Naimark's problem

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1. Main theorem

Naimark's problem asks whether a C^*-algebra with only one unitary equivalence class of nonzero irreducible representations is necessarily isomorphic to an algebra of compact operators. We construct, in ZFC, a counterexample that admits a faithful representation on a separable Hilbert space.

Theorem 1.1. *There exists a unital, simple, infinite-dimensional C^*-algebra A with exactly one unitary equivalence class of nonzero irreducible representations and a unique tracial state $\widetilde{\tau}$. The GNS representation of $\widetilde{\tau}$ is faithful and acts on a separable Hilbert space.*

Moreover, A contains a unital copy $j : C \hookrightarrow A$ of the CAR algebra such that its normalized trace τ has a unique extension among all states of A, namely $\widetilde{\tau}$.

In particular, A is isomorphic to a unital C^*-subalgebra of $\mathcal{B}(\ell^2(\mathbb{N}))$, but is not isomorphic to $\mathcal{K}(K)$ for any Hilbert space K. Here *separably represented* refers to the existence of a faithful representation on a separable Hilbert space, not to norm separability of the algebra.

Akemann and Weaver obtained a counterexample from Jensen's diamond principle [2]. Further constructions under diamond control the number of irreducible classes and the tracial state space [5, 8, 9]. Calderón and Farah replace diamond by a weaker principle together with CH; they leave the existence of a counterexample in ZFC, or under CH alone, open [3, Theorem 5.4 and §8]. Thus the existence of a counterexample, or of one with a unique trace, under additional set-theoretic assumptions is not new. The assertion here combines existence in ZFC with a faithful tracial representation on a separable Hilbert space.

The two main steps are the representation theorem in Theorem 4.1 and the unique-extension argument in Theorem 5.3. The first controls every irreducible representation of a single concrete generated algebra. The second shows that extending the CAR trace to this algebra does not enlarge its cyclic Hilbert space. Both use the same decreasing projections: they isolate pure-state vectors in irreducible representations, whereas their trace tends to zero. Pure-state homogeneity and excision are established results used in the construction. The assertions specific to this report concern the generated algebra and the relation between its irreducible and tracial representations.

Throughout, algebras and Hilbert spaces are complex. Inner products are linear in the second variable, and $\theta_{x,y}z = x\langle y, z\rangle$. A representation is a $*$-homomorphism, and irreducible representations are understood to be nonzero. No separability assumption is imposed on the Hilbert spaces of irreducible representations. A nonzero irreducible representation of a unital algebra is automatically unital. We use the usual GNS correspondence and Schur's lemma; see [4, §§1.10, 3.4–3.6]. Strong limits below are strong operator limits.

2. Pure-state projections in the CAR algebra

Let

$$C = \overline{\bigcup_{n\geq 0} C_n}, \qquad C_n = M_{2^n}(\mathbb{C}), \qquad C_n \longrightarrow C_{n+1}, \quad a \longmapsto a \otimes 1_2.$$

We use the standard CAR model and its normalized trace, as in [4, §2.2 and Corollary 4.1.9]. In particular, C is unital, simple, separable, and infinite-dimensional, and its unique tracial state satisfies

$$\tau(a) = 2^{-n}\,\mathrm{Tr}(a) \qquad (a \in C_n).$$

Let ϕ_o be the product state obtained by choosing the first coordinate at each tensor factor. With compatible matrix units, put $q_n = e^{(n)}_{00}$. Then

$$q_0 = 1, \qquad q_{n+1} \leq q_n, \qquad \phi_o(q_n) = 1, \qquad \tau(q_n) = 2^{-n}. \tag{2.1}$$

The restrictions of ϕ_o to the matrix stages are pure, so ϕ_o is pure by the standard inductive-limit criterion [4, Lemma 5.5.3].

The projections q_n excise the product state. We record the calculation because this particular sequence will also control the trace.

Lemma 2.1. *For every $a \in C$,*

$$\|q_n a q_n - \phi_o(a) q_n\| \longrightarrow 0. \tag{2.2}$$

Proof. The general excision theorem for pure states is due to Akemann–Anderson–Pedersen [1, Proposition 2.2]. For the particular projection sequence chosen here, the assertion follows directly from the matrix-stage calculation below. For $a \in C_m$ and $n \geq m$, rank-one compression in the matrix stage gives $q_n a q_n = \phi_o(a) q_n$. For arbitrary $a \in C$, choose $b \in C_m$ close to a. Since both the projections and the state are contractive,

$$\|q_n a q_n - \phi_o(a) q_n\| \leq 2\|a - b\| \qquad (n \geq m).$$

Density proves the assertion. □

Let I index the unitary equivalence classes of irreducible GNS representations of C. Choose a pure state ϕ_i in each class, with the preceding state at $o \in I$, and write (H_i, π_i, ξ_i) for its GNS triple. Every nonzero irreducible representation is equivalent to one of the π_i: every nonzero vector in an irreducible representation is cyclic, and the GNS uniqueness theorem identifies its representation with that of its vector state [4, Lemma 1.10.9 and Proposition 3.6.5].

We now apply the homogeneity theorem of Kishimoto–Ozawa–Sakai [6, Theorems 1.1 and 2.1]. For pure states of a separable C^*-algebra whose GNS representations have the same kernel, it provides an asymptotically inner automorphism carrying one state to the other; its implementing path may be chosen to start at the identity. Here simplicity makes all the GNS kernels zero. Hence, for each $i \neq o$, choose one such automorphism α_i with

$$\phi_i \circ \alpha_i = \phi_o, \tag{2.3}$$

and take $\alpha_o = \mathrm{id}_C$. Only approximate innerness of these automorphisms will be needed from now on. Define

$$p_{i,n} = \alpha_i(q_n), \qquad f_n = q_n - q_{n+1}, \qquad e_{i,n} = p_{i,n} - p_{i,n+1} = \alpha_i(f_n). \tag{2.4}$$

For every i, the $p_{i,n}$ decrease from 1, while the $e_{i,n}$ are pairwise orthogonal. By Lemma 2.1 and (2.3),

$$\phi_i(p_{i,n}) = 1, \qquad \|p_{i,n} a p_{i,n} - \phi_i(a) p_{i,n}\| \longrightarrow 0 \quad (a \in C). \tag{2.5}$$

Lemma 2.2. *There are elements $w_{i,n} \in C$ such that*

$$w_{i,n}^* w_{i,n} = e_{i,n}, \qquad w_{i,n} w_{i,n}^* = f_n, \qquad w_{o,n} = f_n. \tag{2.6}$$

For these same choices,

$$\tau(p_{i,n}) = 2^{-n} \qquad (i \in I,\ n \geq 0). \tag{2.7}$$

Proof. Approximate innerness gives a unitary g with $\|\alpha_i(f_n) - g f_n g^*\| < 1$. The standard perturbation lemma for projections [4, Lemma 1.5.7(2)] gives a unitary t with $t\alpha_i(f_n)t^* = g f_n g^*$. Thus $w_{i,n} = g^* t \alpha_i(f_n)$ has the required supports. At $i = o$ use f_n itself.

The trace takes equal values on the initial and final projections of a partial isometry [4, Example 4.1.6]. Applying this to (2.6) gives

$$\tau(p_{i,n}) - \tau(p_{i,n+1}) = \tau(q_n) - \tau(q_{n+1}).$$

Both projection sequences start at 1, so telescoping and (2.1) give (2.7). □

We shall also use the following standard consequence of excision. We include the proof to identify the common fixed spaces associated with the chosen projections.

Lemma 2.3. *Let $\sigma : C \to \mathcal{B}(K)$ be a unital representation, and let P_i^σ be the projection onto $\bigcap_n \sigma(p_{i,n})K$. Then*

$$P_i^\sigma \sigma(a) P_i^\sigma = \phi_i(a) P_i^\sigma \qquad (a \in C). \tag{2.8}$$

In π_i, this projection is θ_{ξ_i,ξ_i}; in an irreducible representation inequivalent to π_i, it is zero.

Proof. Monotone convergence of projections gives $\sigma(p_{i,n}) \to P_i^\sigma$ strongly [4, Lemma 3.1.3]. Apply σ to (2.5). Passing to strong limits, using continuity of multiplication on bounded sets [4, Example 3.1.1(3)], proves (2.8).

In π_i, every $\pi_i(p_{i,n})$ fixes ξ_i. If $P = P_i^{\pi_i}$, then

$$P\pi_i(a)\xi_i = \phi_i(a)\xi_i.$$

Cyclicity shows that $PH_i = \mathbb{C}\xi_i$. In any other irreducible representation, a unit vector in the common range would have vector state ϕ_i by (2.8). GNS uniqueness [4, Lemma 1.10.9] would then give equivalence with π_i. This proves the final assertion. □

3. The generated algebra

We use the term *shell* for the difference of two consecutive projections in a decreasing sequence. The following orthogonal-sum calculation constructs the generators and determines their images in an arbitrary representation.

Lemma 3.1. *Let P_n, Q_n be decreasing orthogonal projections on a Hilbert space with $P_0 = Q_0 = 1$. Suppose*

$$W_n^* W_n = P_n - P_{n+1}, \qquad W_n W_n^* = Q_n - Q_{n+1}.$$

Write $P = \mathrm{s-}\lim_n P_n$ and $Q = \mathrm{s-}\lim_n Q_n$. The sums $\sum_{n<N} W_n$ and $\sum_{n<N} W_n^*$ converge strongly to S and S^*, respectively, with

$$S^*S = 1 - P, \qquad SS^* = 1 - Q. \tag{3.1}$$

If V is a unitary satisfying $V(P_n - P_{n+1}) = W_n$, then

$$V = S + D, \quad D = VP = QDP, \quad D^*D = P, \quad DD^* = Q. \tag{3.2}$$

Conversely, if $P = \theta_{\xi,\xi}$ and $Q = \theta_{\eta,\eta}$ for unit vectors ξ, η, then $S + \theta_{\eta,\xi}$ is the unique unitary satisfying these relations and sending ξ to η.

Proof. The limits of the projections exist by monotone convergence [4, Lemma 3.1.3]. Put $S_N = \sum_{n<N} W_n$. The support projections are pairwise orthogonal on both sides, so

$$S_N^* S_N = 1 - P_N, \qquad S_N S_N^* = 1 - Q_N,$$

and, for $M \geq N$,

$$\|(S_M - S_N)x\|^2 = \langle x, (P_N - P_M)x \rangle.$$

This tends to zero as $M, N \to \infty$; the analogous calculation applies to the adjoint sums. The sums are contractions, their limits are adjoints of one another, and continuity of multiplication in the strong operator topology on bounded sets [4, Example 3.1.1(3)] gives (3.1).

If V satisfies the finite relations, then $V(1 - P_N) = S_N$. Hence $V(1 - P) = S$ and $V(1 - P)V^* = 1 - Q$, which prove (3.2). Conversely $S + \theta_{\eta,\xi}$ is unitary because the mixed terms vanish and its two products with its adjoint equal 1. Any other such unitary equals S on $(1 - P)K$ and has the prescribed action on $PK = \mathbb{C}\xi$; it is therefore the same operator. ☐

Form the reduced atomic representation

$$H = \bigoplus_{i \in I} H_i, \qquad \Pi = \bigoplus_{i \in I} \pi_i, \qquad \zeta_i = \iota_i \xi_i,$$

where $\iota_i : H_i \to H$ is the coordinate inclusion. It is faithful, since every π_i is a nonzero representation of the simple algebra C. Let $E_i = \theta_{\zeta_i, \zeta_i}$. By Lemma 2.3,

$$\text{s-}\lim_n \Pi(p_{i,n}) = E_i, \qquad \text{s-}\lim_n \Pi(q_n) = E_o. \tag{3.3}$$

Indeed, the assertion holds on each summand, and uniform boundedness extends it from finite-support vectors to H.

Apply Lemma 3.1 to the operators $\Pi(w_{i,n})$ and define

$$S_i = \text{s-}\lim_N \sum_{n<N} \Pi(w_{i,n}), \qquad R_i = \theta_{\zeta_o, \zeta_i}, \qquad L_i = S_i + R_i. \tag{3.4}$$

Then

$$L_i \in \mathcal{U}(\mathcal{B}(H)), \qquad L_i \zeta_i = \zeta_o, \qquad L_i \Pi(e_{i,n}) = \Pi(w_{i,n}), \qquad L_o = 1. \tag{3.5}$$

The last equality follows from $w_{o,n} = f_n$ and $S_o = 1 - E_o$.

Define the concrete C^*-algebra

$$A = C^*(\Pi(C), \{L_i : i \in I\}) \subseteq \mathcal{B}(H), \qquad j(a) = \Pi(a) \quad (a \in C). \tag{3.6}$$

This fixes the algebra and all its generators. The strong limits E_i and S_i are operators on H; their separate membership in A is not required.

Proposition 3.2. *The algebra A is unital and infinite-dimensional, j is injective, and the inclusion $A \subseteq \mathcal{B}(H)$ is irreducible.*

Proof. Only irreducibility needs proof. Since the π_i are pairwise inequivalent irreducible representations, Schur's lemma and the direct-sum commutant formula [4, Proposition 3.5.2] give

$$\Pi(C)' = \left\{ \bigoplus_i \lambda_i 1_{H_i} : \sup_i |\lambda_i| < \infty \right\}.$$

If T in this commutant also commutes with every L_i, evaluation at ζ_i gives $\lambda_o \zeta_o = T L_i \zeta_i = L_i T \zeta_i = \lambda_i \zeta_o$. Hence every $\lambda_i = \lambda_o$, and $A' = \mathbb{C}1$. This proves irreducibility. ☐

4. All irreducible representations

The preceding proposition only treats the displayed inclusion. The main representation-theoretic assertion is that the same model exhausts every irreducible representation of A.

Theorem 4.1. *Every nonzero irreducible representation $\rho : A \to \mathcal{B}(K)$ is unitarily equivalent to the inclusion $A \subseteq \mathcal{B}(H)$.*

Proof. The representation ρ is unital. Put $\sigma = \rho \circ j$ and $V_i = \rho(L_i)$. Let P_i and $Q = P_o$ be the common-range projections of $\sigma(p_{i,n})$ and $\sigma(q_n)$, respectively. Applying Lemma 3.1 to the represented finite identities in (3.5) gives

$$S_i^\sigma = \mathrm{s}\!-\!\lim_N \sum_{n<N} \sigma(w_{i,n}), \tag{4.1}$$

$$V_i = S_i^\sigma + D_i, \qquad D_i = V_i P_i = Q D_i P_i, \tag{4.2}$$

$$D_i^* D_i = P_i, \qquad D_i D_i^* = Q. \tag{4.3}$$

The adjoint sums converge strongly as well. These strong limits are taken on K after applying ρ to the finite algebraic identities. We do not assert that ρ preserves strong operator limits formed on H.

A nonzero common fixed space. Suppose first that every $P_i = 0$. Then $V_i = S_i^\sigma$. Any closed subspace reducing $\sigma(C)$ reduces the finite sums in (4.1) and their adjoints, and hence also their strong limits. It therefore reduces every V_i, so it reduces $\rho(A)$. Irreducibility of ρ implies that σ is irreducible. It must then be equivalent to one of the π_i. By Lemma 2.3, that representation has a nonzero common fixed vector for the corresponding projection sequence, contradicting $P_i = 0$. Thus some $P_i \neq 0$. Since (4.3) implies $V_i P_i V_i^* = Q$, also $Q \neq 0$.

Choose a unit vector $\eta_o \in QK$ and put

$$\eta_i = V_i^* \eta_o. \tag{4.4}$$

Then $\eta_i \in P_i K$, $\|\eta_i\| = 1$, and $V_i \eta_i = \eta_o$. Since $V_o = 1$, this definition agrees with the chosen vector η_o. The compression formula (2.8) shows that

$$\langle \eta_i, \sigma(a)\eta_i \rangle = \phi_i(a) \qquad (a \in C). \tag{4.5}$$

The cyclic subspaces. Let $M_i = \overline{\sigma(C)\eta_i}$. By GNS uniqueness and purity [4, Lemma 1.10.9 and Proposition 3.6.5], there is a pointed unitary $T_i : H_i \to M_i$ intertwining π_i and $\sigma|_{M_i}$. Distinct M_i are orthogonal: the projection from one to the other is an intertwiner, hence zero by Schur's lemma; compare [4, Lemma 3.5.1]. Consequently the T_i assemble into an isometry $W : H \to K$ satisfying

$$W\zeta_i = \eta_i, \qquad W\Pi(a) = \sigma(a)W \quad (a \in C). \tag{4.6}$$

Its range $M = \overline{\bigoplus_i M_i}$ is closed and reduces $\sigma(C)$.

On M_i, the projection P_i is the rank-one projection onto $\mathbb{C}\eta_i$, while on M_k, $k \neq i$, it vanishes, by Lemma 2.3. To justify restriction of the limiting projections, observe that the projection onto M_k commutes with every $\sigma(p_{i,n})$, hence with their strong limit. It follows, first on finite sums and then by continuity, that

$$P_i M \subseteq \mathbb{C}\eta_i, \qquad QM \subseteq \mathbb{C}\eta_o. \tag{4.7}$$

The space M reduces S_i^σ, since it reduces $\sigma(C)$ and both strong sums. Moreover, (4.2) and (4.7) give

$$D_i M \subseteq \mathbb{C}\eta_o \subseteq M.$$

For the adjoint, $D_i^* = P_i D_i^* Q$ and $D_i^* \eta_o = \eta_i$ give $D_i^* M \subseteq \mathbb{C}\eta_i \subseteq M$. Thus M reduces every V_i, and therefore all of $\rho(A)$. It contains the unit vector η_o, so irreducibility yields $M = K$. In particular, W is unitary.

The additional generators. The intertwining identity for C in (4.6) passes through the finite shell sums and then their strong limits, giving $WS_i = S_i^\sigma W$. It also gives $WE_i = P_i W$ by (3.3). For $x \in H$,

$$WR_i x = \eta_o \langle \zeta_i, x\rangle = V_i P_i W x = D_i W x.$$

Hence $WL_i = V_i W$ for every i. Together with (4.6), norm generation gives $Wa = \rho(a)W$ for all $a \in A$, as required. □

Corollary 4.2. *The algebra A is simple and is not isomorphic to an algebra of compact operators.*

Proof. A nonzero proper closed ideal would give an irreducible representation through the nonzero quotient, using the standard existence of pure states and their irreducible GNS representations [4, §3.6]. Such a representation is not faithful, whereas Theorem 4.1 makes it equivalent to the faithful inclusion. Hence there is no such ideal. Since A is unital and infinite-dimensional, it cannot be isomorphic to $\mathcal{K}(K)$: a compact-operator algebra is unital only when K is finite-dimensional. □

5. The tracial representation

The second part of the construction takes place on this same algebra A. By the state-extension theorem [4, Lemma 1.7.6(3)], choose a state Φ of A with $\Phi \circ j = \tau$. In its GNS triple $(K_\Phi, \rho_\Phi, \Omega_\Phi)$ put

$$\sigma_\Phi = \rho_\Phi \circ j, \qquad M_\Phi = \overline{\sigma_\Phi(C)\Omega_\Phi}.$$

The state-extension theorem does not assert that Φ is tracial. The crucial point is that its full GNS space is already generated by the CAR subalgebra. A related equality of cyclic GNS spaces for finite crossed products appears in [3, Lemma 6.3(2)]. Here the hypotheses are different: the equality will follow from the vanishing of the represented decreasing projections, as proved below.

Lemma 5.1. *Suppose $\sigma : C \to \mathcal{B}(K)$ is unital and a unit vector Ω has vector state τ. For every projection $p \in C$ and every $b \in C$,*

$$\|\sigma(p)\sigma(b)\Omega\|^2 \le \|b\|^2 \tau(p). \tag{5.1}$$

Thus $\sigma(p_{i,n})x \to 0$ and $\sigma(q_n)x \to 0$ for every $x \in \overline{\sigma(C)\Omega}$.

Proof. This is the standard tracial L^2 estimate (see [4, §4.2]) in the form needed here:

$$\|\sigma(p)\sigma(b)\Omega\|^2 = \tau(b^*pb) = \tau(pbb^*p) \le \|b\|^2\tau(p).$$

The last inequality follows from $0 \le pbb^*p \le \|b\|^2 p$. Equation (2.7) now bounds the squared norm by $\|b\|^2 2^{-n}$. Uniform contractivity extends convergence from the cyclic orbit to its closure. □

Proposition 5.2. *For every state extension Φ of τ to A, one has $M_\Phi = K_\Phi$. In this representation,*

$$\rho_\Phi(L_i) = \mathrm{s}\!-\!\lim_N \sum_{n<N} \sigma_\Phi(w_{i,n}), \qquad \rho_\Phi(L_i)^* = \mathrm{s}\!-\!\lim_N \sum_{n<N} \sigma_\Phi(w_{i,n})^*. \tag{5.2}$$

In particular, K_Φ is separable.

Proof. Apply Lemma 3.1 in K_Φ. If P_i and Q are the common-range projections of the represented projection sequences, it gives

$$\rho_\Phi(L_i) = S_i^\Phi + D_i, \qquad D_i = \rho_\Phi(L_i)P_i = QD_iP_i.$$

By Lemma 5.1, both P_i and Q vanish on M_Φ. Hence D_i and $D_i^* = P_i D_i^* Q$ vanish there. The cyclic subspace M_Φ reduces $\sigma_\Phi(C)$, so it reduces both strong sums S_i^Φ and $(S_i^\Phi)^*$. It follows that it reduces every $\rho_\Phi(L_i)$, and thus $\rho_\Phi(A)$.

Since $\Omega_\Phi \in M_\Phi$ and its A-orbit is dense in K_Φ, this proves $M_\Phi = K_\Phi$. The projections P_i and Q consequently vanish on the whole GNS space, proving (5.2). The continuous orbit map from the separable algebra C now has dense range in K_Φ, so this Hilbert space is separable. $\square$

Let $(H_\tau, \pi_\tau, \Omega_\tau)$ be the GNS triple of τ on C. By GNS uniqueness [4, Lemma 1.10.9] and Proposition 5.2, there is a pointed unitary $T_\Phi : H_\tau \to K_\Phi$ intertwining the two C-representations. Consequently

$$\Theta_\Phi(a) = T_\Phi^* \rho_\Phi(a) T_\Phi \quad (a \in A) \tag{5.3}$$

is a representation of A on H_τ, with

$$\Theta_\Phi(j(b)) = \pi_\tau(b), \qquad \Theta_\Phi(L_i) = \mathrm{s}\!-\!\lim_N \sum_{n<N} \pi_\tau(w_{i,n}). \tag{5.4}$$

It is faithful by simplicity of A, and therefore isometric. This constructs a faithful separable representation of the already defined algebra, rather than merely another family of operators satisfying its displayed relations.

Theorem 5.3. *The normalized trace τ of C has a unique state extension $\widetilde{\tau}$ to A. This extension is tracial and faithful, and it is the only tracial state of A.*

Proof. Existence was obtained by state extension. For uniqueness, let Φ and Ψ be two extensions. Their C-representations are cyclic on their full GNS spaces by Proposition 5.2. GNS uniqueness [4, Lemma 1.10.9] therefore gives a unitary $W : K_\Phi \to K_\Psi$ with

$$W\Omega_\Phi = \Omega_\Psi, \qquad W\sigma_\Phi(b) = \sigma_\Psi(b)W \quad (b \in C).$$

Passing through the finite sums and their strong limits in (5.2) gives $W\rho_\Phi(L_i) = \rho_\Psi(L_i)W$. Since these operators and $j(C)$ generate A, the intertwining holds for every $a \in A$. Its matrix coefficient at the cyclic vector gives $\Phi(a) = \Psi(a)$. Denote the unique extension by $\widetilde{\tau}$.

We next prove traciality; it has not been used for $\widetilde{\tau}$ so far. For a unitary $u \in C$, the state

$$a \longmapsto \widetilde{\tau}(j(u)^* a j(u))$$

also extends τ. By uniqueness it equals $\widetilde{\tau}$, whence

$$\widetilde{\tau}(j(u)a) = \widetilde{\tau}(aj(u)) \qquad (a \in A).$$

Since unitaries linearly span a unital C^*-algebra [4, Exercise 1.11.16(4)–(5)], the same equality holds with any $b \in C$ in place of u.

Put $s_{i,N} = \sum_{n<N} w_{i,n}$. We have

$$\widetilde{\tau}(j(s_{i,N})a) = \widetilde{\tau}(aj(s_{i,N})).$$

In the GNS representation of $\widetilde{\tau}$, the operators $\rho(j(s_{i,N}))$ converge strongly to $\rho(L_i)$ by (5.2). Taking matrix coefficients at Ω and $\rho(a)\Omega$ gives

$$\widetilde{\tau}(L_i a) = \widetilde{\tau}(aL_i) \qquad (a \in A).$$

The centralizer

$$\{x \in A : \widetilde{\tau}(xa) = \widetilde{\tau}(ax) \text{ for all } a \in A\}$$

is a norm-closed unital $*$-subalgebra. It contains $j(C)$ and every L_i, and hence equals A. Thus $\widetilde{\tau}$ is tracial.

Any tracial state of A restricts to the unique trace τ of C [4, Corollary 4.1.9], so uniqueness of the state extension proves uniqueness among tracial states. Finally, a tracial state on a simple C^*-algebra is faithful [4, Corollary 4.1.4]. This applies to $\widetilde{\tau}$. $\square$

Proof of Theorem 1.1. Take A and j from (3.6). By Proposition 3.2, Theorem 4.1, and Corollary 4.2, this algebra is unital, simple, and infinite-dimensional, and all its nonzero irreducible representations are equivalent. The unique trace and the stronger state-extension assertion are Theorem 5.3. By Proposition 5.2, its GNS space is separable; its representation is faithful by simplicity. All choices and arguments above are in ZFC. No diamond principle or instance of CH is used. □

6. Consequences

Corollary 6.1. *The algebra A has no nonzero irreducible representation on a separable Hilbert space. Its faithful tracial GNS representation has no nonzero irreducible closed subrepresentation.*

Proof. A separable irreducible representation would, by Theorem 4.1, be faithful and represent the unique irreducible class. Rosenberg's theorem [7, Theorem 4] would then identify A with the compact operators, contrary to Corollary 4.2. A closed subspace of a separable Hilbert space is separable, proving the second assertion. □

Thus the existence of a faithful representation on a separable Hilbert space cannot replace the existence of an *irreducible* representation on such a space in this classical result. The latter is also the hypothesis of [4, Corollary 5.5.6]; the broader formulations in [4, §5.5, p. 155] and [3, abstract and p. 3] omit the irreducibility qualification.

Write $\mathfrak{c} = 2^{\aleph_0}$, and let $\mathrm{dens}(X)$ denote the density character of a topological space X.

Corollary 6.2. *For the constructed algebra A, its faithful tracial GNS space H_τ, and the Hilbert space H of the distinguished irreducible representation,*

$$|A| = \mathrm{dens}(A) = \mathfrak{c}, \qquad H_\tau \textit{ is separable and infinite-dimensional}, \qquad \mathrm{dens}(H) = \mathfrak{c}.$$

Consequently, H_τ has a countably infinite orthonormal basis, whereas every orthonormal basis of H has cardinality $\mathfrak{c}$.

Proof. An operator on the separable space H_τ is determined by countably many matrix coefficients. The faithful representation (5.3) therefore gives $|A| \le \mathfrak{c}$, while scalar multiples of the identity give the reverse inequality. Hence $|A| = \mathfrak{c}$ and $\mathrm{dens}(A) \le \mathfrak{c}$. The reverse density inequality is the lower bound for counterexamples to Naimark's problem due to Akemann–Weaver [2, Proposition 6]. Thus $\mathrm{dens}(A) = \mathfrak{c}$. The space H_τ is separable by Proposition 5.2, and it is infinite-dimensional because it faithfully represents the infinite-dimensional algebra A.

For H, we recall the pure-state argument behind the classical small-representation obstruction [4, Corollary 5.5.6], using the explicit CAR subalgebra. Its diagonal algebra is $C(\{0,1\}^{\mathbb{N}})$ [4, §2.2, p. 54]; choose in it a self-adjoint d with Cantor spectrum. For each $t \in \mathrm{sp}(d)$, evaluation at t extends to a pure state of A [4, Lemma 5.4.1]. In its GNS representation the cyclic vector is a t-eigenvector of d, since the state vanishes on $(d - t1)^2$. By Theorem 4.1, these vectors may all be transported into the fixed representation on H. Distinct eigenvalues of the same self-adjoint operator give an orthogonal, hence uniformly separated, family of cardinality $\mathfrak{c}$. Therefore every dense subset of H has cardinality at least $\mathfrak{c}$, so $\mathrm{dens}(H) \ge \mathfrak{c}$. Conversely, a nonzero vector is cyclic for this irreducible representation; the orbit of a norm-dense subset of A of cardinality $\mathfrak{c}$ is dense in H. Hence $\mathrm{dens}(H) \le \mathfrak{c}$.

Finally, for an infinite-dimensional Hilbert space, the density character equals the cardinality of an orthonormal basis. The last assertions follow from the separability and infinite-dimensionality of H_τ and from $\mathrm{dens}(H) = \mathfrak{c}$. □

Corollary 6.3. *Over ZFC, CH is equivalent to the existence of a counterexample to Naimark's problem of norm density $\aleph_1$.*

Proof. Under CH, Corollary 6.2 supplies such a counterexample. Conversely, the lower bound in [2, Proposition 6] gives $\mathfrak{c} \leq \aleph_1$ from the existence of one of that density, and hence CH. $\square$

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