left\{ J \in S(B) : \bigcap_{L \in \mathcal{U}} L \subseteq J \right\}, \qquad \mathcal{U} \subseteq S(B).$$

These are the geometric structure and its closure, with the norm-one description justified by the support-functional norm theorem.

For $I \in S(A)$, let $C_A(I)$ be its finite-chain component: the points reachable from I by a finite chain in which adjacent points are equal or form a nonclosed two-point set. This uses the linked-pair relation and its generated components, rather than topological connectedness. Give $C_A(I)$ the induced closure

$$c_{C_A(I)}(E) = C_A(I) \cap c_A(E), \qquad E \subseteq C_A(I).$$

For a complex Hilbert space H, let $\mathcal{K}(H)$ be the algebra of compact operators on H. In particular, $c_{\mathcal{K}(K)}$ denotes closure on $S(\mathcal{K}(K))$, where $\mathcal{K}(K)$ is itself the algebra to which the geometric structure construction is applied.

The following conditions are equivalent:

- For every irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$ on a complex Hilbert space H,

  $$\mathcal{K}(H) \subseteq \pi(A). \qquad (1)$$

- For every $I \in S(A)$, there are a nonzero complex Hilbert space K and a bijection $e : C_A(I) \to S(\mathcal{K}(K))$ such that, for every finite subset $F \subseteq C_A(I)$,

$$e(c_{C_A(I)}(F)) = c_{\mathcal{K}(K)}(e(F)), \qquad (2)$$

and, for the same e,

$$e(c_{C_A(I)}(E)) \subseteq c_{\mathcal{K}(K)}(e(E)) \quad (E \subseteq C_A(I)). \qquad (3)$$

Condition (2) says that e is a finite homeomorphism. Condition (3) is its forward continuity for arbitrary subsets; continuity of e^{-1} for arbitrary subsets is not required. Both K and e may depend on I. Here and below, $e(E)$ denotes the image of a set.

Assumptions

The algebra A is nonzero. Representations are $*$-homomorphisms into bounded operators; a unit is not required, and the source does not impose preservation of a unit. The Hilbert spaces are complete over $\mathbb{C}$. Irreducibility includes a nonzero action, but does not require faithfulness. No separability or finite-dimensionality is assumed. The component carries the closure induced from $S(A)$.

Conclusion

Compact containment $\mathcal{K}(H) \subseteq \pi(A)$ in every irreducible image is equivalent to the component comparisons (2)–(3). The arbitrary-set condition concerns only the forward map. Equality with the compact operators and inverse continuity are the stronger conditions in the compact-image equality characterization.

Main citations

The compact-inclusion characterization states the equivalence. The forward-continuity comparison condition requires one nonzero compact target, finite continuity both ways and full continuity only forward. The pure-state comparison theorem and transport to an independently specified Hilbert space supply the representation correspondence. This is the compact-inclusion condition; identifying it with another abstract definition of Type I remains an external classification boundary, not an additional conclusion here.

The finite-homeomorphism fields, closure continuity, and homeomorphism fields specify the map conditions used here. The closure-transport identity expresses a homeomorphism by equality of closures of arbitrary sets.

Lean source signature (exact)

The preceding `variable` declaration introduces A and its nontriviality. The local spectral order is used for positivity and pure states. The `include hA in` retains nonzero A among this theorem's parameters, although the main excerpt does not repeat these ambient binders. Both separate excerpts are shown in the source context below.

GCRCharacterization.lean. The contiguous declaration prefix ends before the proof assignment.

```
theorem forall_irreducible_compact_le_iff_component_compact_comparison :
    (∀ (K : Type u) [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K],
      ∀ pi : NonUnitalCStarRepresentation A K, pi.IsIrreducible →
        ContinuousLinearMap.compactOperatorStarSubalgebra ℂ K ≤ (NonUnitalStarAlgHom.range pi)) ↔
    ∀ I : NormedSpace.GeometricStructureSpace ℂ A,
      ClosureSpace.HasContinuousFiniteCompactComparison.{u, u}
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component I))
```

The component condition uses the following separate exact definition, including its complete right-hand side: GCRCharacterization.lean.

```
def HasContinuousFiniteCompactComparison.{uH, uX} {X : Type uX}
    (c : ClosureSpace X) : Prop :=
  ∃ (H : Type uH) (instN : NormedAddCommGroup H),
    letI := instN
    ∃ instI : InnerProductSpace ℂ H,
      letI := instI
      ∃ instC : CompleteSpace H,
        letI := instC
        ∃ instT : Nontrivial H,
          letI := instT
          ∃ e : FiniteHomeomorph c
              (NormedSpace.geometricStructureClosure ℂ
                (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)),
            c.Continuous (NormedSpace.geometricStructureClosure ℂ
              (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)) e.toEquiv
```

In the source	Mathematical meaning

Exact declaration
Ambient inputs
`variable {A : Type u} [NonUnitalCStarAlgebra A] [hA : Nontrivial A]`
`attribute [local instance]`
`_root_.CStarAlgebra.spectralOrder`
`_root_.CStarAlgebra.spectralOrderedRing`
	The nonzero complex C*-algebra A. The two local instances give its compatible spectral order, used for positivity and pure states. They do not introduce another choice of order. The separate `include hA in` shown below retains nontriviality in this theorem; no unit is required.

In the source	Mathematical meaning
Exact declaration Representation quantifiers `(∀ (K : Type u) [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K], ∀ pi : NonUnitalCStarRepresentation A K, pi.IsIrreducible →`	Every complex Hilbert space H and every irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$. Here source `K` is the representation space called H in the Statement. Its norm, complex inner product and completeness are required by the three bracketed inputs. Irreducibility includes a nonzero action, not faithfulness.
Exact declaration Representation condition `ContinuousLinearMap.compactOperatorStarSubalgebra ℂ K ≤ (NonUnitalStarAlgHom.range pi)) ↔`	The compact operators are contained in the image of this same representation: $$\mathcal{K}(H) \subseteq \pi(A).$$ This is (1), not image equality. The outer ↔ relates the universal representation condition to the component condition on its right.
Exact declaration Component quantifier `∀ I : NormedSpace.GeometricStructureSpace ℂ A,`	Every point $I \in S(A)$. The model space and comparison map required after this quantifier may depend on I.
Exact declaration Comparison applied to the component `ClosureSpace.HasContinuousFiniteCompactComparison. {u, u} ((NormedSpace.geometricStructureClosure ℂ A).subspace ((NormedSpace.geometricStructureClosure ℂ A).component I))`	Apply the displayed comparison definition to the closure induced on $C_A(I)$. `geometricStructureClosure ℂ A` is the closure c_A on $S(A)$; its `component I` is the point set $C_A(I)$. `subspace` makes that set a closure space, with $$c_{C_A(I)}(E) = C_A(I) \cap c_A(E)$$ for $E \subseteq C_A(I)$. This closure is the argument `c` of the definition above, and its underlying set is `X`.
Exact declaration Model and inner product `∃ (H : Type uH) (instN : NormedAddCommGroup H), letI := instN ∃ instI : InnerProductSpace ℂ H, letI := instI`	The condition requires a model space K, called `H` in this definition, with the following structures. `instN` gives its normed additive group, and `instI` its complex inner product. Each following `letI` uses that choice on the same space, first to form `InnerProductSpace ℂ H`, then to interpret the remaining Hilbert-space conditions.
Exact declaration Completeness and nonzero model `∃ instC : CompleteSpace H, letI := instC ∃ instT : Nontrivial H, letI := instT`	`instC` requires this same K to be complete; `instT` requires $K \neq \{0\}$. Their `letI` declarations retain these structures when forming bounded operators, the compact-operator algebra and its geometric closure. They do not choose a second space.

In the source	Mathematical meaning
Exact declaration Finite comparison map `∃ e : FiniteHomeomorph c (NormedSpace.geometricStructureClosure ℂ (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)),`	There exists a finite homeomorphism $e : C_A(I) \to S(\mathcal{K}(K))$. The second closure argument is $c_{\mathcal{K}(K)}$ on the geometric structure of the compact-operator algebra on this same model K, not a closure on the vectors of K. The finite-homeomorphism fields below give $$e(c_{C_A(I)}(F)) = c_{\mathcal{K}(K)}(e(F))$$ for every finite $F \subseteq C_A(I)$, as in (2).
Exact declaration Forward continuity of the same map `c.Continuous (NormedSpace.geometricStructureClosure ℂ (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)) e.toEquiv`	`e.toEquiv` denotes this same bijection e. This condition is $$e(c_{C_A(I)}(E)) \subseteq c_{\mathcal{K}(K)}(e(E))$$ for every $E \subseteq C_A(I)$, as in (3). There is no additional arbitrary-set continuity condition on e^{-1}.
Exact declaration Finite-continuity definition `def FinitelyContinuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop := ∀ s : Set α, s.Finite → f '' c.closure s ⊆ d.closure (f '' s)`	For a map $r : Y \to Z$ between closure spaces, the condition is $$r(c_Y(F)) \subseteq c_Z(r(F))$$ for finite $F \subseteq Y$. Here source `α`, `β`, `c`, `d`, `f` denote Y, Z, c_Y, c_Z, r. The double prime denotes the set image $r(F)$.
Exact declaration Finite-homeomorphism fields `structure FiniteHomeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where finitelyContinuous_toFun : c.FinitelyContinuous d toFun finitelyContinuous_invFun : d.FinitelyContinuous c invFun`	`extends α ≃ β` supplies a bijection e and its inverse. With $Y = C_A(I)$ and $Z = S(\mathcal{K}(K))$, `toFun` and `invFun` are e and e^{-1}. The two fields impose the finite-set condition in the preceding row on these same maps in opposite directions. Together they express the finite closure equality (2).
Exact declaration Continuity definition `def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop := ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`	For a map $r : Y \to Z$, continuity means $$r(c_Y(E)) \subseteq c_Z(r(E))$$ for every $E \subseteq Y$. Source `α`, `β`, `c`, `d`, `f` denote Y, Z, c_Y, c_Z, r; the double prime denotes the set image $r(E)$. Unlike `FinitelyContinuous`, this definition has no finiteness restriction.

Size scope: the principal equivalence fixes the universe of A and quantifies both representation spaces and compact-model spaces in that same universe. The separately cited transport theorem allows an independently specified Hilbert-space universe; it does not change the quantifiers of this declaration.

Exact surrounding source context (separate excerpts)

GCRCharacterization.lean.

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [hA : Nontrivial A]
attribute [local instance] _root_.CStarAlgebra.spectralOrder
  _root_.CStarAlgebra.spectralOrderedRing
```

GCRCharacterization.lean.

```
include hA in
```

Exact declaration for the source guide

```
theorem forall_irreducible_compact_le_iff_component_compact_comparison :
    (∀ (K : Type u) [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K],
      ∀ pi : NonUnitalCStarRepresentation A K, pi.IsIrreducible →
        ContinuousLinearMap.compactOperatorStarSubalgebra ℂ K ≤ (NonUnitalStarAlgHom.range pi))
↔
    ∀ I : NormedSpace.GeometricStructureSpace ℂ A,
      ClosureSpace.HasContinuousFiniteCompactComparison.{u, u}
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component I)) := by
```

Irreducible dimension equal to a fixed number

`MathlibAnnex.Analysis.CStarAlgebra.forall_irreducible_finrank_eq_iff_component_compact_homeomorph`

theorem

Characterizes one fixed irreducible dimension by finite-dimensional compact component models.

Statement

Let A be a nonzero complex C*-algebra, and let $n \in \mathbb{N}$.

For a complex C*-algebra B, write $S(B)$ for its geometric structure space. Its points are the unions of kernels of norm-one bounded complex-linear functionals supporting a maximal proper face of the closed unit ball of B; faces use real convexity. Write c_B for the hull-kernel closure on $S(B)$:

$$c_B(\mathcal{U}) = \left\{ J \in S(B) : \bigcap_{L \in \mathcal{U}} L \subseteq J \right\}, \qquad \mathcal{U} \subseteq S(B).$$

These are the geometric structure and its closure, with the norm-one description justified by the support-functional norm theorem.

For $I \in S(A)$, let $C_A(I)$ be its finite-chain component: the points reachable from I by a finite chain in which adjacent points are equal or form a nonclosed two-point set. This uses the linked-pair relation and its generated components, rather than topological connectedness. Give $C_A(I)$ the induced closure

$$c_{C_A(I)}(E) = C_A(I) \cap c_A(E), \qquad E \subseteq C_A(I).$$

For a complex Hilbert space H, let $\mathcal{K}(H)$ be the algebra of compact operators on H. In particular, $c_{\mathcal{K}(K)}$ denotes closure on $S(\mathcal{K}(K))$, where $\mathcal{K}(K)$ is itself the algebra to which the geometric structure construction is applied.

The following conditions are equivalent:

- For every irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$ on a complex Hilbert space H,

$$H \text{ is finite-dimensional} \quad \text{and} \quad \dim_{\mathbb{C}} H = n. \qquad (1)$$

- For every $I \in S(A)$, there are a nonzero finite-dimensional complex Hilbert space K with $\dim_{\mathbb{C}} K = n$ and a bijection $e : C_A(I) \to S(\mathcal{K}(K))$ satisfying

$$e(c_{C_A(I)}(E)) = c_{\mathcal{K}(K)}(e(E)) \quad (E \subseteq C_A(I)). \qquad (2)$$

The same model K has dimension n and supports the homeomorphism e. The number n is fixed for all components, while K and e may depend on I. Equality (2) expresses continuity of the same e and e^{-1} for arbitrary subsets; $e(E)$ denotes the image of a set.

Assumptions

The algebra A is nonzero; no assumption $n > 0$ is made. Representations are $*$-homomorphisms into bounded operators, without a requirement of unitality. Irreducibility includes a nonzero action, but does not require faithfulness. All Hilbert spaces are complex and complete. Finite-dimensionality is a conclusion on the representation side and a requirement on the model side. It is separate from the numerical dimension equation. No separability hypothesis is added.

Conclusion

Every irreducible representation space is finite-dimensional of the same dimension n exactly when each geometric component has a compact-operator model of that dimension. For nonzero A, both conditions fail at $n = 0$, as specified by the zero-dimension exclusion and the equivalence above. This does not add positivity of n as a hypothesis.

Main citations

The fixed-dimension characterization specializes the general dimension-condition theorem to equality with n. The finite-dimensional compact-model condition retains both finiteness and the numerical condition on the same target. The same-map upgrade theorem supplies full continuity without replacing the finite comparison or its inverse. The independent-universe dimension transport and the exclusion of zero dimension specify those two scope points.

The finite-homeomorphism fields, closure continuity, and homeomorphism fields specify the map conditions used here. The closure-transport identity expresses a homeomorphism by equality of closures of arbitrary sets.

Lean source signature (exact)

The preceding `variable` declaration introduces A and its nontriviality. The local spectral order is used for positivity and pure states. The `include hA in` retains nonzero A among this theorem's parameters, although the main excerpt does not repeat these ambient binders. Both separate excerpts are shown in the source context below.

HomogeneousCharacterization.lean. The contiguous declaration prefix ends before the proof assignment.

```
theorem forall_irreducible_finrank_eq_iff_component_compact_homeomorph (n : ℕ) :
    (∀ (L : Type u) [NormedAddCommGroup L] [InnerProductSpace ℂ L] [CompleteSpace L],
      ∀ pi : NonUnitalCStarRepresentation A L, pi.IsIrreducible →
        FiniteDimensional ℂ L ∧ Module.finrank ℂ L = n) ↔
    ∀ I : NormedSpace.GeometricStructureSpace ℂ A,
      ClosureSpace.HasFiniteDimensionalCompactHomeomorph.{u, u}
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component I)) (fun m => m = n)
```

The component condition uses the following separate exact definition, including its complete right-hand side: HomogeneousCharacterization.lean.

```
def HasFiniteDimensionalCompactHomeomorph.{uH, uX} {X : Type uX}
    (c : ClosureSpace X) (P : ℕ → Prop) : Prop :=
  ∃ (H : Type uH) (instN : NormedAddCommGroup H),
    letI := instN
    ∃ instI : InnerProductSpace ℂ H,
      letI := instI
      ∃ instC : CompleteSpace H,
        letI := instC
        ∃ instT : Nontrivial H,
          letI := instT
          FiniteDimensional ℂ H ∧ P (Module.finrank ℂ H) ∧
            Nonempty (Homeomorph c
              (NormedSpace.geometricStructureClosure ℂ
                (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))
```

In the source	Mathematical meaning

Exact declaration
Ambient inputs
`variable {A : Type u} [NonUnitalCStarAlgebra A] [hA : Nontrivial A]`
`attribute [local instance] _root_.CStarAlgebra.spectralOrder _root_.CStarAlgebra.spectralOrderedRing`
	The nonzero complex C*-algebra A. The two local instances give its compatible spectral order, used for positivity and pure states. They do not introduce another choice of order. The separate `include hA in` shown below retains nontriviality in this theorem; no unit is required.

Exact declaration Representation quantifiers `(∀ (L : Type u) [NormedAddCommGroup L] [InnerProductSpace ℂ L] [CompleteSpace L], ∀ pi : NonUnitalCStarRepresentation A L, pi.IsIrreducible →`	Every complex Hilbert space H and every irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$. Here source `L` is the representation space called H in the Statement. Its norm, complex inner product and completeness are required by the three bracketed inputs. Irreducibility includes a nonzero action, not faithfulness.

In the source	Mathematical meaning
Exact declaration Representation condition `FiniteDimensional ℂ L ∧ Module.finrank ℂ L = n) ↔`	The representation space H is finite-dimensional and satisfies $\dim_{\mathbb{C}} H = n$, as in (1). Both clauses are required. The parameter `(n : ℕ)` in the main declaration fixes one natural number throughout the equivalence.
Exact declaration Component quantifier `∀ I : NormedSpace.GeometricStructureSpace ℂ A,`	Every point $I \in S(A)$. The model space and comparison map required after this quantifier may depend on I. The number n remains fixed.
Exact declaration Comparison applied to the component `ClosureSpace.HasFiniteDimensionalCompactHomeomorph. {u, u} ((NormedSpace.geometricStructureClosure ℂ A).subspace ((NormedSpace.geometricStructureClosure ℂ A).component I)) (fun m => m = n)`	Apply the displayed comparison definition to the closure induced on $C_A(I)$. `geometricStructureClosure ℂ A` is the closure c_A on $S(A)$; its `component I` is the point set $C_A(I)$. `subspace` makes that set a closure space, with $$c_{C_A(I)}(E) = C_A(I) \cap c_A(E)$$ for $E \subseteq C_A(I)$. This closure is the argument `c` of the definition above, and its underlying set is `X`. The final argument `(fun m => m = n)` assigns $P(m)$ the condition $m = n$ on the model dimension.
Exact declaration Model and inner product `∃ (H : Type uH) (instN : NormedAddCommGroup H), letI := instN ∃ instI : InnerProductSpace ℂ H, letI := instI`	The condition requires a model space K, called `H` in this definition, with the following structures. `instN` gives its normed additive group, and `instI` its complex inner product. Each following `letI` uses that choice on the same space, first to form `InnerProductSpace ℂ H`, then to interpret the remaining Hilbert-space conditions.
Exact declaration Completeness and nonzero model `∃ instC : CompleteSpace H, letI := instC ∃ instT : Nontrivial H, letI := instT`	`instC` requires this same K to be complete; `instT` requires $K \neq \{0\}$. Their `letI` declarations retain these structures when forming bounded operators, the compact-operator algebra and its geometric closure. They do not choose a second space.
Exact declaration Dimension of the model `FiniteDimensional ℂ H ∧ P (Module.finrank ℂ H) ∧`	This same model K must be finite-dimensional. Substituting $P(m) \equiv (m = n)$ gives $\dim_{\mathbb{C}} K = n$. The conjunction separately retains finite-dimensionality and the numerical equation. It does not replace them by the `finrank` equation alone.
Exact declaration Homeomorphism existence `Nonempty (Homeomorph c (NormedSpace.geometricStructureClosure ℂ (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))`	`Nonempty` requires an element of this `Homeomorph` type: one bijection $e : C_A(I) \to S(\mathcal{K}(K))$ with both continuity fields below. Its target closure is $c_{\mathcal{K}(K)}$ on the geometric structure of the compact-operator algebra on the same model K, not on K itself.

In the source	Mathematical meaning
Exact declaration Homeomorphism and forward field `structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where continuous_toFun : c.Continuous d toFun`	`extends α ≃ β` supplies the bijection, extracted as `e.toEquiv`. Its forward map `toFun` is e. In this application, $c = c_{C_A(I)}$ and $d = c_{\mathcal{K}(K)}$, so the first field says $$e(c_{C_A(I)}(E)) \subseteq c_{\mathcal{K}(K)}(e(E))$$ for every $E \subseteq C_A(I)$.
Exact declaration Inverse field of the same homeomorphism `continuous_invFun : d.Continuous c invFun`	`invFun`, also written `e.toEquiv.symm`, is the inverse of that same e. The second field says $$e^{-1}(c_{\mathcal{K}(K)}(F)) \subseteq c_{C_A(I)}(e^{-1}(F))$$ for every $F \subseteq S(\mathcal{K}(K))$. Together the two fields give the arbitrary-set closure equality (2).
Exact declaration Continuity definition `def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop := ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`	For a map $r : Y \to Z$, continuity means $$r(c_Y(E)) \subseteq c_Z(r(E))$$ for every $E \subseteq Y$. Source `α`, `β`, `c`, `d`, `f` denote Y, Z, c_Y, c_Z, r; the double prime denotes the set image $r(E)$. Unlike `FinitelyContinuous`, this definition has no finiteness restriction.

Size scope: the principal equivalence fixes the universe of A and quantifies both representation spaces and compact-model spaces in that same universe. The separately cited transport theorem allows an independently specified Hilbert-space universe; it does not change the quantifiers of this declaration.

Exact surrounding source context (separate excerpts)

HomogeneousCharacterization.lean.

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [hA : Nontrivial A]
attribute [local instance] _root_.CStarAlgebra.spectralOrder
  _root_.CStarAlgebra.spectralOrderedRing
```

HomogeneousCharacterization.lean.

```
include hA in
```

Exact declaration for the source guide

```
theorem forall_irreducible_finrank_eq_iff_component_compact_homeomorph (n : ℕ) :
    (∀ (L : Type u) [NormedAddCommGroup L] [InnerProductSpace ℂ L] [CompleteSpace L],
      ∀ pi : NonUnitalCStarRepresentation A L, pi.IsIrreducible →
        FiniteDimensional ℂ L ∧ Module.finrank ℂ L = n) ↔
    ∀ I : NormedSpace.GeometricStructureSpace ℂ A,
      ClosureSpace.HasFiniteDimensionalCompactHomeomorph.{u, u}
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component I)) (fun m => m = n) :=
```

Irreducible dimension at most a fixed number

`MathlibAnnex.Analysis.CStarAlgebra.forall_irreducible_finrank_le_iff_component_compact_homeomorph`

theorem

Characterizes a uniform finite dimension bound by full compact component models.

Statement

Let A be a nonzero complex C*-algebra, and let $n \in \mathbb{N}$.

For a complex C*-algebra D, write $S(D)$ for its geometric structure space. Its points are the unions of kernels of norm-one bounded complex-linear functionals supporting a maximal proper face of the closed unit ball of D; faces use real convexity. Write c_D for the hull–kernel closure on $S(D)$:

$$c_D(E) = \left\{ I \in S(D) : \bigcap_{J \in E} J \subseteq I \right\}, \qquad E \subseteq S(D).$$

These are the geometric structure definition and its closure, with the norm-one description justified by the norm-one support characterization.

For $I \in S(A)$, let $C_A(I)$ be its finite-chain component: the points reachable from I by a finite chain in which adjacent points are equal or form a nonclosed two-point set. Give $C_A(I)$ the induced closure

$$c_{C_A(I)}(E) = C_A(I) \cap c_A(E), \qquad E \subseteq C_A(I).$$

For a complex Hilbert space H, let $\mathcal{K}(H)$ be the algebra of compact operators on H. In particular, $c_{\mathcal{K}(K)}$ denotes closure on $S(\mathcal{K}(K))$, where $\mathcal{K}(K)$ is itself the algebra to which the geometric structure construction is applied.

The following conditions are equivalent:

- Every irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$ on a complex Hilbert space has H finite-dimensional and $\dim_{\mathbb{C}} H \le n$.
- For every $I \in S(A)$, there are a nonzero finite-dimensional complex Hilbert space K with $\dim_{\mathbb{C}} K \le n$ and a bijection $e : C_A(I) \to S(\mathcal{K}(K))$ such that

$$e(c_{C_A(I)}(E)) = c_{\mathcal{K}(K)}(e(E)) \qquad (E \subseteq C_A(I)). \quad (1)$$

Here $c_{\mathcal{K}(K)}$ is the closure on the geometric structure space of the compact-operator algebra on this same K. Thus e is a homeomorphism of closure spaces. The model K and the map e may depend on I; the bound n is common to all components.

Assumptions

The algebra A is nonzero; no assumption $n > 0$ is made. Representations are $*$-homomorphisms into bounded operators, without a requirement of unitality. Irreducibility includes a nonzero action, but does not require faithfulness. All Hilbert spaces are complex and complete. The compatible order on A supplies positivity and pure states. Finite-dimensionality is a conclusion on the representation side and a requirement on the model side. It is separate from the numerical dimension bound. No separability hypothesis is added.

Conclusion

A uniform finite bound on the dimensions of all irreducible representations is equivalent to the compact-operator component models in (1), with the same bound. For nonzero A, both sides are false at $n = 0$, by the cited zero-bound exclusions. This does not add $n > 0$ as an input assumption.

Main citations

The bounded-dimension characterization specializes the general dimension-predicate theorem to the upper bound n. The finite-dimensional compact-model predicate keeps finiteness and the bound on the same target. The same-map full-continuity theorem preserves the finite comparison and its inverse. The representation-side zero-bound exclusion and component-side zero-bound exclusion cover $n = 0$.

Lean source signature (exact)

Exact source and declaration. Only the literal signature is displayed here; the linked source retains the proof.

```
theorem forall_irreducible_finrank_le_iff_component_compact_homeomorph (n : ℕ) :
    (∀ (L : Type u) [NormedAddCommGroup L] [InnerProductSpace ℂ L] [CompleteSpace L],
      ∀ pi : NonUnitalCStarRepresentation A L, pi.IsIrreducible →
        FiniteDimensional ℂ L ∧ Module.finrank ℂ L ≤ n) ↔
    ∀ I : NormedSpace.GeometricStructureSpace ℂ A,
      ClosureSpace.HasFiniteDimensionalCompactHomeomorph.{u, u}
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component I)) (fun m => m ≤ n)
```

The principal equivalence fixes the universe of A and quantifies both representation spaces and compact-model spaces in that same universe. The dimension transport to an independently specified Hilbert space is a separate theorem, not a quantifier over all universes in this equivalence.

In the source	Mathematical meaning

Ambient algebra and compatible order
`universe u w`
`variable {A : Type u} [NonUnitalCStarAlgebra A] [hA : Nontrivial A]`
`attribute [local instance] ContinuousLinearMap.isClosed_compactOperatorStarSubalgebra`
`  _root_.CStarAlgebra.spectralOrder _root_.CStarAlgebra.spectralOrderedRing`
	The nonzero complex C*-algebra A is fixed. The local instances supply its compatible order and closedness of the compact-operator algebra. The universe of representation and model spaces is the universe of A.

Effective nonzero hypothesis `include hA in`	The preceding nontriviality instance for A is included as an input to this theorem.

Fixed bound and irreducible representations
`theorem forall_irreducible_finrank_le_iff_component_compact_homeomorph (n : ℕ) :`
`    (∀ (L : Type u) [NormedAddCommGroup L] [InnerProductSpace ℂ L] [CompleteSpace L],`
`      ∀ pi : NonUnitalCStarRepresentation A L, pi.IsIrreducible →`
	The natural number n is fixed. For every complete complex Hilbert space H (called L here), every irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$ is tested. Nonunital notation does not require A to lack an identity.

Finite dimension and upper bound `FiniteDimensional ℂ L ∧ Module.finrank ℂ L ≤ n) ↔`	The two requirements are finite-dimensionality of H and $\dim_{\mathbb{C}} H \leq n$. A finite-rank value alone is not substituted for finite-dimensionality.

Component and applied comparison
`∀ I : NormedSpace.GeometricStructureSpace ℂ A,`
`ClosureSpace.HasFiniteDimensionalCompactHomeomorph.{u, u}`
`    ((NormedSpace.geometricStructureClosure ℂ A).subspace`
`      ((NormedSpace.geometricStructureClosure ℂ A).component I)) (fun m => m ≤ n) :=`
	For every geometric point $I \in S(A)$, the ambient closure c_A is restricted to $C_A(I)$. This induced closure is the argument of the compact-model condition. The function of m sets the dimension condition to $m \leq n$ for the same model.

Closure and dimension-condition inputs
`def HasFiniteDimensionalCompactHomeomorph.{uH, uX} {X : Type uX}`
`    (c : ClosureSpace X) (P : ℕ → Prop) : Prop :=`
	Here $X = C_A(I)$ and the closure argument is $c_{C_A(I)}$; the condition on natural numbers is $P(m) \iff m \leq n$.

Model and inner product
`∃ (H : Type uH) (instN : NormedAddCommGroup H),`
`  letI := instN`
`  ∃ instI : InnerProductSpace ℂ H,`
`    letI := instI`
	There exist a model space K (called H in this definition), a normed additive structure and a complex inner product on it. The first local instance makes the normed structure available to the inner-product type; the second makes that same inner product available below.

Completeness and nontriviality
`∃ instC : CompleteSpace H,`
`  letI := instC`
`  ∃ instT : Nontrivial H,`
`    letI := instT`
	The same model K is complete and nonzero. Each local instance supplies the chosen structure to the subsequent types; these are not four different spaces.

In the source	Mathematical meaning
Dimension and one homeomorphism `FiniteDimensional ℂ H ∧ P (Module.finrank ℂ H) ∧ Nonempty (Homeomorph c (NormedSpace.geometricStructureClosure ℂ (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))`	The same model is finite-dimensional with $\dim_{\mathbb{C}} K \le n$. The final type is inhabited, so there exists a homeomorphism $e : C_A(I) \to S(\mathcal{K}(K))$. The algebra $\mathcal{K}(K)$ consists of compact operators on this same K. Both K and e may depend on I; n is fixed.
Homeomorphism and its inverse `structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where continuous_toFun : c.Continuous d toFun continuous_invFun : d.Continuous c invFun`	The structure extends one bijection e; its two continuity fields use that map and its inverse. For all $E \subseteq C_A(I)$ and $F \subseteq S(\mathcal{K}(K))$, $$e(c_{C_A(I)}(E)) \subseteq c_{\mathcal{K}(K)}(e(E)),$$ $$e^{-1}(c_{\mathcal{K}(K)}(F)) \subseteq c_{C_A(I)}(e^{-1}(F)).$$ These conditions give the closure-preserving equality in the Statement.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
universe u w
variable {A : Type u} [NonUnitalCStarAlgebra A] [hA : Nontrivial A]
attribute [local instance] ContinuousLinearMap.isClosed_compactOperatorStarSubalgebra
  _root_.CStarAlgebra.spectralOrder _root_.CStarAlgebra.spectralOrderedRing
```

Exact surrounding source:

```
include hA in
```

Related exact source for the reading guide

Closure and dimension-condition inputs · Fixed source

```
def HasFiniteDimensionalCompactHomeomorph.{uH, uX} {X : Type uX}
    (c : ClosureSpace X) (P : ℕ → Prop) : Prop :=
  ∃ (H : Type uH) (instN : NormedAddCommGroup H),
    letI := instN
    ∃ instI : InnerProductSpace ℂ H,
      letI := instI
      ∃ instC : CompleteSpace H,
        letI := instC
        ∃ instT : Nontrivial H,
          letI := instT
          FiniteDimensional ℂ H ∧ P (Module.finrank ℂ H) ∧
            Nonempty (Homeomorph c
              (NormedSpace.geometricStructureClosure ℂ
                (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))
```

Component and applied comparison · Fixed source

```
    ∀ I : NormedSpace.GeometricStructureSpace ℂ A,
      ClosureSpace.HasFiniteDimensionalCompactHomeomorph.{u, u}
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component I)) (fun m => m ≤ n) :=
```

Homeomorphism and its inverse · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```

Compact equality in every irreducible image

`MathlibAnnex.Analysis.CStarAlgebra.forall_irreducible_range_eq_compact_iff_component_compact_homeomorph`

theorem

Characterizes equality with the compact operators by full component homeomorphisms.

Statement

Let A be a nonzero complex C*-algebra.

For a complex C*-algebra B, write $S(B)$ for its geometric structure space. Its points are the unions of kernels of norm-one bounded complex-linear functionals supporting a maximal proper face of the closed unit ball of B; faces use real convexity. Write c_B for the hull-kernel closure on $S(B)$:

$$c_B(\mathcal{U}) = \left\{ J \in S(B) : \bigcap_{L \in \mathcal{U}} L \subseteq J \right\}, \qquad \mathcal{U} \subseteq S(B).$$

These are the geometric structure and its closure, with the norm-one description justified by the support-functional norm theorem.

For $I \in S(A)$, let $C_A(I)$ be its finite-chain component: the points reachable from I by a finite chain in which adjacent points are equal or form a nonclosed two-point set. This uses the linked-pair relation and its generated components, rather than topological connectedness. Give $C_A(I)$ the induced closure

$$c_{C_A(I)}(E) = C_A(I) \cap c_A(E), \qquad E \subseteq C_A(I).$$

For a complex Hilbert space H, let $\mathcal{K}(H)$ be the algebra of compact operators on H. In particular, $c_{\mathcal{K}(K)}$ denotes closure on $S(\mathcal{K}(K))$, where $\mathcal{K}(K)$ is itself the algebra to which the geometric structure construction is applied.

The following conditions are equivalent:

- For every irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$ on a complex Hilbert space H,

$$\pi(A) = \mathcal{K}(H). \tag{1}$$

- For every $I \in S(A)$, there are a nonzero complex Hilbert space K and a bijection $e : C_A(I) \to S(\mathcal{K}(K))$ satisfying

$$e(c_{C_A(I)}(E)) = c_{\mathcal{K}(K)}(e(E)) \quad (E \subseteq C_A(I)). \tag{2}$$

Thus e is a homeomorphism of closure spaces: the same e and its inverse e^{-1} are continuous for arbitrary subsets. Both K and e may depend on I. The equality in (2) applies to all subsets, not only finite ones; $e(E)$ denotes the image of a set.

Assumptions

The algebra A is nonzero. Representations are $*$-homomorphisms into bounded operators; a unit is not required, and the source does not impose preservation of a unit. The Hilbert spaces are complete over $\mathbb{C}$. Irreducibility includes a nonzero action, but does not require faithfulness. Neither separability nor finite-dimensionality is assumed. The component closure is induced from $S(A)$.

Conclusion

The CCR image condition $\pi(A) = \mathcal{K}(H)$ is equivalent to homeomorphisms of the components onto compact-operator geometric structure spaces. The image equality is inside $\mathcal{B}(H)$. In contrast, the forward-continuity criterion characterizes only $\mathcal{K}(H) \subseteq \pi(A)$.

Main citations

The compact-image equality characterization states the equivalence. The full compact-homeomorphism condition requires full continuity in both directions. The pure-state equality theorem and equality transport to an independently specified Hilbert space apply to the same representation images. The forward-only characterization instead detects inclusion, not equality.

The finite-homeomorphism fields, closure continuity, and homeomorphism fields specify the map conditions used here. The closure-transport identity expresses a homeomorphism by equality of closures of arbitrary sets.

Lean source signature (exact)

The preceding `variable` declaration introduces A and its nontriviality. The local spectral order is used for positivity and pure states. The `include hA in` retains nonzero A among this theorem's parameters, although the main excerpt does not repeat these ambient binders. Both separate excerpts are shown in the source context below.

GCRCharacterization.lean. The contiguous declaration prefix ends before the proof assignment.

```
theorem forall_irreducible_range_eq_compact_iff_component_compact_homeomorph :
    (∀ (K : Type u) [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K],
      ∀ pi : NonUnitalCStarRepresentation A K, pi.IsIrreducible →
        (NonUnitalStarAlgHom.range pi) = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ K) ↔
    ∀ I : NormedSpace.GeometricStructureSpace ℂ A,
      ClosureSpace.HasCompactOperatorHomeomorph.{u, u}
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component I))
```

The component condition uses the following separate exact definition, including its complete right-hand side: GCRCharacterization.lean.

```
def HasCompactOperatorHomeomorph.{uH, uX} {X : Type uX}
    (c : ClosureSpace X) : Prop :=
  ∃ (H : Type uH) (instN : NormedAddCommGroup H),
    letI := instN
    ∃ instI : InnerProductSpace ℂ H,
      letI := instI
      ∃ instC : CompleteSpace H,
        letI := instC
        ∃ instT : Nontrivial H,
          letI := instT
          Nonempty (Homeomorph c
            (NormedSpace.geometricStructureClosure ℂ
              (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))
```

In the source	Mathematical meaning

Exact declaration
Ambient inputs
`variable {A : Type u} [NonUnitalCStarAlgebra A] [hA : Nontrivial A]`
`attribute [local instance] _root_.CStarAlgebra.spectralOrder _root_.CStarAlgebra.spectralOrderedRing`
	The nonzero complex C*-algebra A. The two local instances give its compatible spectral order, used for positivity and pure states. They do not introduce another choice of order. The separate `include hA in` shown below retains nontriviality in this theorem; no unit is required.

Exact declaration Representation quantifiers `(∀ (K : Type u) [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K], ∀ pi : NonUnitalCStarRepresentation A K, pi.IsIrreducible →`	Every complex Hilbert space H and every irreducible $*$-representation $\pi : A \to \mathcal{B}(H)$. Here source `K` is the representation space called H in the Statement. Its norm, complex inner product and completeness are required by the three bracketed inputs. Irreducibility includes a nonzero action, not faithfulness.
Exact declaration Representation condition `(NonUnitalStarAlgHom.range pi) = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ K) ↔`	The image of this same representation is exactly the compact-operator algebra: $$\pi(A) = \mathcal{K}(H).$$ This is (1). The outer ↔ relates the universal representation condition to the component condition on its right.
Exact declaration Component quantifier `∀ I : NormedSpace.GeometricStructureSpace ℂ A,`	Every point $I \in S(A)$. The model space and comparison map required after this quantifier may depend on I.

In the source	Mathematical meaning
Exact declaration Comparison applied to the component `ClosureSpace.HasCompactOperatorHomeomorph.{u, u} ((NormedSpace.geometricStructureClosure ℂ A).subspace ((NormedSpace.geometricStructureClosure ℂ A).component I))`	Apply the displayed comparison definition to the closure induced on $C_A(I)$. `geometricStructureClosure ℂ A` is the closure c_A on $S(A)$; its `component I` is the point set $C_A(I)$. `subspace` makes that set a closure space, with $$c_{C_A(I)}(E) = C_A(I) \cap c_A(E)$$ for $E \subseteq C_A(I)$. This closure is the argument `c` of the definition above, and its underlying set is `X`.
Exact declaration Model and inner product `∃ (H : Type uH) (instN : NormedAddCommGroup H), letI := instN ∃ instI : InnerProductSpace ℂ H, letI := instI`	The condition requires a model space K, called `H` in this definition, with the following structures. `instN` gives its normed additive group, and `instI` its complex inner product. Each following `letI` uses that choice on the same space, first to form `InnerProductSpace ℂ H`, then to interpret the remaining Hilbert-space conditions.
Exact declaration Completeness and nonzero model `∃ instC : CompleteSpace H, letI := instC ∃ instT : Nontrivial H, letI := instT`	`instC` requires this same K to be complete; `instT` requires $K \neq \{0\}$. Their `letI` declarations retain these structures when forming bounded operators, the compact-operator algebra and its geometric closure. They do not choose a second space.
Exact declaration Homeomorphism existence `Nonempty (Homeomorph c (NormedSpace.geometricStructureClosure ℂ (ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)))`	`Nonempty` requires an element of this `Homeomorph` type: one bijection $e : C_A(I) \to S(\mathcal{K}(K))$ with both continuity fields below. Its target closure is $c_{\mathcal{K}(K)}$ on the geometric structure of the compact-operator algebra on the same model K, not on K itself.
Exact declaration Homeomorphism and forward field `structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where continuous_toFun : c.Continuous d toFun`	`extends α ≃ β` supplies the bijection, extracted as `e.toEquiv`. Its forward map `toFun` is e. In this application, $c = c_{C_A(I)}$ and $d = c_{\mathcal{K}(K)}$, so the first field says $$e(c_{C_A(I)}(E)) \subseteq c_{\mathcal{K}(K)}(e(E))$$ for every $E \subseteq C_A(I)$.
Exact declaration Inverse field of the same homeomorphism `continuous_invFun : d.Continuous c invFun`	`invFun`, also written `e.toEquiv.symm`, is the inverse of that same e. The second field says $$e^{-1}(c_{\mathcal{K}(K)}(F)) \subseteq c_{C_A(I)}(e^{-1}(F))$$ for every $F \subseteq S(\mathcal{K}(K))$. Together the two fields give the arbitrary-set closure equality (2).

In the source	Mathematical meaning

Exact declaration
Continuity definition
`def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=`
`∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`
	For a map $r : Y \to Z$, continuity means
	$$r(c_Y(E)) \subseteq c_Z(r(E))$$
	for every $E \subseteq Y$. Source `α`, `β`, `c`, `d`, `f` denote Y, Z, c_Y, c_Z, r; the double prime denotes the set image $r(E)$. Unlike `FinitelyContinuous`, this definition has no finiteness restriction.

Size scope: the principal equivalence fixes the universe of A and quantifies both representation spaces and compact-model spaces in that same universe. The separately cited transport theorem allows an independently specified Hilbert-space universe; it does not change the quantifiers of this declaration.

Exact surrounding source context (separate excerpts)

GCRCharacterization.lean.

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [hA : Nontrivial A]
attribute [local instance] _root_.CStarAlgebra.spectralOrder
  _root_.CStarAlgebra.spectralOrderedRing
```

GCRCharacterization.lean.

```
include hA in
```

Exact declaration for the source guide

```
theorem forall_irreducible_range_eq_compact_iff_component_compact_homeomorph :
    (∀ (K : Type u) [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K],
      ∀ pi : NonUnitalCStarRepresentation A K, pi.IsIrreducible →
        (NonUnitalStarAlgHom.range pi) = ContinuousLinearMap.compactOperatorStarSubalgebra ℂ K)
↔
    ∀ I : NormedSpace.GeometricStructureSpace ℂ A,
      ClosureSpace.HasCompactOperatorHomeomorph.{u, u}
        ((NormedSpace.geometricStructureClosure ℂ A).subspace
          ((NormedSpace.geometricStructureClosure ℂ A).component I)) := by
```

A geometric-structure homeomorphism transports the irreducible spectrum

`MathlibAnnex.Analysis.CStarAlgebra.irreducibleSpectrumHomeomorphOfGeometricStructure`

def

Transports the irreducible spectrum along a homeomorphism of geometric structure spaces.

Statement

Let $A, B \neq \{0\}$ be complex C*-algebras, possibly without identities. Suppose

$$e : (S(A), c_A) \longrightarrow (S(B), c_B)$$

is a homeomorphism of their geometric structure **closure spaces**. Both e and its inverse are continuous for the specified closure operators, with continuity tested on every subset.

The map e determines an ordinary homeomorphism

$$H_e : \widehat{A} \longrightarrow \widehat{B}$$

between the independent irreducible spectra with their kernel-induced ordinary topologies.

Definition

For $D = A, B$, $S(D)$ denotes kernels of extreme points of the closed unit ball of the Banach dual, with real convexity, and $c_D(\mathcal{U}) = \{I : \bigcap_{J \in \mathcal{U}} J \subseteq I\}$. The extreme-functional kernel characterization gives its geometric identification. Let $\mathrm{GS}(D)$ be the component space and $\Phi_D : \widehat{D} \to \mathrm{GS}(D)$ the specified spectral homeomorphism.

The induced component homeomorphism is $G(e) : \mathrm{GS}(A) \to \mathrm{GS}(B)$, with $C \mapsto e(C)$ and inverse $D \mapsto e^{-1}(D)$. Full closure continuity gives ordinary continuity for the specified topologies.

The defining map and its inverse, with composition read from right to left, are

$$H_e = \Phi_B^{-1} \circ G(e) \circ \Phi_A, \qquad H_e^{-1} = \Phi_A^{-1} \circ G(e)^{-1} \circ \Phi_B.$$

Thus the three stages go from $\widehat{A}$ to $\mathrm{GS}(A)$, then $\mathrm{GS}(B)$, then $\widehat{B}$; the last stage uses the inverse of Φ_B.

Assumptions

The source requires nonzero complex C*-algebras A, B, each with its compatible order, and the specified full closure-space homeomorphism e. The underlying Hilbert spaces of irreducible representations are not fixed as additional inputs; the independent spectra use their pure-state

GNS presentation. The map e has domain $S(A)$, whose points are subsets of A; it is not a map $A \to B$ and no linear or multiplicative property on algebra elements is assumed.

Conclusion

The output is this specific ordinary homeomorphism between the independent irreducible spectra. It uses the given full closure-space homeomorphism and the two fixed spectral comparisons. The input e acts on geometric points, which are subsets of algebras, not on algebra elements or representation operators.

Main citations

- Exact full definition: `irreducibleSpectrumHomeomorphOfGeometricStructure`.
- The ordinary spectral comparison: the two endpoint homeomorphisms and their exact topologies.
- Normed/GeometricStructure/SpectrumMap.lean, `geometricSpectrumHomeomorph`, `geometricSpectrumHomeomorph_val`, `geometricSpectrumHomeomorph_toGeometricSpectrum`, and `image_geometricSpectrumClosure`: the induced component map and full closure transport.
- The concrete geometric-spectrum closure: the component-intersection closure used by this construction.
- Ryotaro Tanaka, arXiv:2504.05551v1, Corollary 5.13: the spectral consequence of a geometric-structure homeomorphism.
- Component-map construction and full closure continuity.

Lean source signature (exact)

The full definition below is copied from the pinned source.

```
def irreducibleSpectrumHomeomorphOfGeometricStructure : IrreducibleSpectrum A ≃ₜ IrreducibleSpectrum B :=
  (irreducibleSpectrumHomeomorph A).trans
    (((NormedSpace.geometricSpectrumHomeomorph e).toTopologicalHomeomorph
      (isTopologizable_geometricSpectrumClosure A) (isTopologizable_geometricSpectrumClosure B)).trans
        (irreducibleSpectrumHomeomorph B).symm)
```

In the source	Mathematical meaning
The two ambient algebras and independent universes **Exact declaration**	The independently specified nonzero complex C*-algebras A, B with compatible orders; no units are required.
Exact declaration `e` in the surrounding scope	The supplied full closure-space homeomorphism of geometric points, not a map on algebra elements.

In the source	Mathematical meaning
Exact declaration `(irreducibleSpectrumHomeomorph A).trans`	The first stage is $\Phi_A : \widehat{A} \to \mathrm{GS}(A)$.
Exact declaration `NormedSpace.geometricSpectrumHomeomorph e`	The middle stage is the component map $C \mapsto e(C)$ with its inverse.
Exact declaration `.toTopologicalHomeomorph`	The existing topologizability results give ordinary continuity of that same component map.
Exact declaration `(irreducibleSpectrumHomeomorph B).symm`	The final stage is $\Phi_B^{-1} : \mathrm{GS}(B) \to \widehat{B}$, giving the displayed composite H_e.

Exact surrounding source context (separate excerpts)

Source lines 27–27:

```
variable {A : Type u} [NonUnitalCStarAlgebra A] [PartialOrder A] [StarOrderedRing A] [Nontrivial A]
```

Source lines 28–28:

```
variable {B : Type v} [NonUnitalCStarAlgebra B] [PartialOrder B] [StarOrderedRing B] [Nontrivial B]
```

Source lines 29–30:

```
variable (e : ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A)
  (NormedSpace.geometricStructureClosure ℂ B))
```

GSS rigidity from a full bounded-operator model

`MathlibAnnex.Analysis.CStarAlgebra.nonempty_starAlgEquiv_iff_nonempty_geometricStructure_homeomorph_of_exists_boundedOperatorEquiv`

theorem

Obtains algebraic rigidity from an actual full bounded-operator model on either side.

Statement

Let A, B be complex C*-algebras. For a complex C*-algebra D, write $S(D)$ for its geometric structure space. Its points are the unions of kernels of norm-one bounded complex-linear functionals supporting a maximal proper face of the closed unit ball of D; faces use real convexity. Write c_D for its hull–kernel closure:

$$c_D(E) = \left\{ I \in S(D) : \bigcap_{J \in E} J \subseteq I \right\}, \qquad E \subseteq S(D).$$

These are the geometric structure definition and its closure, with the norm-one description justified by the norm-one support characterization.

Let $\mathcal{B}(H)$ denote the algebra of all bounded complex-linear operators on a complex Hilbert space H. Assume that at least one of A, B is $*$-isomorphic to $\mathcal{B}(H)$ for some such H. Then A and B are $*$-isomorphic if and only if $S(A)$ and $S(B)$ are homeomorphic as closure spaces. Here a closure-space homeomorphism is a bijection $e : S(A) \to S(B)$ satisfying

$$e(c_A(E)) = c_B(e(E)), \qquad E \subseteq S(A).$$

The equality is required for every subset, not just finite subsets.

Assumptions

The bounded-operator model is required on at least one side. It is an isomorphism onto all of $\mathcal{B}(H)$, not onto a proper subalgebra. The Hilbert space is complex and complete; it and the algebras may be zero. No nonzero, unitality, separability, countability, or finite-dimensionality hypothesis is imposed.

Conclusion

The equivalence concerns existence of a $*$-algebra isomorphism and existence of a closure-space homeomorphism. It does not require the isomorphism to induce a prescribed homeomorphism, or assert uniqueness.

Main citations

The either-side bounded-operator model theorem uses the specified left-model equivalence and specified right-model equivalence. The full-range model predicate requires faithfulness and range equal to all bounded operators. The zero-model equivalence retains the zero case. No abstract type-I factor classification is claimed.

Lean source signature (exact)

Exact source and declaration. Only the literal signature is displayed here; the linked source retains the proof.

```
theorem
nonempty_starAlgEquiv_iff_nonempty_geometricStructure_homeomorph_of_exists_boundedOperatorEquiv
    (hModel :
      (∃ (L : Type w) (normed : NormedAddCommGroup L),
        letI : NormedAddCommGroup L := normed
        ∃ inner : InnerProductSpace ℂ L,
          letI : InnerProductSpace ℂ L := inner
          ∃ complete : CompleteSpace L,
            letI : CompleteSpace L := complete
            Nonempty (A ≃⋆ₐ[ℂ] (L →L[ℂ] L))) ∨
      (∃ (L : Type z) (normed : NormedAddCommGroup L),
        letI : NormedAddCommGroup L := normed
        ∃ inner : InnerProductSpace ℂ L,
          letI : InnerProductSpace ℂ L := inner
          ∃ complete : CompleteSpace L,
            letI : CompleteSpace L := complete
            Nonempty (B ≃⋆ₐ[ℂ] (L →L[ℂ] L)))) :
    Nonempty (A ≃⋆ₐ[ℂ] B) ↔
      Nonempty (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A)
        (NormedSpace.geometricStructureClosure ℂ B))
```

In the source	Mathematical meaning

Two C*-algebras
`variable {A : Type u} [NonUnitalCStarAlgebra A]`
`variable {B : Type v} [NonUnitalCStarAlgebra B]`
	The inputs A, B are arbitrary complex C*-algebras, including zero. Their size levels are independent. The source type allows unital algebras but does not require a unit.

Normed model on the A side
`(∃ (L : Type w) (normed : NormedAddCommGroup L),`
`  letI : NormedAddCommGroup L := normed`
	This alternative requires a normed additive group, called `L` in the source and H in the Statement. The first `letI` installs this chosen normed structure on the same space for the inner-product type that follows.

Complex inner product on the A model
`∃ inner : InnerProductSpace ℂ L,`
`  letI : InnerProductSpace ℂ L := inner`
	The same model must have a compatible complex inner product. This `letI` installs that inner product, so subsequent Hilbert-space operator types use this structure, not a separately chosen one.

In the source	Mathematical meaning

Complete A model and isomorphism
`∃ complete : CompleteSpace L,`
`  letI : CompleteSpace L := complete`
`  Nonempty (A ≃⋆ₐ[ℂ] (L →L[ℂ] L))) ∨`
	The model must be complete; the last `letI` installs this completeness for its operator algebra. `Nonempty` requires a $*$-isomorphism $A \to \mathcal{B}(H)$ onto the entire operator algebra. The two model branches are alternatives; their spaces are quantified separately and may be zero.

Normed model on the B side
`(∃ (L : Type z) (normed : NormedAddCommGroup L),`
`  letI : NormedAddCommGroup L := normed`
	This alternative requires a normed additive group, called `L` in the source and H in the Statement. The first `letI` installs this chosen normed structure on the same space for the inner-product type that follows.

Complex inner product on the B model
`∃ inner : InnerProductSpace ℂ L,`
`  letI : InnerProductSpace ℂ L := inner`
	The same model must have a compatible complex inner product. This `letI` installs that inner product, so subsequent Hilbert-space operator types use this structure, not a separately chosen one.

Complete B model and isomorphism
`∃ complete : CompleteSpace L,`
`  letI : CompleteSpace L := complete`
`  Nonempty (B ≃⋆ₐ[ℂ] (L →L[ℂ] L))) :`
	The model must be complete; the last `letI` installs this completeness for its operator algebra. `Nonempty` requires a $*$-isomorphism $B \to \mathcal{B}(H)$ onto the entire operator algebra. The two model branches are alternatives; their spaces are quantified separately and may be zero.

Existence of a star-isomorphism `Nonempty (A ≃⋆ₐ[ℂ] B) ↔`	`Nonempty` says that the type of $*$-algebra isomorphisms $A \to B$ has an element. The outer equivalence relates this existence to the existence in the next row, not to a choice of a prescribed map.

Existence of a closure-space homeomorphism
`Nonempty (ClosureSpace.Homeomorph`
`(NormedSpace.geometricStructureClosure ℂ A)`
`  (NormedSpace.geometricStructureClosure ℂ B)) :=`
`by`
	The complete arguments are c_A on $S(A)$ and c_B on $S(B)$. `Nonempty` requires one homeomorphism $e : S(A) \to S(B)$ between these closure spaces.

The same map and inverse
`structure Homeomorph (c : ClosureSpace α) (d :`
`ClosureSpace β) extends α ≃ β where`
`  continuous_toFun : c.Continuous d toFun`
`  continuous_invFun : d.Continuous c invFun`
	The inherited equivalence supplies the bijection e and its inverse e^{-1}. The first field requires, for every $E \subseteq S(A)$,
	$$e(c_A(E)) \subseteq c_B(e(E)).$$
	The second requires, for every $F \subseteq S(B)$,
	$$e^{-1}(c_B(F)) \subseteq c_A(e^{-1}(F)).$$
	Both concern the same e and together express the closure equality in the Statement.

In the source	Mathematical meaning

Continuity for arbitrary subsets
`def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=`
`  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`
	For a map $r : (X, c) \to (Y, d)$, continuity means
	$$r(c(E)) \subseteq d(r(E)) \quad (E \subseteq X).$$
	In the preceding row this definition is applied first to $r = e$ and then to $r = e^{-1}$, with the domain and target closures interchanged.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
universe u v w z

variable {A : Type u} [NonUnitalCStarAlgebra A]
variable {B : Type v} [NonUnitalCStarAlgebra B]
variable {H : Type w} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
variable {K : Type z} [NormedAddCommGroup K] [InnerProductSpace ℂ K] [CompleteSpace K]

local instance instSpectralOrderBoundedOperatorModel : PartialOrder A :=
  _root_.CStarAlgebra.spectralOrder A
local instance instSpectralOrderedRingBoundedOperatorModel : StarOrderedRing A :=
  _root_.CStarAlgebra.spectralOrderedRing A
```

The model spaces in the two alternatives have the fixed independent source levels `w` and `z`; the algebra levels are `u` and `v`. The source names for those model spaces are local to their respective existential branches. No additional ambient Hilbert space is an input to this equivalence.

Related exact source for the reading guide

Existence of a closure-space homeomorphism · Fixed source

```
      Nonempty (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A)
        (NormedSpace.geometricStructureClosure ℂ B)) := by
```

Continuity for arbitrary subsets · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

The same map and inverse · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```

GSS rigidity when either algebra has an elementary model

`MathlibAnnex.Analysis.CStarAlgebra.nonempty_starAlgEquiv_iff_nonempty_geometricStructure_homeomorph_of_exists_compactOperatorEquiv`

theorem

Obtains algebraic rigidity from an elementary compact-operator model on either side.

Statement

Let A, B be complex C*-algebras. For a complex C*-algebra D, write $S(D)$ for its geometric structure space. Its points are the unions of kernels of norm-one bounded complex-linear functionals supporting a maximal proper face of the closed unit ball of D; faces use real convexity. Write c_D for its hull–kernel closure:

$$c_D(E) = \left\{ I \in S(D) : \bigcap_{J \in E} J \subseteq I \right\}, \qquad E \subseteq S(D).$$

These are the geometric structure definition and its closure, with the norm-one description justified by the norm-one support characterization.

Let $\mathcal{K}(H)$ denote the algebra of compact operators on a complex Hilbert space H. Assume that at least one of A, B is $*$-isomorphic to $\mathcal{K}(H)$ for some such H. Then A and B are $*$-isomorphic if and only if $S(A)$ and $S(B)$ are homeomorphic as closure spaces. Here a closure-space homeomorphism is a bijection $e : S(A) \to S(B)$ satisfying

$$e(c_A(E)) = c_B(e(E)), \qquad E \subseteq S(A).$$

The equality is required for every subset, not just finite subsets.

Assumptions

The compact-operator model is required on at least one side. It is an isomorphism onto the whole algebra $\mathcal{K}(H)$, not merely a representation with compact values. The Hilbert space is complex and complete; it and the algebras may be zero. No unitality, separability, countability, or finite-dimensionality hypothesis is imposed.

Conclusion

The equivalence concerns existence of a $*$-algebra isomorphism and existence of a closure-space homeomorphism. It does not require the isomorphism to induce a prescribed homeomorphism, or assert uniqueness.

Main citations

The either-side elementary-model theorem uses rigidity for a specified compact-operator equivalence. The given-action compact rigidity theorem applies to the representation on that same model space, including the zero case.

Lean source signature (exact)

Exact source and declaration. Only the literal signature is displayed here; the linked source retains the proof.

```
theorem
nonempty_starAlgEquiv_iff_nonempty_geometricStructure_homeomorph_of_exists_compactOperatorEquiv
    (hModel :
      (∃ (H : Type w) (normed : NormedAddCommGroup H),
        letI : NormedAddCommGroup H := normed
        ∃ inner : InnerProductSpace ℂ H,
          letI : InnerProductSpace ℂ H := inner
          ∃ complete : CompleteSpace H,
            letI : CompleteSpace H := complete
            Nonempty (A ≃⋆ₐ[ℂ] ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)) ∨
      (∃ (K : Type z) (normed : NormedAddCommGroup K),
        letI : NormedAddCommGroup K := normed
        ∃ inner : InnerProductSpace ℂ K,
          letI : InnerProductSpace ℂ K := inner
          ∃ complete : CompleteSpace K,
            letI : CompleteSpace K := complete
            Nonempty (B ≃⋆ₐ[ℂ] ContinuousLinearMap.compactOperatorStarSubalgebra ℂ K))) :
    Nonempty (A ≃⋆ₐ[ℂ] B) ↔
      Nonempty (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A)
        (NormedSpace.geometricStructureClosure ℂ B))
```

In the source	Mathematical meaning

Two C*-algebras
`variable {A : Type u} {B : Type v}`
`variable [NonUnitalCStarAlgebra A]`
`[NonUnitalCStarAlgebra B]`
	The inputs A, B are arbitrary complex C*-algebras, including zero. Their size levels are independent. The source type allows unital algebras but does not require a unit.

Normed model on the A side
`(∃ (H : Type w) (normed : NormedAddCommGroup H),`
`letI : NormedAddCommGroup H := normed`
	This alternative requires a normed additive group, called `H` in the source and H in the Statement. The first `letI` installs this chosen normed structure on the same space for the inner-product type that follows.

Complex inner product on the A model
`∃ inner : InnerProductSpace ℂ H,`
`letI : InnerProductSpace ℂ H := inner`
	The same model must have a compatible complex inner product. This `letI` installs that inner product, so subsequent Hilbert-space operator types use this structure, not a separately chosen one.

In the source	Mathematical meaning

Complete A model and isomorphism
`∃ complete : CompleteSpace H,`
`  letI : CompleteSpace H := complete`
`  Nonempty (A ≃⋆ₐ[ℂ] ContinuousLinearMap.compactOperatorStarSubalgebra ℂ H)) ∨`
	The model must be complete; the last `letI` installs this completeness for its operator algebra. `Nonempty` requires a $*$-isomorphism $A \to \mathcal{K}(H)$ onto the entire operator algebra. The two model branches are alternatives; their spaces are quantified separately and may be zero.

Normed model on the B side
`(∃ (K : Type z) (normed : NormedAddCommGroup K),`
`  letI : NormedAddCommGroup K := normed`
	This alternative requires a normed additive group, called `K` in the source and H in the Statement. The first `letI` installs this chosen normed structure on the same space for the inner-product type that follows.

Complex inner product on the B model
`∃ inner : InnerProductSpace ℂ K,`
`  letI : InnerProductSpace ℂ K := inner`
	The same model must have a compatible complex inner product. This `letI` installs that inner product, so subsequent Hilbert-space operator types use this structure, not a separately chosen one.

Complete B model and isomorphism
`∃ complete : CompleteSpace K,`
`  letI : CompleteSpace K := complete`
`  Nonempty (B ≃⋆ₐ[ℂ] ContinuousLinearMap.compactOperatorStarSubalgebra ℂ K))) :`
	The model must be complete; the last `letI` installs this completeness for its operator algebra. `Nonempty` requires a $*$-isomorphism $B \to \mathcal{K}(H)$ onto the entire operator algebra. The two model branches are alternatives; their spaces are quantified separately and may be zero.

Existence of a star-isomorphism `Nonempty (A ≃⋆ₐ[ℂ] B) ↔`	`Nonempty` says that the type of $*$-algebra isomorphisms $A \to B$ has an element. The outer equivalence relates this existence to the existence in the next row, not to a choice of a prescribed map.

Existence of a closure-space homeomorphism
`Nonempty (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A)`
`  (NormedSpace.geometricStructureClosure ℂ B)) :=`
`by`
	The complete arguments are c_A on $S(A)$ and c_B on $S(B)$. `Nonempty` requires one homeomorphism $e : S(A) \to S(B)$ between these closure spaces.

The same map and inverse
`structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where`
`  continuous_toFun : c.Continuous d toFun`
`  continuous_invFun : d.Continuous c invFun`
	The inherited equivalence supplies the bijection e and its inverse e^{-1}. The first field requires, for every $E \subseteq S(A)$,
	$$e(c_A(E)) \subseteq c_B(e(E)).$$
	The second requires, for every $F \subseteq S(B)$,
	$$e^{-1}(c_B(F)) \subseteq c_A(e^{-1}(F)).$$
	Both concern the same e and together express the closure equality in the Statement.

In the source	Mathematical meaning

Continuity for arbitrary subsets
`def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=`
`  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)`
	For a map $r : (X, c) \to (Y, d)$, continuity means
	$$r(c(E)) \subseteq d(r(E)) \quad (E \subseteq X).$$
	In the preceding row this definition is applied first to $r = e$ and then to $r = e^{-1}$, with the domain and target closures interchanged.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
universe u v w z

variable {A : Type u} {B : Type v}
variable [NonUnitalCStarAlgebra A] [NonUnitalCStarAlgebra B]
```

The model spaces in the two alternatives have the fixed independent source levels `w` and `z`; the algebra levels are `u` and `v`. The source names for those model spaces are local to their respective existential branches. No additional ambient Hilbert space is an input to this equivalence.

Related exact source for the reading guide

Existence of a closure-space homeomorphism · Fixed source

```
      Nonempty (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A)
        (NormedSpace.geometricStructureClosure ℂ B)) := by
```

Continuity for arbitrary subsets · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

The same map and inverse · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```

One-sided compact representability gives GSS rigidity

`MathlibAnnex.Analysis.CStarAlgebra.nonempty_starAlgEquiv_iff_nonempty_geometricStructure_homeomorph_of_exists_isCompactlyRepresented`

theorem

Obtains algebraic rigidity when either side admits a faithful compact action.

Statement

Let A, B be complex C*-algebras. For a complex C*-algebra D, write $S(D)$ for its geometric structure space. Its points are the unions of kernels of norm-one bounded complex-linear functionals supporting a maximal proper face of the closed unit ball of D; faces use real convexity. Write c_D for its hull–kernel closure:

$$c_D(E) = \left\{ I \in S(D) : \bigcap_{J \in E} J \subseteq I \right\}, \qquad E \subseteq S(D).$$

These are the geometric structure definition and its closure, with the norm-one description justified by the norm-one support characterization. Let $\mathcal{K}(H)$ denote the algebra of compact operators on a complex Hilbert space H.

Assume that at least one of A, B admits a faithful $*$-representation π on a complex Hilbert space H whose image is contained in $\mathcal{K}(H)$. Then A and B are $*$-isomorphic if and only if $S(A)$ and $S(B)$ are homeomorphic as closure spaces. Here a closure-space homeomorphism is a bijection $e : S(A) \to S(B)$ satisfying

$$e(c_A(E)) = c_B(e(E)), \qquad E \subseteq S(A).$$

The equality is required for every subset, not just finite subsets.

Assumptions

A faithful compact-valued representation is required on at least one side. All Hilbert spaces are complex and complete. The algebras and representation spaces may be zero. Representations need not be unital; no separability or countability assumption is imposed.

Conclusion

The equivalence concerns existence of a $*$-algebra isomorphism and existence of a closure-space homeomorphism. It does not require the isomorphism to induce a prescribed homeomorphism, or assert uniqueness.

Main citations

The either-side compact rigidity theorem is the stated equivalence. Rigidity for a given compact action allows an independent Hilbert universe and uses the canonical compact c0-decomposition. The existential representation comparison links the existential condition to the same reduced atomic action.

Lean source signature (exact)

Exact source and declaration. Only the literal signature is displayed here; the linked source retains the proof.

```
theorem
nonempty_starAlgEquiv_iff_nonempty_geometricStructure_homeomorph_of_exists_isCompactlyRepresent
ed
    (hCompact :
      (∃ (K : Type u) (normed : NormedAddCommGroup K),
        letI : NormedAddCommGroup K := normed
        ∃ inner : InnerProductSpace ℂ K,
          letI : InnerProductSpace ℂ K := inner
          ∃ complete : CompleteSpace K,
            letI : CompleteSpace K := complete
            ∃ pi : NonUnitalCStarRepresentation A K, pi.IsCompactlyRepresented) ∨
      (∃ (K : Type v) (normed : NormedAddCommGroup K),
        letI : NormedAddCommGroup K := normed
        ∃ inner : InnerProductSpace ℂ K,
          letI : InnerProductSpace ℂ K := inner
          ∃ complete : CompleteSpace K,
            letI : CompleteSpace K := complete
            ∃ pi : NonUnitalCStarRepresentation B K, pi.IsCompactlyRepresented)) :
    Nonempty (A ≃⋆ₐ[ℂ] B) ↔
      Nonempty (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A)
        (NormedSpace.geometricStructureClosure ℂ B))
```

In the source	Mathematical meaning

Two C*-algebras
`variable {A : Type u} {B : Type v}`
`variable [NonUnitalCStarAlgebra A]`
`[NonUnitalCStarAlgebra B]`
	The inputs A, B are arbitrary complex C*-algebras, including zero. Their size levels are independent. The nonunital representation type imposes no requirement that either algebra lack a unit.

Representation space on the A side
`(∃ (K : Type u) (normed : NormedAddCommGroup K),`
`letI : NormedAddCommGroup K := normed`
`∃ inner : InnerProductSpace ℂ K,`
`letI : InnerProductSpace ℂ K := inner`
	The left alternative requires a space with a norm and compatible complex inner product. The source calls it `K`; it is the space H of the Statement. Each `letI` makes the chosen structure on that same space available in the next type.

Complete faithful compact action on A
`∃ complete : CompleteSpace K,`
`letI : CompleteSpace K := complete`
`∃ pi : NonUnitalCStarRepresentation A K,`
`pi.IsCompactlyRepresented) ∨`
	The next existential requires completeness of that space. Its `letI` makes the Hilbert-space representation type available. There must then exist a faithful $\pi : A \to \mathcal{B}(H)$ with $\pi(A) \subseteq \mathcal{K}(H)$. The final disjunction permits the alternative on the B side.

In the source	Mathematical meaning
Representation space on the B side `(∃ (K : Type v) (normed : NormedAddCommGroup K), letI : NormedAddCommGroup K := normed ∃ inner : InnerProductSpace ℂ K, letI : InnerProductSpace ℂ K := inner`	The right alternative has its own existentially quantified normed space and compatible complex inner product, again installed by `letI`. It is not required to be the representation space of the A alternative.
Complete faithful compact action on B `∃ complete : CompleteSpace K, letI : CompleteSpace K := complete ∃ pi : NonUnitalCStarRepresentation B K, pi.IsCompactlyRepresented)) :`	This space must be complete, and must admit a faithful $\pi : B \to \mathcal{B}(H)$ with $\pi(B) \subseteq \mathcal{K}(H)$. The two complete representation conditions are alternatives, not simultaneous requirements.
Meaning of compact representability `def NonUnitalCStarRepresentation.IsCompactlyRepresented (pi : NonUnitalCStarRepresentation A H) : Prop := Function.Injective pi ∧ ∀ a : A, IsCompactOperator (pi a)`	For whichever algebra is represented, injectivity is faithfulness and the second conjunct is compactness of every represented element. Surjectivity onto all of $\mathcal{K}(H)$ is not required.
Existence of a star-isomorphism `Nonempty (A ≃⋆ₐ[ℂ] B) ↔`	`Nonempty` asserts that the type of $*$-algebra isomorphisms $A \to B$ has an element. The equivalence relates this existence to the existence in the next row; it selects neither a unique isomorphism nor a prescribed one.
Existence of a closure-space homeomorphism `Nonempty (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A) (NormedSpace.geometricStructureClosure ℂ B)) := by`	The two full arguments are the geometric closures c_A on $S(A)$ and c_B on $S(B)$. `Nonempty` requires one homeomorphism $e : S(A) \to S(B)$ between these closure spaces.
The same map and inverse `structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where continuous_toFun : c.Continuous d toFun continuous_invFun : d.Continuous c invFun`	The inherited equivalence supplies the bijection e and its inverse e^{-1}. The two fields require, for every $E \subseteq S(A)$ and $F \subseteq S(B)$ respectively, $$e(c_A(E)) \subseteq c_B(e(E)),$$ $$e^{-1}(c_B(F)) \subseteq c_A(e^{-1}(F)).$$ Both use the same e; together they give the closure equality in the Statement.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
universe u v w

variable {A : Type u} {B : Type v}
variable [NonUnitalCStarAlgebra A] [NonUnitalCStarAlgebra B]
variable {H : Type w} [NormedAddCommGroup H] [InnerProductSpace ℂ H] [CompleteSpace H]
```

The two existential Hilbert spaces have the fixed size levels of their respective algebras (u for A, v for B). The independently sized space in the cited fixed-action theorem does not change these quantifiers.

Related exact source for the reading guide

Existence of a closure-space homeomorphism · Fixed source

```
      Nonempty (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A)
        (NormedSpace.geometricStructureClosure ℂ B)) := by
```

Meaning of compact representability · Fixed source

```
def NonUnitalCStarRepresentation.IsCompactlyRepresented
    (pi : NonUnitalCStarRepresentation A H) : Prop :=
  Function.Injective pi ∧ ∀ a : A, IsCompactOperator (pi a)
```

The same map and inverse · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```

GSS rigidity when either algebra is finite-dimensional

`MathlibAnnex.Analysis.CStarAlgebra.nonempty_starAlgEquiv_iff_nonempty_geometricStructure_homeomorph_of_finiteDimensional_or`

theorem

Obtains algebraic rigidity when either algebra is finite-dimensional.

Statement

Let A, B be complex C*-algebras. For a complex C*-algebra D, write $S(D)$ for its geometric structure space. Its points are the unions of kernels of norm-one bounded complex-linear functionals supporting a maximal proper face of the closed unit ball of D; faces use real convexity. Write c_D for its hull–kernel closure:

$$c_D(E) = \left\{ I \in S(D) : \bigcap_{J \in E} J \subseteq I \right\}, \qquad E \subseteq S(D).$$

These are the geometric structure definition and its closure, with the norm-one description justified by the norm-one support characterization.

Assume that at least one of A, B is finite-dimensional as a complex vector space. Then A and B are $*$-isomorphic if and only if $S(A)$ and $S(B)$ are homeomorphic as closure spaces. Here a closure-space homeomorphism is a bijection $e : S(A) \to S(B)$ satisfying

$$e(c_A(E)) = c_B(e(E)), \qquad E \subseteq S(A).$$

The equality is required for every subset, not just finite subsets.

Assumptions

Finite-dimensionality is required on at least one side, not on both as an input. The algebras may be zero. No nonzero, unitality, separability, or prescribed dimension hypothesis is imposed.

Conclusion

The equivalence concerns existence of a $*$-algebra isomorphism and existence of a closure-space homeomorphism. It does not require the isomorphism to induce a prescribed homeomorphism, or assert uniqueness.

Main citations

The either-side finite-dimensional theorem uses the finite separating GNS representation theorem and rigidity for a given compact action. The finite separating family and compact action are supplied by these results, not additional hypotheses.

Lean source signature (exact)

Exact source and declaration. Only the literal signature is displayed here; the linked source retains the proof.

```
theorem
nonempty_starAlgEquiv_iff_nonempty_geometricStructure_homeomorph_of_finiteDimensional_or
    (hfinite : FiniteDimensional ℂ A ∨ FiniteDimensional ℂ B) :
    Nonempty (A ≃⋆ₐ[ℂ] B) ↔
      Nonempty (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A)
        (NormedSpace.geometricStructureClosure ℂ B))
```

In the source	Mathematical meaning

Two C*-algebras
`variable {A : Type u} {B : Type v}`
`variable [NonUnitalCStarAlgebra A]`
`[NonUnitalCStarAlgebra B]`
	The inputs A, B are arbitrary complex C*-algebras, including zero. Their size levels are independent. The source type allows unital algebras but does not require a unit.

Finite-dimensionality on either side
`(hfinite : FiniteDimensional ℂ A ∨`
`FiniteDimensional ℂ B) :`
	The disjunction requires $\dim_{\mathbb{C}} A < \infty$ or $\dim_{\mathbb{C}} B < \infty$. One side suffices, and no particular dimension or positivity condition is imposed.

Existence of a star-isomorphism `Nonempty (A ≃⋆ₐ[ℂ] B) ↔`	`Nonempty` says that the type of $*$-algebra isomorphisms $A \to B$ has an element. The outer equivalence relates this existence to the existence in the next row, not to a choice of a prescribed map.

Existence of a closure-space homeomorphism
`Nonempty (ClosureSpace.Homeomorph`
`(NormedSpace.geometricStructureClosure ℂ A)`
`  (NormedSpace.geometricStructureClosure ℂ B)) :=`
`by`
	The complete arguments are c_A on $S(A)$ and c_B on $S(B)$. `Nonempty` requires one homeomorphism $e : S(A) \to S(B)$ between these closure spaces.

The same map and inverse
`structure Homeomorph (c : ClosureSpace α) (d :`
`ClosureSpace β) extends α ≃ β where`
`  continuous_toFun : c.Continuous d toFun`
`  continuous_invFun : d.Continuous c invFun`
	The inherited equivalence supplies the bijection e and its inverse e^{-1}. The first field requires, for every $E \subseteq S(A)$,
	$$e(c_A(E)) \subseteq c_B(e(E)).$$
	The second requires, for every $F \subseteq S(B)$,
	$$e^{-1}(c_B(F)) \subseteq c_A(e^{-1}(F)).$$
	Both concern the same e and together express the closure equality in the Statement.

Continuity for arbitrary subsets
`def Continuous (c : ClosureSpace α) (d :`
`ClosureSpace β) (f : α → β) : Prop :=`
`  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f ''`
`s)`
	For a map $r : (X, c) \to (Y, d)$, continuity means
	$$r(c(E)) \subseteq d(r(E)) \quad (E \subseteq X).$$
	In the preceding row this definition is applied first to $r = e$ and then to $r = e^{-1}$, with the domain and target closures interchanged.

Exact surrounding source context (separate excerpts)

Exact surrounding source:

```
universe u v w z

variable {A : Type u} {B : Type v}
variable [NonUnitalCStarAlgebra A] [NonUnitalCStarAlgebra B]
```

Related exact source for the reading guide

Existence of a closure-space homeomorphism · Fixed source

```
      Nonempty (ClosureSpace.Homeomorph (NormedSpace.geometricStructureClosure ℂ A)
        (NormedSpace.geometricStructureClosure ℂ B)) := by
```

Continuity for arbitrary subsets · Fixed source

```
def Continuous (c : ClosureSpace α) (d : ClosureSpace β) (f : α → β) : Prop :=
  ∀ s : Set α, f '' c.closure s ⊆ d.closure (f '' s)
```

The same map and inverse · Fixed source

```
structure Homeomorph (c : ClosureSpace α) (d : ClosureSpace β) extends α ≃ β where
  continuous_toFun : c.Continuous d toFun
  continuous_invFun : d.Continuous c invFun
```