Naimark's problem: Prior-Art Search

Naimark's problem ▪ R1 ▪ Search cutoff: 20 September 2026

Status of this version. This note records a bounded AI-assisted literature search concerning the accompanying Brief Report. No equivalent result was identified in the sources examined. This is not a novelty certificate, peer review, or independent verification of the proposed proof.

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1. Claim boundary

The comparison concerns the following claim in the accompanying Brief Report, *Naimark's problem*: in ZFC there exists a unital, simple, infinite-dimensional complex C^*-algebra with exactly one unitary equivalence class of nonzero irreducible representations and a faithful representation on a separable Hilbert space. The same algebra is claimed to have a unique tracial state whose GNS representation is separable and faithful. The normalized trace of its specified CAR subalgebra is claimed to have a unique extension among all states of the larger algebra.

Here *separably represented* means faithfully represented on a separable Hilbert space; it does not mean that the algebra is norm separable or that the representation is irreducible. All irreducible representations are counted, without a restriction on the size of their Hilbert spaces. The unconditional assertion is distinguished both from constructions under additional set-theoretic assumptions and from examples having two or more irreducible equivalence classes.

The claimed norm-density and Hilbert-dimension conclusions, and the consequence under CH, are corollaries of this construction and established lower bounds. They are not independent claims to a new cardinal-arithmetic method. The manuscript's mathematical claims and its formal-verification status are separate from the literature comparison recorded here.

2. Problem lineage and closest inspected results

The classical boundary. Rosenberg's account of Naimark's question and his Theorem 4 concern a C^*-algebra acting *irreducibly* on a separable Hilbert space [8, §5, pp. 528–530]. If it has only one irreducible equivalence class, it is the algebra of all compact operators on that space. This theorem is a cited input, not a result the Brief Report purports to disprove or re-establish.

Constructions under additional axioms. Akemann–Weaver construct a counterexample using Jensen's $\diamondsuit_{\aleph_1}$ [2]. Their Proposition 6 gives the lower bound $\mathfrak{c} = 2^{\aleph_0}$ for its norm density; Corollary 7 establishes independence of the existence of an $\aleph_1$-generated counterexample. That cardinality-restricted independence statement is not an independence theorem for unrestricted existence.

Farah–Hirshberg give further conditional examples with controlled numbers of irreducible equivalence classes and additional properties [6]. Vaccaro studies the tracial state spaces of such examples [9, 10]. In particular, the existence of a counterexample with a unique trace under additional axioms is not new. The 2020 corrigendum replaces an auxiliary argument whose original proof did not cover all extreme traces; the original article and the corrigendum must be read together.

The closest general construction. Calderón–Farah's published Theorem 5.4 uses $\diamondsuit^{\mathrm{Cohen}}$+CH, and their consistency result allows Jensen's full diamond principle to fail [4]. Section 8 explicitly leaves construction in ZFC alone, or in ZFC with CH alone, open. Their Theorem B, a special case of Theorem 7.1, also gives separably represented examples with a prescribed finite number $m \geq 2$ of irreducible equivalence classes. Neither result gives the unconditional one-class construction under comparison here.

3. The representation hypothesis

The exact hypothesis matters in this comparison. Farah's first edition, Corollary 5.5.6 on p. 158, assumes an *irreducible* representation on a Hilbert space of density character less than $\mathfrak{c}$ [5]. Its proof places an orthogonal family of vectors in one irreducible GNS representation. A faithful reducible representation is not asserted to contain those vectors.

The introductory discussion on p. 155 of the same book uses the broader wording *faithfully represented*. Related broader formulations occur in Akemann–Weaver's introduction and in the abstract and p. 3 of the published Calderón–Farah article [2, 4, 5]. The supplied originals have been checked at these locations. The Brief Report's strengthened claim, if established, contradicts that broader exclusion, not Rosenberg's theorem or Farah's explicitly stated irreducible-representation corollary. No inference about the authors' intentions is made.

Nor does the claim contradict Calderón–Farah's Proposition 7.2, which assumes that *all* irreducible representations act on separable Hilbert spaces. The Brief Report claims the opposite for its algebra: its faithful tracial representation is separable, but none of its nonzero irreducible representations is separable.

This last distinction, by itself, is not presented as a new phenomenon. For example, a II_1 factor with separable predual has a faithful separable standard representation, whereas its representations on separable Hilbert spaces are normal [3, Theorem 6.5]. A nonzero normal irreducible representation would identify its image both as a type II_1 von Neumann algebra and as all of $\mathcal{B}(K)$, which is impossible. This application of the cited theorem gives the representation-size distinction without a one-class conclusion. The conjunction with a unique irreducible equivalence class and unconditional existence is the comparison target.

4. Methods and attribution

Pure-state homogeneity, excision, and standard GNS arguments are established inputs. The equal-GNS-kernel homogeneity theorem of Kishimoto–Ozawa–Sakai includes asymptotically inner automorphisms and paths starting at the identity [7, Theorems 1.1 and 2.1]; these features are not attributed to the present project. Akemann–Anderson–Pedersen's excision theorem is likewise cited [1, Proposition 2.2]. The Brief Report retains a short calculation for its specified CAR projections rather than a new proof of general excision.

The central issue is control of the representations of the *final* algebra. Adjoining operators to identify initially chosen states does not by itself dispose of the states of the enlarged algebra; this difficulty is explicit in Akemann–Weaver and in Weaver's exposition [2, 11]. The proposed construction instead fixes one concrete generated algebra and uses its finite projection relations inside an arbitrary irreducible representation. Its second argument extends the CAR trace to this same algebra and proves that the cyclic Hilbert space does not enlarge. A related cyclic-space identity for finite crossed products is already present in Calderón–Farah, Lemma 6.3(2); the Brief Report cites it while proving the statement needed under its different hypotheses [4].

No matching prior theorem for the combined conclusion was identified. This does not imply that each lemma or proof device is new: the standard projection, GNS, and tracial arguments remain attributed to the literature.

5. Search record and limitations

The record combines the earlier search with the subsequent reading of the supplied Rosenberg article, Vaccaro corrigendum, Farah first edition, and Calderón–Farah journal version. Backward and selected forward citation searches started from KOS, Akemann–Weaver, and Calderón–Farah; they included conditional refinements, special-class positive results, representation-size restrictions, and adjacent geometric and foundational work. Sources included arXiv, publisher and author pages, MathNet, J-STAGE, and ordinary web search. A targeted web update and proof-citation check were made on the cutoff date.

Recorded candidate works or editions	30
Records with relevant primary passages inspected	19
Partial text, abstract, or preview only	4
Bibliographic or upstream-citation leads	7

These are evidence levels for candidate records, not counts of independently verified proofs. The detailed ledger preserves the earlier readings and the later changes of status. An abstract-only or inaccessible source is not used as resolved negative evidence.

The mandatory original-text requests recorded in the earlier note have been resolved. Farah's 2026 second edition has been checked only through publisher information and previews; its full revised chapters are not claimed as inspected. Other incomplete leads remain listed for follow-up. No systematic MathSciNet or zbMATH export, exhaustive traversal of the citation graph, or independent human audit of the candidate proof was performed.

6. Bounded finding and update rule

No equivalent result was identified in the sources examined through 20 September 2026. The closest inspected general constructions require additional set-theoretic hypotheses. This is a record of the sources examined, not a guarantee that no earlier result exists and not a certification that the Brief Report proves its claim.

Any newly identified close result reopens the comparison. If public release occurs more than 30 days after the cutoff, a delta search is required unless the older cutoff remains prominent and that state is explicitly accepted. Updates must identify a successor note rather than overwrite the historical search record.

References

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