Sphere Rigidity

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1. Main statements

Let X be a real normed space, and write

$$B_X = \{x \in X : \|x\|_X \le 1\}, \qquad S_X = \{x \in X : \|x\|_X = 1\}.$$

We regard S_X as a metric subspace of X, with

$$d_X(x,y) = \|x-y\|_X \qquad (x, y \in S_X). \tag{1.1}$$

This distance is sometimes called the chord metric, but we shall use the explicit formula (1.1). A *surjective sphere isometry* is a surjective map $\Delta : S_X \to S_Y$ such that

$$\|\Delta(x) - \Delta(y)\|_Y = \|x-y\|_X \qquad (x, y \in S_X). \tag{1.2}$$

Such a map is automatically injective and hence is a metric isomorphism between the two unit spheres.

Theorem 1.1 (Rigidity of unit spheres)**.** *Let X and Y be finite-dimensional real normed spaces. If there exists a surjective sphere isometry $\Delta : S_X \to S_Y$, then there exists a surjective real-linear isometry $T : X \to Y$.*

Corollary 1.2 (Classification by the unit-sphere metric)**.** *For finite-dimensional real normed spaces X and Y, the following assertions are equivalent:*

(i) the metric spaces (S_X, d_X) and (S_Y, d_Y) are isometric;

(ii) there exists a surjective real-linear isometry from X onto Y.

Remark 1.3 (Object rigidity versus marked rigidity)**.** The linear isometry supplied by Theorem 1.1 is not asserted to extend the prescribed map Δ, nor is it asserted to be unique or canonical. The theorem determines the linear isometry class of the ambient space from the metric isometry class of its unit sphere, but it does not solve Tingley's marked extension problem. The object-level question is stated in [7, Section 5].

2. Preliminary reductions and coordinate models

Throughout this section, X and Y are finite-dimensional real normed spaces and $\Delta : S_X \to S_Y$ is a surjective sphere isometry.

2.1 The zero-dimensional case

Lemma 2.1. *For a normed space X, the unit sphere S_X is empty if and only if $X = \{0\}$.*

Proof. If $X = \{0\}$, then no vector has norm 1, and hence $S_X = \varnothing$. Conversely, if $x \in X$ is nonzero, then $x/\|x\|_X \in S_X$. $\square$

Lemma 2.2 (The zero-dimensional case)**.** *If either X or Y is zero-dimensional, then both spaces are zero-dimensional and there is a unique surjective real-linear isometry from X onto Y.*

Proof. Suppose first that $X = \{0\}$. Then $S_X = \varnothing$, so surjectivity of Δ implies $S_Y = \varnothing$. By Lemma 2.1, we have $Y = \{0\}$. Conversely, if $Y = \{0\}$, then $S_Y = \varnothing$. A map into the empty set can exist only when its domain is empty, so the given map $\Delta : S_X \to S_Y$ forces $S_X = \varnothing$ and hence $X = \{0\}$. The unique map between the two zero-dimensional spaces is linear, bijective, and norm preserving. $\square$

Henceforth, until the zero-dimensional case is reinstated in Section 9, both spaces are assumed to be nonzero.

2.2 The radial extension

Define the radial extension of Δ by

$$\widehat{\Delta}(0) = 0, \qquad \widehat{\Delta}(x) = \|x\|_X \,\Delta\!\left(\frac{x}{\|x\|_X}\right) \quad (x \neq 0). \tag{2.1}$$

It preserves the radial parameter:

$$\left\|\widehat{\Delta}(x)\right\|_Y = \|x\|_X \qquad (x \in X). \tag{2.2}$$

Proposition 2.3 (Radial bi-Lipschitz extension)**.** *The map $\widehat{\Delta} : X \to Y$ is bijective, maps B_X onto B_Y, and satisfies*

$$\left\|\widehat{\Delta}(x) - \widehat{\Delta}(y)\right\|_Y \le 3\,\|x - y\|_X \qquad (x, y \in X).$$

Its inverse is the radial extension of Δ^{-1} and satisfies the analogous estimate.

Proof. Let $\widehat{\Delta^{-1}}$ denote the radial extension of Δ^{-1}. The definitions show directly that

$$\widehat{\Delta^{-1}} \circ \widehat{\Delta} = \mathrm{id}_X, \qquad \widehat{\Delta} \circ \widehat{\Delta^{-1}} = \mathrm{id}_Y,$$

so $\widehat{\Delta}$ is bijective. We prove the Lipschitz estimate. Since the desired bound is symmetric in x and y, interchange them if necessary and write $x = ru$ and $y = sv$, where $r \ge s \ge 0$ and the unit vectors are chosen whenever the corresponding radius is positive. If $s = 0$, then (2.2) gives

$$\left\|\widehat{\Delta}(x) - \widehat{\Delta}(y)\right\|_Y = r = \|x - y\|_X .$$

Assume that $s > 0$. Using (1.2), we obtain

$$\begin{aligned}\left\|\widehat{\Delta}(x) - \widehat{\Delta}(y)\right\|_Y &\le \|r\Delta(u) - s\Delta(u)\|_Y + \|s\Delta(u) - s\Delta(v)\|_Y \\ &= (r - s) + s\,\|u - v\|_X .\end{aligned}$$

Moreover,

$$s\,\|u - v\|_X \le \|ru - sv\|_X + (r - s) = \|x - y\|_X + (r - s),$$

and the reverse triangle inequality gives

$$r - s = |\|x\|_X - \|y\|_X| \leq \|x - y\|_X .$$

Combining the three estimates yields the factor 3. The same argument applied to Δ^{-1} gives the estimate for the inverse. Finally, (2.2) and surjectivity imply $\widehat{\Delta}(B_X) = B_Y$. ☐

2.3 Recovery of the dimension

We use Hausdorff measure and Hausdorff dimension with respect to the metric under discussion. The definitions and the behavior of Hausdorff measure under Lipschitz maps are recalled, for example, in [4, Sections 2.1 and 2.4.1]; see also [3, Section 6.1 and Exercise 6.1.10]. For completeness, for $s > 0$, $\delta > 0$, and a subset A of a metric space, set

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} U_i)^s : A \subseteq \bigcup_{i=1}^{\infty} U_i,\ \operatorname{diam} U_i \leq \delta\right\},$$

$\mathcal{H}^s(A) = \lim_{\delta\downarrow 0} \mathcal{H}^s_\delta(A)$, and

$$\dim_{\mathcal{H}} A = \inf\{s > 0 : \mathcal{H}^s(A) = 0\}.$$

Lemma 2.4 (Bi-Lipschitz invariance of Hausdorff dimension)**.** *If $f : A \to B$ is a bi-Lipschitz bijection between metric spaces, then*

$$\dim_{\mathcal{H}} A = \dim_{\mathcal{H}} B.$$

Proof. Suppose first that f is L-Lipschitz, enlarging L so that $L \geq 1$. Let (U_i) be a countable cover of A. Replacing each U_i by $U_i \cap A$ does not increase its diameter, and $(f(U_i \cap A))$ covers $f(A)$. Moreover,

$$\operatorname{diam} f(U_i \cap A) \leq L \operatorname{diam}(U_i \cap A) \leq L \operatorname{diam} U_i.$$

Consequently, for every $s > 0$ and $\delta > 0$,

$$\mathcal{H}^s_{L\delta}(f(A)) \leq L^s \mathcal{H}^s_\delta(A).$$

Letting $\delta \downarrow 0$ gives $\dim_{\mathcal{H}} f(A) \leq \dim_{\mathcal{H}} A$. Applying the same argument to the Lipschitz inverse gives the reverse inequality. ☐

Lemma 2.5 (Dimension recovery)**.** *The existence of Δ implies*

$$\dim_{\mathbb{R}} X = \dim_{\mathbb{R}} Y.$$

Proof. By Proposition 2.3, the unit balls B_X and B_Y are bi-Lipschitz equivalent. Suppose that $\dim_{\mathbb{R}} X = n$. Choose a linear isomorphism $e : \mathbb{R}^n \to X$ and pull the norm of X back to a norm r on $\mathbb{R}^n$. Equivalence of r with the Euclidean norm gives constants $a, b > 0$ such that

$$aB_2^n \subseteq e^{-1}(B_X) \subseteq bB_2^n.$$

The identity map between the Euclidean and r-metric structures on $\mathbb{R}^n$ is bi-Lipschitz. A nonempty Euclidean ball has Hausdorff dimension n, because $\mathcal{H}^n$ is a fixed positive multiple of Lebesgue measure on $\mathbb{R}^n$, whereas every subset of $\mathbb{R}^n$ has Hausdorff dimension at most n; see [4, Sections 2.2 and 2.4.1]. The map e identifies B_X, with its norm metric, isometrically with $e^{-1}(B_X)$ equipped with the metric induced by r. The preceding inclusions and Lemma 2.4 therefore give $\dim_{\mathcal{H}} B_X = n$. The same argument applied to Y gives $\dim_{\mathcal{H}} B_Y = \dim_{\mathbb{R}} Y$. Since Proposition 2.3 makes the two unit balls bi-Lipschitz equivalent, their Hausdorff dimensions agree, and hence so do the vector-space dimensions. ☐

2.4 Two norms on a common coordinate space

Set

$$n = \dim_{\mathbb{R}} X = \dim_{\mathbb{R}} Y \geq 1, \qquad E = \mathbb{R}^n,$$

and choose linear isomorphisms

$$e_X : E \longrightarrow X, \qquad e_Y : E \longrightarrow Y.$$

Pull the two norms back to E by setting

$$p(x) = \|e_X x\|_X, \qquad q(x) = \|e_Y x\|_Y. \tag{2.3}$$

For any norm r on E, we use the notation

$$B_r = \{x \in E : r(x) \leq 1\}, \qquad S_r = \{x \in E : r(x) = 1\}, \qquad v_r = \mathcal{L}^n(B_r),$$

where $\mathcal{L}^n$ denotes Lebesgue measure on E.

Conjugating Δ by e_X and e_Y gives a surjective map

$$\delta : S_p \longrightarrow S_q \tag{2.4}$$

satisfying

$$q(\delta(u) - \delta(v)) = p(u - v) \qquad (u, v \in S_p).$$

We equip E with its standard Euclidean inner product and use the associated Lebesgue measure, Fréchet derivatives, distributional derivatives, and convolutions. Linear maps $E \to \mathbb{R}^N$ are represented by matrices whose rows are the coordinate functionals of the output. We write $\|\cdot\|_2$ and $\|\cdot\|_\infty$ for the Euclidean and supremum norms on coordinate spaces. Since all norms on a finite-dimensional space are equivalent, for each norm r on E there exist constants $0 < \lambda_r \leq \Lambda_r$ such that

$$\lambda_r \|x\|_\infty \leq r(x) \leq \Lambda_r \|x\|_\infty \qquad (x \in E). \tag{2.5}$$

These choices affect only auxiliary estimates; all metric statements continue to be expressed in terms of p and q.

3. Plücker contraction bodies

Fix a norm r on $E = \mathbb{R}^n$, where $n \geq 1$, and an integer $N \geq n$. We identify ℓ_∞^N with $\mathbb{R}^N$ equipped with the supremum norm.

3.1 Maximal-minor data

Let $\mathcal{I}(n, N)$ be the set of increasing n-element subsets of $\{1, \ldots, N\}$, and let

$$\mathcal{P}_{n,N} = \mathbb{R}^{\mathcal{I}(n,N)}$$

be the corresponding Plücker-coordinate space. If $A : E \to \mathbb{R}^N$ is linear, we write $A = (a_1, \ldots, a_N)$ for its ordered row functionals. For $I \in \mathcal{I}(n, N)$, the map $A_I : E \to \mathbb{R}^n$ is obtained by retaining the rows indexed by I in increasing order. All maximal-minor signs below refer to this fixed order.

Definition 3.1 (Normalized Plücker vector)**.** For a linear map $A : E \to \mathbb{R}^N$, define its maximal-minor vector by

$$\mathbf{m}(A) = \left(\det A_I\right)_{I \in \mathcal{I}(n,N)} \in \mathcal{P}_{n,N}.$$

The normalized Plücker vector of A relative to the norm r is

$$\Pi_r(A) = v_r \mathbf{m}(A). \tag{3.1}$$

Definition 3.2 (Plücker contraction body)**.** A linear map $A : E \to \ell_\infty^N$ is called an r*-contraction* if

$$\|Ax\|_\infty \le r(x) \qquad (x \in E).$$

We define the signed raw Plücker set and the Plücker contraction body by

$$\mathcal{R}_r^N = \{\pm\Pi_r(A) : A \text{ is an } r\text{-contraction}\}, \qquad \mathcal{K}_r^N = \operatorname{conv}(\mathcal{R}_r^N) \subset \mathcal{P}_{n,N}. \tag{3.2}$$

The word *body* is part of the name; no full-dimensionality of $\mathcal{K}_r^N$ is assumed.

Proposition 3.3 (Compactness)**.** *The set $\mathcal{R}_r^N$ is nonempty, compact, and centrally symmetric. Consequently, $\mathcal{K}_r^N$ is a nonempty compact centrally symmetric convex subset of $\mathcal{P}_{n,N}$.*

Proof. The zero map is an r-contraction, so $\mathcal{R}_r^N$ is nonempty. The set of r-contractions is closed in the finite-dimensional space $\mathcal{L}(E, \mathbb{R}^N)$. It is also bounded: if $A = (a_1, \dots, a_N)$ is written by rows, then each a_j belongs to the dual unit ball of (E, r), and hence the matrix entries of A are uniformly bounded by equivalence of norms. Thus the contraction set is compact. The map $A \mapsto \Pi_r(A)$ is polynomial in the matrix entries and therefore continuous, so its image is compact. Adjoining the negative image gives the compact centrally symmetric set $\mathcal{R}_r^N$. Let $d = \dim \mathcal{P}_{n,N}$. By Carathéodory's theorem, every point of the convex hull is a convex combination of at most $d+1$ points of $\mathcal{R}_r^N$. Write $\Delta_d = \{(\alpha_0, \dots, \alpha_d) \in [0,1]^{d+1} : \sum_j \alpha_j = 1\}$. Hence $\mathcal{K}_r^N$ is the image of the compact space $\Delta_d \times (\mathcal{R}_r^N)^{d+1}$ under the continuous barycenter map, and is compact; see [6, Theorem 3.1 and Corollary 3.1] for the finite-dimensional theorem. □

3.2 Boundary extensions

Fix a q-contraction $A : (E, q) \to \ell_\infty^N$. Then

$$g_A = A \circ \delta : S_p \longrightarrow \ell_\infty^N$$

is 1-Lipschitz with respect to the metric induced by p: indeed,

$$\|g_A(u) - g_A(v)\|_\infty \le q(\delta(u) - \delta(v)) = p(u - v).$$

Write $g_A = (h_1, \dots, h_N)$. Each h_j is 1-Lipschitz on (S_p, p). Since $n \ge 1$, the compact sphere S_p is nonempty and each h_j is bounded. McShane's extension theorem [9] gives the finite-valued extension

$$F_{A,j}(x) = \inf_{u \in S_p} \big(h_j(u) + p(x - u)\big) \qquad (x \in E).$$

For $x \in S_p$, the Lipschitz inequality gives $h_j(u) + p(x-u) \ge h_j(x)$ for every $u \in S_p$, while $u = x$ gives equality. Thus $F_{A,j} = h_j$ on S_p. Setting $F_A = (F_{A,1}, \dots, F_{A,N})$, we obtain

$$F_A : E \longrightarrow \ell_\infty^N. \tag{3.3}$$

The map agrees with g_A on S_p and is 1-Lipschitz with respect to p. By equivalence of norms, F_A is also Euclidean-Lipschitz. Rademacher's theorem (see [10] and [4, Section 3.1.2]) therefore gives Fréchet differentiability almost everywhere. At every differentiability point,

$$\|DF_A(x)h\|_\infty \le p(h) \qquad (h \in E),$$

so $DF_A(x)$ is a p-contraction.

3.3 Averaged derivative minors

Definition 3.4 (Plücker average)**.** Let $F : E \to \ell_\infty^N$ be Lipschitz. Its Plücker average over B_r is

$$\mathcal{A}_r(F) = \frac{1}{v_r} \int_{B_r} \Pi_r(DF(x))\,\mathrm{d}x = \int_{B_r} \mathbf{m}(DF(x))\,\mathrm{d}x \in \mathcal{P}_{n,N}. \tag{3.4}$$

The integral is taken coordinatewise in the finite-dimensional space $\mathcal{P}_{n,N}$. We use a measurable representative of the weak derivative of F, which agrees almost everywhere with the classical derivative, and assign the value 0 on the remaining null set; this convention does not change the integral.

A Euclidean-Lipschitz map has an essentially bounded measurable derivative, and every maximal minor is a polynomial of degree n in the matrix entries. Hence the integrand in (3.4) is measurable and essentially bounded on the bounded set B_r, so the integral is well defined.

Lemma 3.5 (Averages in closed convex sets)**.** *Let K be a nonempty closed convex subset of a finite-dimensional real vector space, let μ be a probability measure, and let h be an integrable map satisfying $h(\omega) \in K$ for μ-almost every ω. Then*

$$\int h\ \mathrm{d}\mu \in K.$$

Proof. If the integral did not belong to K, the finite-dimensional separation theorem would provide a linear functional λ and a number α such that $\lambda \le \alpha$ on K but $\lambda(\int h\ \mathrm{d}\mu) > \alpha$. Since $h \in K$ almost everywhere,

$$\lambda\Big(\int h\ \mathrm{d}\mu\Big) = \int \lambda(h)\ \mathrm{d}\mu \le \alpha,$$

a contradiction. □

Proposition 3.6 (The average lies in the contraction body)**.** *If $F : E \to \ell_\infty^N$ is 1-Lipschitz with respect to r, then*

$$\mathcal{A}_r(F) \in \mathcal{K}_r^N.$$

In particular, the extension F_A from (3.3) satisfies

$$\mathcal{A}_p(F_A) \in \mathcal{K}_p^N.$$

Proof. At almost every differentiability point, $DF(x)$ is an r-contraction, and therefore

$$\Pi_r(DF(x)) \in \mathcal{R}_r^N \subseteq \mathcal{K}_r^N.$$

The measure $v_r^{-1}\mathbf{1}_{B_r}\mathcal{L}^n$ is a probability measure. Apply Lemma 3.5 to the closed convex set $\mathcal{K}_r^N$. □

4. A boundary-trace theorem for maximal minors

Throughout this section, Lipschitz constants, derivatives, convolutions, and Lebesgue integrals refer to the Euclidean structure fixed in Section 2.

Fix $N \ge n$, set $Z = \mathbb{R}^N$, and let $I \in \mathcal{I}(n, N)$. Denote by $P_I : Z \to \mathbb{R}^n$ the coordinate projection onto the components indexed by I. At points where a Lipschitz map $F : E \to Z$ is differentiable, define the selected Jacobian

$$J_I F(x) = \det D(P_I \circ F)(x). \tag{4.1}$$

Its value on the exceptional null set is irrelevant.

4.1 Compactly supported perturbations

For the null-Lagrangian property of minors and the Piola identity, see [1, Theorem 3.4 and p. 136] and [8]. Proofs of the forms used here follow.

Lemma 4.1 (Differential Piola identity)**.** *Let $g \in C^2(E;E)$, with the convention $(Dg)_{ij} = \partial_j g_i$. If $C_i(Dg)$ denotes the ith row of the cofactor matrix of Dg, then*

$$\operatorname{div} C_i(Dg) = 0 \qquad (1 \le i \le n).$$

Proof. If $n = 1$, then the cofactor matrix of Dg is the constant matrix (1), and the assertion is immediate. Assume that $n \ge 2$ and fix i. With the convention that $\varepsilon_{j_1 \dots j_n}$ denotes the alternating symbol, the jth component of the ith cofactor row can be written as

$$C_{ij}(Dg) = \frac{1}{(n-1)!} \sum_{\substack{i_2,\dots,i_n \\ j_2,\dots,j_n}} \varepsilon_{i i_2 \cdots i_n} \varepsilon_{j j_2 \cdots j_n} \prod_{a=2}^{n} \partial_{j_a} g_{i_a}.$$

When ∂_j is applied and the result is summed over j, each term contains one mixed second derivative $\partial_j \partial_{j_a} g_{i_a}$. Interchanging the two column indices j and j_a leaves that second derivative unchanged and reverses the sign of the second alternating tensor. The terms therefore cancel in pairs. Hence

$$\sum_{j=1}^{n} \partial_j C_{ij}(Dg) = 0.$$

This is the row form of the classical Piola identity for the convention $(Dg)_{ij} = \partial_j g_i$. □

Lemma 4.2 (Change of one output coordinate)**.** *Let $g \in C^2(E;E)$, let $\phi \in C^1_c(E)$, and fix $i \in \{1, \dots, n\}$. Let $g^{(i,\phi)}$ be obtained from g by replacing its ith component g_i with $g_i + \phi$. Then*

$$\int_E \left(\det Dg^{(i,\phi)} - \det Dg\right) \mathrm{d}x = 0.$$

Proof. Expansion of the determinant along the modified row gives

$$\det Dg^{(i,\phi)} - \det Dg = \sum_{j=1}^{n} (\partial_j \phi) C_{ij}(Dg).$$

By Lemma 4.1, the right-hand side is

$$\operatorname{div}\left(\phi C_i(Dg)\right).$$

The vector field $\phi C_i(Dg)$ is continuously differentiable and compactly supported, so the integral of its divergence over E vanishes. □

Proposition 4.3 (Smooth compact perturbations)**.** *Let $G, U : E \to Z$ be smooth, and suppose that U has compact support. Then, for every $I \in \mathcal{I}(n, N)$, the difference $J_I(G+U) - J_I(G)$ is compactly supported and integrable, and*

$$\int_E \left(J_I(G+U) - J_I(G)\right) \mathrm{d}x = 0.$$

Proof. Write $P_I \circ U = (u_1, \dots, u_n)$ and set

$$H_0 = P_I \circ G, \qquad H_k = H_{k-1} + u_k e_k \quad (1 \le k \le n).$$

Thus $H_n = P_I \circ (G+U)$. Each u_k is smooth and compactly supported. Applying Lemma 4.2 to the pair (H_{k-1}, u_k) gives

$$\int_E (\det DH_k - \det DH_{k-1})\,\mathrm{d}x = 0.$$

The difference is supported in $\operatorname{supp} u_k$, and is therefore integrable. Summing over k telescopes from H_0 to H_n and proves the claim. □

Write $\|P\|_{\mathrm{F}}$ for the Frobenius norm of a matrix P. Since the determinant is a homogeneous polynomial of degree n, there exists a constant $C_n > 0$ such that

$$|\det P - \det Q| \le C_n \|P - Q\|_{\mathrm{F}} \left(\|P\|_{\mathrm{F}}^{n-1} + \|Q\|_{\mathrm{F}}^{n-1}\right). \tag{4.2}$$

Indeed, replace the columns of Q by those of P one at a time, expand the resulting telescoping sum by multilinearity, and bound each mixed determinant by a dimension-dependent product of Euclidean column norms.

Lemma 4.4 (Strong L^n continuity of the determinant)**.** *Let $K \subset E$ be measurable with finite measure, with $n \ge 1$. Let $P_k, P : K \to \mathbb{R}^{n\times n}$ be matrix fields. If $P_k \to P$ strongly in $L^n(K; \mathbb{R}^{n\times n})$, then*

$$\det P_k \longrightarrow \det P \quad \textit{strongly in } L^1(K).$$

The same conclusion holds for any fixed $n \times n$ submatrix of rectangular matrix fields converging strongly in $L^n(K)$.

Proof. For $n = 1$, the determinant is linear. Suppose that $n \ge 2$. By (4.2) and Hölder's inequality,

$$\begin{aligned}\|\det P_k - \det P\|_{L^1(K)} &\le C \|P_k - P\|_{L^n(K)} \\ &\qquad \cdot \left(\|P_k\|_{L^n(K)}^{n-1} + \|P\|_{L^n(K)}^{n-1}\right).\end{aligned}$$

The second factor is bounded because $P_k \to P$ in $L^n(K)$, whereas the first tends to zero. □

Proposition 4.5 (Lipschitz compact perturbations)**.** *Let $G, U : E \to Z$ be Lipschitz, and suppose that U has compact support. Then, for every $I \in \mathcal{I}(n, N)$, the difference $J_I(G+U) - J_I(G)$ is integrable, and*

$$\int_E (J_I(G+U) - J_I(G))\,\mathrm{d}x = 0.$$

Proof. Choose a compact set K_0 containing $\operatorname{supp} U$ in its interior, and then a compact set K with $K_0 \subset \operatorname{int} K$. Let ρ_η be a standard mollifier with $\eta > 0$ smaller than the distance from K_0 to $E \setminus K$, and set

$$G_\eta = G * \rho_\eta, \qquad U_\eta = U * \rho_\eta.$$

Convolution is linear, so $(G+U) * \rho_\eta = G_\eta + U_\eta$, and $\operatorname{supp} U_\eta \subset K$. By Proposition 4.3,

$$\int_E (J_I(G_\eta + U_\eta) - J_I(G_\eta))\,\mathrm{d}x = 0.$$

Outside K, both U_η and its derivative vanish, so the integrand is supported in K.

Lipschitz maps belong locally to $W^{1,\infty}$. Hence the standard approximation theorem for mollifiers gives, as $\eta \downarrow 0$,

$$DG_\eta \to DG, \qquad DU_\eta \to DU$$

strongly in $L^n(K)$; see, for example, [4, Section 4.2.1]. Consequently $D(G_\eta + U_\eta) \to D(G+U)$ strongly in $L^n(K)$. Applying Lemma 4.4 to the selected square submatrices of these two convergences yields

$$J_I(G_\eta + U_\eta) - J_I(G_\eta) \longrightarrow J_I(G+U) - J_I(G)$$

strongly in $L^1(K)$. Passing to the limit in the displayed integral proves the result. Notice that only the compactly supported difference is integrated; neither individual whole-space determinant need be integrable. □

4.2 Patching across a norm sphere

Let r be a norm on E.

Lemma 4.6 (Zero extension)**.** *Let $u : E \to Z$ be Lipschitz and satisfy $u = 0$ on S_r. Define*

$$u_0(x) = \begin{cases} u(x), & x \in B_r, \\ 0, & x \notin B_r. \end{cases}$$

Then u_0 is globally Lipschitz and has compact support.

Proof. Only pairs of points on opposite sides of S_r require attention. Let $x \in B_r$ and $y \notin B_r$. By continuity of r along the segment and convexity of B_r, the line segment from x to y meets S_r at a point z. Since $u(z) = 0$ and z lies between x and y,

$$\|u_0(x) - u_0(y)\|_2 = \|u(x) - u(z)\|_2 \le \mathrm{Lip}(u)\,\|x - z\|_2 \le \mathrm{Lip}(u)\,\|x - y\|_2\,.$$

The remaining cases follow directly from the Lipschitz property of u. Finally, $\operatorname{supp} u_0 \subseteq B_r$, and B_r is compact. ☐

Lemma 4.7 (The unit sphere is null)**.** *If $n \ge 1$, then $\mathcal{L}^n(S_r) = 0$.*

Proof. For $0 < t < 1$, one has $tB_r \subset \operatorname{int} B_r$ and

$$S_r \subset B_r \setminus tB_r.$$

Therefore

$$\mathcal{L}^n(S_r) \le \mathcal{L}^n(B_r) - \mathcal{L}^n(tB_r) = (1 - t^n)v_r.$$

Letting $t \uparrow 1$ gives the result. ☐

Theorem 4.8 (Boundary-trace invariance)**.** *Let r be a norm on $E = \mathbb{R}^n$ with $n \ge 1$, and let $F, G : E \to \ell_\infty^N$ be Lipschitz maps satisfying*

$$F = G \quad \text{on } S_r.$$

Then, for every $I \in \mathcal{I}(n, N)$,

$$\int_{B_r} J_I F(x)\,\mathrm{d}x = \int_{B_r} J_I G(x)\,\mathrm{d}x. \tag{4.3}$$

Equivalently,

$$\mathcal{A}_r(F) = \mathcal{A}_r(G).$$

Proof. Set $u = F - G$, and let u_0 be the zero extension of $u|_{B_r}$ provided by Lemma 4.6. Define $H = G + u_0$. Then $H = F$ on B_r, $H = G$ on $E \setminus B_r$, and $H - G$ is a compactly supported Lipschitz map. By Proposition 4.5,

$$\int_E (J_I H - J_I G)\,\mathrm{d}x = 0.$$

The integrand vanishes almost everywhere outside B_r and equals $J_I F - J_I G$ on the interior of B_r. Since $S_r = \partial B_r$ is null by Lemma 4.7, this proves (4.3). Applying the identity to every maximal minor gives equality of the Plücker averages. ☐

5. Orientation of the radial extension

Let

$$f = \widehat{\delta} : E \longrightarrow E$$

be the radial extension of the sphere isometry $\delta : S_p \to S_q$, and let $g = \widehat{\delta^{-1}}$ be its inverse. Set

$$U = \operatorname{int} B_p, \qquad V = \operatorname{int} B_q.$$

By Proposition 2.3, the restrictions $f|_U : U \to V$ and $g|_V : V \to U$ are inverse bi-Lipschitz homeomorphisms. Combining the lower Lipschitz estimate for f with equivalence of the Euclidean and norm metrics, we obtain a constant $a > 0$ such that

$$a \|x - x'\|_2 \le \|f(x) - f(x')\|_2 \qquad (x, x' \in E). \tag{5.1}$$

Choose a Borel measurable representative of Df that agrees almost everywhere with the classical derivative, assigning the value 0 on a Borel null set containing the exceptional points, and write

$$J_f(x) = \det Df(x).$$

The function J_f is measurable and essentially bounded on bounded sets.

5.1 Nondegeneracy and signed change of variables

Lemma 5.1 (Almost-everywhere nondegeneracy)**.** *At almost every point $x \in U$, the derivative $Df(x)$ exists and is invertible. In particular, $J_f(x) \neq 0$ for almost every $x \in U$.*

Proof. Rademacher's theorem gives differentiability almost everywhere. Let x be a differentiability point outside the chosen exceptional null set. Applying (5.1) to x and $x + tv$, dividing by $|t|$, and letting $t \to 0$, we obtain

$$a \|v\|_2 \le \|Df(x)v\|_2 \qquad (v \in E).$$

Thus $Df(x)$ is injective. Its domain and codomain both have dimension n, so it is invertible. ☐

Define the source sign by

$$\varepsilon(x) = \begin{cases} 1, & J_f(x) \ge 0, \\ -1, & J_f(x) < 0. \end{cases}$$

Then ε is Borel measurable, takes values in $\{-1, 1\}$, and satisfies

$$J_f(x) = \varepsilon(x)\, |J_f(x)|$$

for every x. By Lemma 5.1, J_f is nonzero for almost every $x \in U$. Transport it to the target by setting

$$\sigma(y) = \varepsilon(g(y)) \qquad (y \in V). \tag{5.2}$$

The inverse map g is continuous, so σ is Borel measurable and bounded by 1 in absolute value. The values assigned on exceptional null sets will not affect any integral.

Lemma 5.2 (Lipschitz images of null sets)**.** *Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz. If $A \subseteq \mathbb{R}^n$ has Lebesgue measure zero, then $h(A)$ has Lebesgue measure zero.*

Proof. If h is L-Lipschitz, the covering argument used in Lemma 2.4 gives

$$\mathcal{H}^n(h(A)) \le L^n \mathcal{H}^n(A).$$

On $\mathbb{R}^n$, $\mathcal{H}^n$ is a fixed positive multiple of Lebesgue measure; see [4, Sections 2.2 and 2.4.1]. Hence $\mathcal{L}^n(A) = 0$ implies $\mathcal{L}^n(h(A)) = 0$. ☐

Lemma 5.3 (Signed change of variables)**.** *Let $\varphi : V \to \mathbb{R}$ be a measurable representative of an element of $L^1(V)$. Then both sides below are absolutely integrable and*

$$\int_U \varphi(f(x))J_f(x)\,\mathrm{d}x = \int_V \varphi(y)\sigma(y)\,\mathrm{d}y. \tag{5.3}$$

The value of either side depends only on the L^1 class of φ.

Proof. Since $f^{-1}(N) = g(N)$ is null whenever $N \subset V$ is null, by Lemma 5.2 applied to the Lipschitz map g, the composition $\varphi \circ f$ is Lebesgue measurable even for a Lebesgue measurable representative φ. The area formula for the injective Lipschitz map $f|_U : U \to V$, applied first to the nonnegative function $|\varphi|$, gives

$$\int_U |\varphi(f(x))|\,|J_f(x)|\,\mathrm{d}x = \int_V |\varphi(y)|\,\mathrm{d}y < \infty. \tag{5.4}$$

Thus the left side of (5.3) is absolutely integrable. Let $G \subset U$ be a measurable full-measure set on which f is differentiable and J_f is nonzero. Since $f(U) = V$, the complement of $f(G)$ in V is contained in $f(U \setminus G)$ and is therefore null by Lemma 5.2. For $x \in G$,

$$\sigma(f(x)) = \varepsilon(g(f(x))) = \varepsilon(x), \qquad J_f(x) = \varepsilon(x)|J_f(x)|.$$

Since $|\sigma| = 1$, the function $\psi = \varphi\sigma$ belongs to $L^1(V)$. Applying the area formula to the positive and negative parts of ψ gives

$$\int_U \psi(f(x))|J_f(x)|\,\mathrm{d}x = \int_V \psi(y)\,\mathrm{d}y.$$

On G the integrand on the left is $\varphi(f(x))J_f(x)$, and the complement of G is null. This proves (5.3). Equation (5.4) also shows that changing φ on a null set changes neither integral, because the preimage of that null set has zero $|J_f|\mathcal{L}^n$-measure. This is the injective case of the weighted change-of-variables formula [4, Theorem 3.9]; see also [5, 3.2.3(1)]. ☐

5.2 A weak Piola identity

For $1 \le i \le n$, let $P_i : \mathbb{R}^n \to \mathbb{R}^n$ be the rank-one coordinate projection

$$P_i(z) = z_i e_i,$$

where (e_i) is the standard basis of $\mathbb{R}^n$.

Lemma 5.4 (Weak Piola identity)**.** *For every vector field $W \in C^1_c(V;\mathbb{R}^n)$,*

$$\int_U (\operatorname{div} W)(f(x))J_f(x)\,\mathrm{d}x = 0. \tag{5.5}$$

Proof. Because $\operatorname{supp} W$ is compact and contained in the open set V, its distance from $E \setminus V$ is positive. Hence W vanishes on an open neighborhood of ∂V, and extension by zero defines a field in $C^1_c(E;\mathbb{R}^n)$; we use the same symbol for this extension. Fix $i \in \{1,\dots,n\}$ and define

$$u_i(x) = P_i(W(f(x))).$$

The zero extension of W has bounded derivative and is therefore globally Lipschitz. Since f is Lipschitz, so is u_i. Moreover,

$$\operatorname{supp} u_i \subseteq f^{-1}(\operatorname{supp} W) = g(\operatorname{supp} W).$$

The set on the right is compact and contained in U, because $g : V \to U$ is continuous. Thus u_i is a compactly supported Lipschitz perturbation of f, with support compactly contained in U.

At almost every point where f is differentiable, the ordinary chain rule gives

$$D(f + u_i)(x) = \big(I + P_i \circ DW(f(x))\big)Df(x).$$

The matrix $P_i \circ DW(f(x))$ has only its ith row possibly nonzero. Expansion in that row gives

$$\det\big(I + P_i \circ DW(f(x))\big) = 1 + \partial_i W_i(f(x)).$$

Consequently,

$$\det D(f + u_i)(x) - \det Df(x) = \partial_i W_i(f(x))J_f(x)$$

for almost every x. Apply Proposition 4.5 to the full determinant of f and the compactly supported perturbation u_i. Because $\operatorname{supp} u_i \subset U$, the determinant difference vanishes almost everywhere on $E \setminus U$. Hence

$$\int_U \partial_i W_i(f(x))J_f(x)\,\mathrm{d}x = 0.$$

Summing over i proves (5.5). □

Applying the signed change-of-variables formula to the weak Piola identity gives

$$\int_V \sigma(y)\operatorname{div} W(y)\,\mathrm{d}y = 0 \qquad (W \in C_c^1(V; \mathbb{R}^n)). \tag{5.6}$$

Equivalently, every distributional partial derivative of σ vanishes on V.

Lemma 5.5 (Functions with vanishing weak gradient)**.** *Let $\Omega \subset \mathbb{R}^n$ be connected and open. If $h \in L^1_{\mathrm{loc}}(\Omega)$ has zero distributional gradient, then h is almost everywhere equal to a constant on Ω.*

Proof. If Ω is empty, the conclusion is immediate. Assume that Ω is nonempty. Let B be a ball of positive radius whose closure is contained in Ω, and choose a standard mollifier ρ_η with η smaller than the distance from B to $\partial\Omega$. Since distributional differentiation commutes with convolution,

$$\nabla(h * \rho_\eta) = 0$$

on B. Thus every mollification is constant on B; write this constant as c_η. Since $h * \rho_\eta \to h$ in $L^1(B)$, the constant functions c_η form a Cauchy net in $L^1(B)$ and therefore the scalars c_η converge. It follows that h is almost everywhere equal on B to their limit. If two relatively compact balls overlap, their constants agree because the intersection has positive measure. Finally, every connected open subset of $\mathbb{R}^n$ is polygonally connected: the points reachable from a fixed point by polygonal paths form a nonempty subset that is both open and closed in Ω. Covering a polygonal path by finitely many overlapping relatively compact balls shows that all such balls have the same constant. Choose a countable cover of Ω by relatively compact balls. The union of their exceptional null sets is null, so h equals that constant almost everywhere on Ω. □

Because V is convex, it is connected. Since $|\sigma| = 1$ almost everywhere, (5.6) and Lemma 5.5 yield a sign

$$s_\delta \in \{-1, 1\}$$

such that $\sigma = s_\delta$ almost everywhere on V.

Theorem 5.6 (Orientation of the radial extension)**.** *There exists $s_\delta \in \{-1, 1\}$ such that*

$$\int_{B_p} \det Df(x)\,\mathrm{d}x = s_\delta v_q. \tag{5.7}$$

Moreover, for every integer $N \geq n$ and every linear map $A : E \to \ell_\infty^N$,

$$\mathcal{A}_p(A \circ f) = s_\delta \Pi_q(A). \tag{5.8}$$

Proof. The preceding argument shows that the transported sign is almost everywhere equal to s_δ. Since V has finite measure, Lemma 5.3 applied to the constant function 1 gives

$$\int_U J_f(x)\,\mathrm{d}x = s_\delta \mathcal{L}^n(V) = s_\delta v_q.$$

By Lemma 4.7, the boundary of B_p is null, so the same identity holds over the closed ball B_p. Fix $I \in \mathcal{I}(n, N)$. At differentiability points of f, the chain rule and multiplicativity of the determinant give

$$\det D(A_I \circ f)(x) = \det(A_I) \det Df(x).$$

Integrating and using (5.7) yields

$$\int_{B_p} \det D(A_I \circ f)(x)\,\mathrm{d}x = s_\delta v_q \det(A_I),$$

which is the Ith coordinate of (5.8). Since I was arbitrary, the vector identity follows. □

6. Metric invariance of the Plücker contraction bodies

Theorem 6.1 (Plücker-body invariance)**.** *For every integer $N \geq n$,*

$$\mathcal{K}_p^N = \mathcal{K}_q^N.$$

Proof. Fix a q-contraction $A : E \to \ell_\infty^N$. Let F_A be the coordinatewise McShane extension from (3.3), and set

$$G_A = A \circ \widehat{\delta}.$$

The map G_A is Lipschitz because A is a q-contraction and $\widehat{\delta}$ is Lipschitz. Since $\widehat{\delta} = \delta$ on S_p, the maps F_A and G_A agree on S_p. Hence Theorems 4.8 and 5.6 give

$$\mathcal{A}_p(F_A) = \mathcal{A}_p(G_A) = s_\delta \Pi_q(A).$$

By Proposition 3.6, the left-hand side belongs to $\mathcal{K}_p^N$. Central symmetry of $\mathcal{K}_p^N$ removes the sign, and therefore

$$\Pi_q(A) \in \mathcal{K}_p^N.$$

The same is true for $-\Pi_q(A)$. Taking all raw Plücker points and then their convex hull yields

$$\mathcal{K}_q^N \subseteq \mathcal{K}_p^N.$$

Applying the same argument to the inverse sphere isometry $\delta^{-1} : S_q \to S_p$ gives the reverse inclusion. □

7. Finite recovery from the contraction bodies

Let p and q be the norms on $E = \mathbb{R}^n$ obtained in Section 2, and assume that

$$\mathcal{K}_p^N = \mathcal{K}_q^N \qquad \text{for every } N \geq n. \tag{7.1}$$

7.1 Almost maximal dual frames

For a norm r on E, let

$$B_{r^*} = \{b \in E^* : |b(x)| \le r(x) \text{ for every } x \in E\}$$

be the closed unit ball of the dual norm. Equivalently,

$$r(x) = \max_{b \in B_{r^*}} |b(x)| \qquad (x \in E). \tag{7.2}$$

The inequality $\max_{b \in B_{r^*}} |b(x)| \le r(x)$ follows from the definition of B_{r^*}. For $x \neq 0$, define a norm-one functional on $\mathbb{R}x$ by $b(tx) = tr(x)$ and extend it to E by the Hahn–Banach theorem; this gives equality. See, for example, [2, Corollary III.6.5].

An ordered n-tuple $B = (b_1, \ldots, b_n) \in (B_{r^*})^n$ will be called an *r-admissible row system.* We also write B for the corresponding $n \times n$ matrix. If $s \in E^*$ and $1 \le i \le n$, then $B[i \leftarrow s]$ denotes the matrix obtained from B by replacing its ith row by s.

Define

$$d_r = \max\left\{|\det B| : B \in (B_{r^*})^n\right\}. \tag{7.3}$$

The maximum exists because $(B_{r^*})^n$ is compact. It is positive: by (2.5), the rows $\lambda_r e_1^*, \ldots, \lambda_r e_n^*$ belong to B_{r^*}, and their determinant is $\lambda_r^n > 0$.

Lemma 7.1 (Uniform inverse bound near the maximum)**.** *Fix $0 < \eta < d_r$. There is a constant $K = K(r, \eta) \ge 1$ such that, whenever B is an r-admissible row system satisfying*

$$|\det B| \ge d_r - \eta,$$

then B is invertible and

$$r(B^{-1}c) \le K \|c\|_\infty \qquad (c \in \mathbb{R}^n). \tag{7.4}$$

Proof. The determinant lower bound is positive, so B is invertible. Let $s \in B_{r^*}$. If $\alpha = sB^{-1}$ is regarded as a row vector, then $s = \alpha B$. Multilinearity in the replaced row gives

$$\det B[i \leftarrow s] = \alpha_i \det B.$$

Consequently, for every column vector $c \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} c_i \det B[i \leftarrow s] = \det B \sum_{i=1}^{n} c_i \alpha_i = \det B\, s(B^{-1}c). \tag{7.5}$$

This is the row form of Cramer's rule. Every matrix $B[i \leftarrow s]$ is again r-admissible, and therefore

$$|\det B[i \leftarrow s]| \le d_r.$$

Take $s = \lambda_r e_j^*$. From (7.5) we obtain

$$\lambda_r (d_r - \eta) \left|(B^{-1}c)_j\right| \le n d_r \|c\|_\infty \qquad (1 \le j \le n).$$

Hence

$$\left\|B^{-1}c\right\|_\infty \le \frac{n d_r}{\lambda_r (d_r - \eta)} \|c\|_\infty .$$

Using the upper comparison in (2.5), we may take

$$K = \max\left\{1,\, \Lambda_r \frac{n d_r}{\lambda_r (d_r - \eta)}\right\}.$$

This constant is at least 1 and satisfies (7.4). ☐

Lemma 7.2 (Finite coefficient net)**.** *Fix $0 < \eta < d_r$, let $K \geq 1$ be as in Lemma 7.1, and let $\rho > 0$. There exists a finite set*

$$C \subset \left\{ c \in \mathbb{R}^n : K^{-1} \leq \|c\|_\infty \leq 1 \right\}$$

with the following property: for every r-admissible row system B satisfying $|\det B| \geq d_r - \eta$ and every $x \in S_r$, there exists $c \in C$ such that

$$r(x - B^{-1}c) \leq K\rho. \tag{7.6}$$

Proof. For $x \in S_r$, put $z = Bx$. Since every row of B belongs to B_{r^*},

$$\|z\|_\infty \leq r(x) = 1.$$

On the other hand, $x = B^{-1}z$, and (7.4) gives

$$1 = r(x) \leq K \|z\|_\infty .$$

Thus z belongs to the compact annulus

$$\mathcal{A}_K = \left\{ c \in \mathbb{R}^n : K^{-1} \leq \|c\|_\infty \leq 1 \right\}.$$

Choose a finite ρ-net $C \subset \mathcal{A}_K$. There is $c \in C$ with $\|Bx - c\|_\infty \leq \rho$, and (7.4) applied to $Bx - c$ yields

$$r(x - B^{-1}c) = r\big(B^{-1}(Bx - c)\big) \leq K\rho.$$

□

7.2 Satellite rows and an almost-norming exposed face

Proposition 7.3 (An almost-norming contraction on an exposed face)**.** *Let r be a norm on E and let $0 < \varepsilon < 1$. Then there exist an integer $N \geq n$, an r-contraction*

$$A : E \longrightarrow \ell_\infty^N,$$

and a linear functional $\ell \in \mathcal{P}_{n,N}^*$ such that:

(i) $\|Ax\|_\infty > 1 - \varepsilon$ *for every $x \in S_r$;*

(ii) the first n rows of A form an invertible matrix;

(iii) $\Pi_r(A)$ *belongs to the raw exposed slice*

$$\left\{ z \in \mathcal{R}_r^N : \ell(z) = \max_{w \in \mathcal{K}_r^N} \ell(w) \right\};$$

(iv) every point in this raw exposed slice has a representation $\pm\Pi_r(A')$ for which A' also satisfies (i) *and* (ii)*.*

Proof. Step 1: choosing a maximizing configuration. Choose $0 < \eta < d_r$, let K be as in Lemma 7.1, and choose $\rho > 0$ so small that

$$2K\rho < \varepsilon. \tag{7.7}$$

Let C be a finite coefficient net as in Lemma 7.2, and fix an ordering $C = \{c^{(1)}, \ldots, c^{(m)}\}$. A configuration consists of an r-admissible base row system $B = (b_1, \ldots, b_n)$ and one satellite row $s_c \in B_{r^*}$ for each $c \in C$. Thus the configuration space is the compact product $(B_{r^*})^{n+m}$.

For $w > 0$, define

$$P_w(B, (s_c)_{c\in C}) = w \det B + \sum_{c\in C} \sum_{i=1}^{n} c_i \det B[i \leftarrow s_c]. \tag{7.8}$$

Each replacement matrix is again r-admissible, so every replacement determinant has absolute value at most d_r. Since $\|c\|_\infty \le 1$ for $c \in C$, the satellite sum has absolute value at most mnd_r; let M_C be any uniform bound with this property. Choose w so that

$$M_C < w\eta, \tag{7.9}$$

and choose a configuration $(B, (s_c))$ maximizing $|P_w|$.

Compare it with a determinant-maximizing base system and zero satellite rows. The latter configuration has polynomial absolute value wd_r, whereas

$$|P_w(B, (s_c))| \le w\,|\det B| + M_C.$$

Therefore

$$wd_r \le w\,|\det B| + M_C < w\,|\det B| + w\eta,$$

so $|\det B| > d_r - \eta$. In particular, B is invertible and the estimate in Lemma 7.1 applies.

Step 2: the satellite rows are norming. Fix $c \in C$ and keep all variables other than s_c fixed. By (7.5), the dependence on s_c is

$$R + \det B\, s_c(B^{-1}c)$$

for a constant R. Put $x_c = B^{-1}c$. The evaluation map $s \mapsto s(x_c)$ is continuous and linear on the compact convex symmetric set B_{r^*}. By (7.2), its maximum is $r(x_c)$; symmetry gives the minimum $-r(x_c)$. Its image is therefore exactly the interval

$$[-r(x_c), r(x_c)].$$

The vector c is nonzero because it lies in the coefficient annulus, and hence $x_c \ne 0$. Thus $r(x_c) > 0$, and the affine function $t \mapsto R + \det(B)t$ is nonconstant on this interval. The convex function $|R + \det(B)t|$ can attain its maximum on the interval only at an endpoint. Since the chosen configuration is a global maximizer even when only s_c is varied, it follows that

$$\left|s_c(B^{-1}c)\right| = r(B^{-1}c) \qquad (c \in C). \tag{7.10}$$

Step 3: the resulting contraction is almost norming. Stack the base rows and satellite rows to obtain a map

$$A : E \longrightarrow \ell_\infty^{n+|C|}.$$

All its rows lie in B_{r^*}, so A is an r-contraction. Let $x \in S_r$, and choose $c \in C$ so that (7.6) holds. Put $x_c = B^{-1}c$. Using (7.10), contractivity of s_c, and the reverse triangle inequality for r, we find

$$\begin{aligned}
|s_c(x)| &\ge |s_c(x_c)| - |s_c(x - x_c)| \\
&\ge r(x_c) - r(x - x_c) \\
&\ge r(x) - 2r(x - x_c) \\
&\ge 1 - 2K\rho > 1 - \varepsilon.
\end{aligned}$$

This proves (i), and the determinant estimate for B proves (ii).

Step 4: encoding the construction by a Plücker functional. It remains to relate the construction to an exposed slice. Enumerate $C = \{c^{(1)}, \dots, c^{(m)}\}$ and let the satellite row corresponding to $c^{(\nu)}$ have index $j_\nu = n + \nu$. Put

$$I_0 = \{1, \dots, n\}, \qquad I_{\nu,i} = I_0 \setminus \{i\} \cup \{j_\nu\}.$$

The rows selected by $I_{\nu,i}$ occur in increasing order, so the satellite row is last and

$$\det A'_{I_{\nu,i}} = (-1)^{n-i} \det B'[i \leftarrow s'_{c(\nu)}].$$

Consequently the functional

$$\ell_0(z) = w z_{I_0} + \sum_{\nu=1}^{m} \sum_{i=1}^{n} (-1)^{n-i} c_i^{(\nu)} z_{I_{\nu,i}}$$

satisfies

$$\ell_0(\Pi_r(A')) = v_r P_w(A')$$

for every configuration A'. Thus the satellite polynomial is not merely analogous to a Plücker functional; it is exactly one after the fixed row-order signs are inserted. Let $M = \max |P_w|$ on the compact configuration space. The comparison configuration gives $M \geq w d_r > 0$, so $P_w(A) \neq 0$. Multiply ℓ_0 by $\operatorname{sgn} P_w(A)$ to obtain ℓ. For every raw Plücker point $\tau \Pi_r(A')$, where $\tau \in \{-1, 1\}$,

$$\ell(\tau \Pi_r(A')) \leq v_r |P_w(A')| \leq v_r M = \ell(\Pi_r(A)).$$

The first inequality uses only $\tau \operatorname{sgn} P_w(A) P_w(A') \leq |P_w(A')|$. A linear functional has the same maximum on a set and on its convex hull. Thus $\Pi_r(A)$ maximizes ℓ on $\mathcal{K}_r^N$, proving (iii).

Step 5: stability on the exposed raw slice. Finally, let z be any raw maximizer of ℓ, and write $z = \tau \Pi_r(A')$ with $\tau \in \{-1, 1\}$. Since $\ell(z) = v_r M$, both inequalities in the preceding display are equalities. In particular, $|P_w(A')| = M$. Hence the row configuration underlying A' is again an absolute maximizer of the same satellite polynomial. The determinant-gap argument and the one-row endpoint argument apply to this configuration without change, proving (iv). ☐

7.3 A common raw Plücker point

Lemma 7.4 (Finite lexicographic extraction)**.** *Let R_1 and R_2 be nonempty compact subsets of a finite-dimensional real vector space, and suppose that*

$$\operatorname{conv} R_1 = \operatorname{conv} R_2 =: K.$$

For every linear functional ℓ, there exists a point

$$z \in R_1 \cap R_2$$

that maximizes ℓ on K.

Proof. Because R_1 and R_2 are compact and the ambient space is finite-dimensional, K is compact; hence all maxima used below exist. Let

$$F_0 = \left\{ x \in K : \ell(x) = \max_K \ell \right\}$$

be the exposed face selected by ℓ. We claim that

$$\operatorname{conv}(R_i \cap F_0) = F_0 \qquad (i = 1, 2). \tag{7.11}$$

Indeed, take $x \in F_0$. Since $K = \operatorname{conv} R_i$, Carathéodory's theorem writes x as a finite convex combination of points of R_i. The value of ℓ at each of those points is at most $\max_K \ell$, and their weighted average equals $\ell(x) = \max_K \ell$. Every point carrying positive weight must therefore lie in F_0, which proves (7.11).

Choose a basis $\lambda_1, \dots, \lambda_d$ of the dual of the ambient vector space. Inductively, suppose that a nonempty compact face F_{k-1} has been obtained and that

$$\mathrm{conv}(R_i \cap F_{k-1}) = F_{k-1} \qquad (i = 1, 2).$$

The sets $R_i \cap F_{k-1}$ are compact. Let F_k be the subset of F_{k-1} on which λ_k is maximal. It is a nonempty compact exposed face of F_{k-1}. Applying the convex-combination argument from (7.11) to the compact generators $R_i \cap F_{k-1}$ gives

$$\mathrm{conv}(R_i \cap F_k) = F_k \qquad (i = 1, 2).$$

After d steps, any two points of F_d agree under every member of the dual basis, and hence are equal. Thus $F_d = \{z\}$ is a singleton. Since $\mathrm{conv}(R_i \cap \{z\}) = \{z\}$, neither intersection can be empty, and therefore $z \in R_1 \cap R_2$. Finally, $F_d \subseteq F_0$, so z still maximizes ℓ on K. □

7.4 Recovery in a fixed leading chart

Lemma 7.5 (Factorization from a common Plücker vector)**.** *Let*

$$T : (E, p) \to \ell_\infty^N, \qquad A : (E, q) \to \ell_\infty^N$$

be contractions. Suppose that, for some signs $\sigma_p, \sigma_q \in \{-1, 1\}$,

$$\sigma_p v_p \mathbf{m}(T) = \sigma_q v_q \mathbf{m}(A), \tag{7.12}$$

and that the first n rows of T and A form invertible matrices T_0 and A_0. Define

$$L = A_0^{-1} T_0.$$

Then L is invertible,

$$AL = T, \tag{7.13}$$

and

$$v_p \left|\det L\right| = v_q. \tag{7.14}$$

If, in addition,

$$\|Ay\|_\infty > 1 - \varepsilon \qquad (y \in S_q),$$

then

$$(1 - \varepsilon) q(Lx) \le p(x) \qquad (x \in E). \tag{7.15}$$

Proof. Step 1: recovery of all rows. The leading rows satisfy $A_0 L = T_0$. Let a_j and t_j be the jth rows of A and T, where $j > n$. For $1 \le k \le n$, denote by $A_0[k \leftarrow a_j]$ and $T_0[k \leftarrow t_j]$ the corresponding row replacements. Let

$$I_{j,k} = \{1, \dots, n\} \setminus \{k\} \cup \{j\},$$

written in increasing order. Since $j > n$, the row a_j appears last in the selected submatrix $A_{I_{j,k}}$. Moving it from the last position to row k requires $n - k$ transpositions, and therefore

$$\det A_{I_{j,k}} = (-1)^{n-k} \det A_0[k \leftarrow a_j].$$

The identical formula holds for T. Divide the $I_{j,k}$ coordinate of (7.12) by its nonzero leading coordinate. The common row-order sign, the global signs σ_p, σ_q, and the volume normalizations v_p, v_q all cancel, leaving

$$\frac{\det A_0[k \leftarrow a_j]}{\det A_0} = \frac{\det T_0[k \leftarrow t_j]}{\det T_0} \qquad (1 \le k \le n). \tag{7.16}$$

By the row form of Cramer's rule, the left side of (7.16) is the kth coordinate of the row vector $a_j A_0^{-1}$, and the right side is the kth coordinate of $t_j T_0^{-1}$. Hence

$$a_j A_0^{-1} = t_j T_0^{-1},$$

and multiplication by T_0 gives $a_j L = t_j$. Together with $A_0 L = T_0$, this proves $AL = T$.

Step 2: the determinant identity. The leading coordinate of (7.12) is

$$\sigma_p v_p \det T_0 = \sigma_q v_q \det A_0.$$

Since $T_0 = A_0 L$, cancellation of the nonzero number $\det A_0$ gives

$$\sigma_p v_p \det L = \sigma_q v_q.$$

Taking absolute values proves (7.14).

Step 3: the norm estimate. Finally, let $x \in E$. If $Lx = 0$, then (7.15) is immediate. Otherwise, apply the lower estimate for A to $Lx/q(Lx) \in S_q$. Using $AL = T$ and contractivity of T, we obtain

$$(1-\varepsilon)q(Lx) < \|ALx\|_\infty = \|Tx\|_\infty \le p(x),$$

which implies (7.15). □

Theorem 7.6 (Finite recovery)**.** *Assume* (7.1)*. For every $\varepsilon > 0$, there exists a bijective linear map $L_\varepsilon : E \to E$ such that*

$$(1-\varepsilon)q(L_\varepsilon x) \le p(x) \qquad (x \in E) \tag{7.17}$$

and

$$v_p \, |\det L_\varepsilon| = v_q. \tag{7.18}$$

Proof. It suffices to consider $0 < \varepsilon < 1$. Apply Proposition 7.3 to the target norm q. We obtain an integer N, a functional $\ell \in \mathcal{P}^*_{n,N}$, and a raw exposed slice. Every point in this slice is represented by a q-contraction that is $(1-\varepsilon)$-norming on S_q and has a nonzero leading minor.

By (7.1),

$$\operatorname{conv} \mathcal{R}_p^N = \mathcal{K}_p^N = \mathcal{K}_q^N = \operatorname{conv} \mathcal{R}_q^N.$$

Apply Lemma 7.4 to $R_1 = \mathcal{R}_p^N$, $R_2 = \mathcal{R}_q^N$, and the functional ℓ. We obtain a common raw Plücker point z maximizing ℓ. First choose a target representation

$$z = \sigma_q v_q \mathbf{m}(A)$$

using property (iv) of Proposition 7.3, so that A is a q-contraction, is $(1-\varepsilon)$-norming on S_q, and has a nonzero leading minor. Next choose any source representation

$$z = \sigma_p v_p \mathbf{m}(T),$$

where T is a p-contraction. The leading coordinate of the common vector z is nonzero in the target representation and hence also in the source representation, so the leading blocks of both A and T are invertible.

The common raw Plücker point therefore yields a target almost-norming contraction and a source contraction with the same normalized maximal minors. Lemma 7.5 produces an invertible linear map L_ε satisfying (7.17) and (7.18). If $\varepsilon \ge 1$, apply the result with any fixed $\varepsilon_0 \in (0,1)$; the desired norm inequality is then automatic because $1-\varepsilon \le 0$. □

8. Compactness and exact recovery of the norm

Assume throughout the section that

$$\mathcal{K}_p^N = \mathcal{K}_q^N \qquad (N \geq n).$$

For $k \geq 0$, set $\varepsilon_k = (k+2)^{-1}$ and choose a map $L_k = L_{\varepsilon_k}$ from Theorem 7.6.

Proposition 8.1 (Limit recovery)**.** *There exists a bijective linear map $L : E \to E$ such that*

$$q(Lx) \leq p(x) \qquad (x \in E) \tag{8.1}$$

and

$$v_p \, |\det L| = v_q. \tag{8.2}$$

Proof. Compactness of the recovery maps. Since $\varepsilon_k \leq 1/2$, (7.17) gives

$$q(L_k x) \leq \frac{1}{1-\varepsilon_k} p(x) \leq 2p(x) \qquad (x \in E).$$

Using (2.5) for p and q, we obtain

$$\|L_k x\|_\infty \leq \frac{2\Lambda_p}{\lambda_q} \|x\|_\infty .$$

Thus (L_k) is bounded in the finite-dimensional operator space $\mathcal{L}(E, E)$. After passing to a subsequence, it converges in operator norm to a linear map L.

Passage to the limit. For each fixed $x \in E$, continuity of q and (7.17) yield

$$q(Lx) = \lim_{k\to\infty} q(L_k x) \leq \lim_{k\to\infty} \frac{p(x)}{1-\varepsilon_k} = p(x),$$

which proves (8.1). Since the determinant is a polynomial in the matrix entries,

$$v_p \, |\det L| = \lim_{k\to\infty} v_p \, |\det L_k| = v_q.$$

This is (8.2). The unit ball B_q has nonempty interior, hence $v_q > 0$. Therefore $\det L \neq 0$, and L is bijective. ☐

The inequality (8.1) is equivalent to

$$L(B_p) \subseteq B_q. \tag{8.3}$$

Lemma 8.2 (Equal-volume convex inclusion)**.** *Let $K \subseteq C$ be compact convex subsets of $\mathbb{R}^n$, and suppose that C has nonempty interior. If*

$$\mathcal{L}^n(K) = \mathcal{L}^n(C),$$

then $K = C$.

Proof. The set C contains an open ball and has positive volume. Equality of the volumes therefore implies that K is nonempty. Assume that the inclusion is proper and choose $y \in C \setminus K$. Since K is compact,

$$d = \operatorname{dist}(y, K) > 0.$$

Choose $x_0 \in \operatorname{int} C$ and $r > 0$ such that $B_2(x_0, r) \subset C$. For $0 < t < 1$, set

$$y_t = (1-t)x_0 + ty.$$

Convexity gives

$$B_2(y_t, (1-t)r) = (1-t)B_2(x_0, r) + ty \subset C,$$

so $y_t \in \operatorname{int} C$. For t sufficiently close to 1, we also have $\|y_t - y\|_2 < d/2$, and hence $\operatorname{dist}(y_t, K) > d/2$. Therefore the ball

$$B_2\left(y_t, \min\left\{\frac{(1-t)r}{2}, \frac{d}{4}\right\}\right)$$

is contained in $C \setminus K$. It has positive Lebesgue measure, contradicting $\mathcal{L}^n(K) = \mathcal{L}^n(C)$. □

Proposition 8.3 (Recovery of the unit ball)**.** *The map L from Proposition 8.1 satisfies*

$$L(B_p) = B_q. \tag{8.4}$$

Consequently,

$$q(Lx) = p(x) \qquad (x \in E). \tag{8.5}$$

Proof. The set $L(B_p)$ is compact and convex, and (8.3) gives $L(B_p) \subseteq B_q$. Since L is invertible and B_p is Borel, the linear case of the change-of-variables formula [4, Theorem 3.9] gives

$$\mathcal{L}^n(L(B_p)) = |\det L|\ \mathcal{L}^n(B_p) = |\det L|\ v_p = v_q = \mathcal{L}^n(B_q),$$

where (8.2) is used in the third equality. Since B_q has nonempty interior, Lemma 8.2 proves (8.4).

For completeness, recall that every norm r is the Minkowski functional of its closed unit ball:

$$r(x) = \inf\{t > 0 : x \in tB_r\}. \tag{8.6}$$

Indeed, $x \in tB_r$ is equivalent to $r(x) \le t$. Linearity of L and (8.4) therefore give, for every $x \in E$,

$$\begin{aligned} q(Lx) &= \inf\{t > 0 : Lx \in tB_q\} \\ &= \inf\{t > 0 : Lx \in L(tB_p)\} \\ &= \inf\{t > 0 : x \in tB_p\} = p(x), \end{aligned}$$

where injectivity of L is used in the third equality. This proves (8.5). □

Theorem 8.4 (Coordinate rigidity)**.** *If*

$$\mathcal{K}_p^N = \mathcal{K}_q^N \qquad \textit{for every } N \ge n,$$

then there exists a surjective linear isometry

$$L : (E, p) \longrightarrow (E, q).$$

Proof. By Proposition 8.1, there is a bijective linear map $L : E \to E$ satisfying (8.1) and (8.2). Proposition 8.3 then gives $q(Lx) = p(x)$ for every $x \in E$. □

9. Proof of the main theorem

Theorem 9.1 (Rigidity of two norms on a common coordinate space)**.** *Let p and q be norms on $E = \mathbb{R}^n$, with $n \ge 1$. Suppose that there is a surjective map*

$$\delta : S_p \longrightarrow S_q$$

such that

$$q(\delta(u) - \delta(v)) = p(u - v) \qquad (u, v \in S_p).$$

Then (E, p) and (E, q) are linearly isometric.

Proof. By Theorem 6.1, the Plücker contraction bodies of p and q agree for every $N \geq n$. Theorem 8.4 therefore yields a surjective linear isometry from (E, p) onto (E, q). ☐

Proof of Theorem 1.1. If either X or Y is zero-dimensional, the conclusion follows from the two-direction argument in Lemma 2.2. Assume that both spaces are nonzero. By Lemma 2.5, they have the same positive dimension. Choose the coordinate isomorphisms e_X, e_Y and the norms p, q from (2.3). The original map Δ induces the sphere isometry $\delta : S_p \to S_q$ in (2.4). By Theorem 9.1, there is a surjective linear isometry

$$L : (E, p) \longrightarrow (E, q).$$

Consequently,

$$T = e_Y \circ L \circ e_X^{-1} : X \longrightarrow Y$$

is a surjective real-linear isometry. ☐

Proof of Corollary 1.2. If (S_X, d_X) and (S_Y, d_Y) are isometric, Theorem 1.1 gives a surjective real-linear isometry from X onto Y. Conversely, the restriction of a surjective linear isometry $T : X \to Y$ to S_X is a surjective map onto S_Y and satisfies (1.2). ☐

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