

Brouwer’s Fixed-Point Theorem from Commuting Operators

Version 1

Prepared: 16 October 2026

Publication responsibility: Ryotaro Tanaka

This note presents an operator-and-polynomial proof of Brouwer’s fixed-point theorem and a direct passage to Schauder’s compact-image theorem. It makes no historical priority claim for this proof arrangement.

Publication responsibility covers release, versioning and corrections; it does not assert historical priority for this proof arrangement.

Abstract

We prove the cube form of Brouwer’s fixed-point theorem using commuting operators, polynomial reduction, and an odd-power perturbation of a polynomial approximation to $f - \text{id}$. An elementary growth estimate confines every real solution of the perturbed system to the cube. We then pass directly from cubes to Schauder’s theorem for a continuous self-map of a closed convex set whose image has compact closure. The connecting construction uses a finite net and a cube of convex coefficients. The compact-domain form of Schauder and the usual finite-dimensional form of Brouwer follow as corollaries. We explain the algebraic identities and fixed-point estimates in detail. No quotient-ring language, general Gröbner basis theory, or topological degree is needed.

1. What we assume, and where the proof is going

We use elementary finite-dimensional linear algebra, the intermediate value theorem, and basic facts about compact metric spaces: finite nets, continuous images, attainment of extrema, and convergent subsequences. The one approximation result we take as an external input is the following.

Polynomial approximation. *Let $C = [-1, 1]^n$ and let $g : C \rightarrow \mathbb{R}$ be continuous. For every $\eta > 0$ there is a real polynomial Q in n variables such that*

$$\sup_{x \in C} |Q(x) - g(x)| < \eta.$$

This is a direct consequence of the real Stone–Weierstrass theorem: the restrictions of real polynomials to C form a subalgebra of $C(C, \mathbb{R})$ containing the constants and separating points. A reader who already knows multivariate polynomial approximation may use that result directly; no other part of Stone–Weierstrass is used. Approximation of a vector-valued map will mean approximation of its finitely many coordinates.

Our first goal is the cube version of Brouwer’s theorem.

Theorem 1 (Brouwer: cube form). *Let $n \geq 1$ and $C = [-1, 1]^n$. Every continuous map $f : C \rightarrow C$ has a fixed point. The same holds on $[0, 1]^n$ by an affine change of coordinates.*

The proof has three steps.

1. Commuting operators on an odd-dimensional real space have a common real eigenvector.
2. Polynomial reduction turns a system $x_i^N = q_i(x)$, with N odd and $\deg q_i < N$, into commuting operators. Their common eigenvalues provide a real solution of the system.

3. We approximate $f - \text{id}$ by polynomials. After choosing an odd exponent using their degrees and coefficients, a polynomial system has a real solution which an elementary growth estimate forces inside the cube. Its fixed-point residual can be made arbitrarily small, and compactness makes the minimum residual zero.

The algebraic spaces and operators may change from one approximation to the next; no convergence of these operators will be needed.

Section 6 then uses Theorem 1 on a cube of coefficients to prove Schauder’s theorem in a real normed space. Only the image of the map needs compact closure; the closed convex domain need not be bounded, and the ambient space need not be complete. The compact-domain form of Schauder and the general finite-dimensional form of Brouwer are deduced afterward in Section 7. Thus the proof of Schauder uses only the cube theorem already proved, not the general convex-set version of Brouwer.

2. A common real eigenvector

An individual real operator in odd dimension has a real eigenvalue, because its characteristic polynomial has odd degree. Commutativity lets us make the eigenvector common to the whole family.

Lemma 2. *Let V be a real vector space of positive odd dimension. If the linear maps $T_1, \dots, T_n : V \rightarrow V$ commute pairwise, then there are $v \neq 0$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that*

$$T_i v = \lambda_i v \quad (1 \leq i \leq n).$$

Proof. Choose a common invariant subspace $W \subset V$ whose dimension is positive and odd, and as small as possible. Such a subspace exists, since V itself is one candidate.

Fix i . The characteristic polynomial of $T_i|_W$ has odd degree, so it has a real zero λ_i . Set

$$S = T_i|_W - \lambda_i I_W.$$

Then $\ker S \neq \{0\}$. Suppose that $S \neq 0$. Its kernel and image are both nonzero proper subspaces of W . Moreover, both are invariant under every T_j . Indeed, commutativity gives

$$S(T_j w) = T_j(Sw),$$

so that

$$w \in \ker S \implies T_j w \in \ker S, \quad T_j(Sw) = S(T_j w) \in \text{im } S.$$

But rank–nullity gives

$$\dim \ker S + \dim \text{im } S = \dim W.$$

The right-hand side is odd, so one of the two terms on the left is odd. This contradicts the minimal choice of W . Thus $S = 0$.

This works for every i , and hence $T_i|_W = \lambda_i I_W$ for every i . Any nonzero $v \in W$ is the required common eigenvector. $\square$

Neither diagonalizability nor simplicity of any eigenvalue was used. This principle and the kernel–image argument occur in Derksen [1, Lemmas 3–4]; the minimal odd-dimensional invariant-subspace argument also appears in Shekhtman [2, proof of Theorem 4.5].

3. From polynomial reduction to a real solution

Throughout this section, $\mathbb{R}[X_1, \dots, X_n]$ denotes the polynomial ring in n commuting formal indeterminates over $\mathbb{R}$. Upper-case X_i denote formal indeterminates, whereas lower-case x_i denote real numbers at which the polynomials are evaluated. The total degree of a nonzero polynomial is the largest sum of exponents occurring in one of its monomials; the zero polynomial is permitted in every upper degree bound.

Lemma 3. *Let $n \geq 1$ and let N be a positive odd integer. Suppose that $q_1, \dots, q_n \in \mathbb{R}[X_1, \dots, X_n]$ have total degree less than N . Then the system*

$$x_i^N = q_i(x_1, \dots, x_n) \quad (1 \leq i \leq n)$$

has a solution in $\mathbb{R}^n$.

The proof is a concrete way of replacing polynomial multiplication by linear operators on a finite-dimensional space. We spell out the consistency of this replacement, since that is the main point.

The reduction rule and its uniqueness

Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, write $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Define the ordinary polynomial subspace

$$E = \text{span}_{\mathbb{R}}\{X^\alpha : 0 \leq \alpha_i < N \text{ for every } i\}. \quad (1)$$

For each i , the exponent α_i can be chosen independently from $0, 1, \dots, N-1$. Thus there are N^n displayed monomials. Distinct monomials are linearly independent in the polynomial ring, and they span E by definition; hence they form a basis and

$$\dim E = N^n,$$

which is odd.

Whenever a monomial contains a factor X_i^N , replace that factor by q_i and expand linearly:

$$mX_i^N \mapsto mq_i. \quad (2)$$

Every monomial on the right has smaller total degree than the monomial on the left. Each replacement produces finitely many terms, and along every branch of the reduction the degree strictly decreases. Thus the resulting reduction tree is finite. A terminal monomial has no exponent at least N , so it is one of the basis monomials in (1); conversely, any monomial outside that list contains some factor X_i^N and can still be reduced. Hence every branch terminates in E .

We must check that the answer does not depend on which factor is reduced first. Use induction on the total degree of the starting monomial. If two choices concern the same variable, their first results agree. If they concern distinct variables $i \neq j$, write the monomial as $mX_i^N X_j^N$. The two choices can be compared explicitly:

$$\begin{aligned} mX_i^N X_j^N &\mapsto mq_i X_j^N \mapsto mq_i q_j, \\ mX_i^N X_j^N &\mapsto mX_i^N q_j \mapsto mq_i q_j. \end{aligned} \quad (3)$$

The second arrows mean that we reduce the displayed factor in each term of the expansion. All monomials after the first arrow have smaller degree than the starting one. By the induction

hypothesis, their eventual reductions are independent of subsequent choices. Since both paths in (3) lead to the same polynomial, the two first choices have the same final result. This proves uniqueness.

Let $\text{NF}(p) \in E$ denote this unique reduced polynomial, extended linearly from monomials to all polynomials. In particular,

$$\text{NF}(h) = h \quad (h \in E). \quad (4)$$

For example, if $n = 2$, $N = 3$, $q_1 = Y$ and $q_2 = X$, then the rules are $X^3 \mapsto Y$ and $Y^3 \mapsto X$. The overlapping reduction is simply

$$X^3Y^3 \mapsto Y^4 \mapsto XY, \quad X^3Y^3 \mapsto X^4 \mapsto XY.$$

The argument above is the general version of this consistency check.

Why reduction is compatible with multiplication

Uniqueness means that a permitted replacement does not change the final normal form. Explicitly, for any monomial m ,

$$\text{NF}(mX_i^N) = \text{NF}(mq_i).$$

By linearity, the same is true when the replacement is made within a polynomial sum; collecting or cancelling like terms causes no problem.

For any two polynomials a, b we have

$$\boxed{\text{NF}(a \text{NF}(b)) = \text{NF}(ab)}. \quad (5)$$

Write $\mapsto^*$ for a finite sequence of permitted reductions. To see (5) first for a monomial a , take a reduction $b \mapsto^* \text{NF}(b)$. Multiplying every step by a still gives valid replacements of factors X_i^N , so

$$ab \mapsto^* a \text{NF}(b) \mapsto^* \text{NF}(a \text{NF}(b)).$$

This is a complete reduction of ab . But the final normal form of ab is unique, so its endpoint must be $\text{NF}(ab)$. In other words, the lifted reduction path and any direct reduction of ab meet at the same normal form. This proves (5) for monomial a . For general a , expand it into monomials and use linearity.

The map NF is a linear projection onto E ; it is not asserted to preserve the ordinary polynomial product. The outer reduction on the left of (5) is essential.

Multiplication becomes commuting operators

Define linear maps $T_i : E \rightarrow E$ by

$$T_i h = \text{NF}(X_i h). \quad (6)$$

Equation (5) gives, for every $h \in E$,

$$\begin{aligned} T_i T_j h &= \text{NF}(X_i \text{NF}(X_j h)) \\ &= \text{NF}(X_i X_j h) \\ &= \text{NF}(X_j \text{NF}(X_i h)) = T_j T_i h. \end{aligned} \quad (7)$$

Thus the operators commute. Applying (5) repeatedly, and then using linearity, yields

$$T_1^{\alpha_1} \cdots T_n^{\alpha_n} h = \text{NF}(X^\alpha h), \quad p(T_1, \dots, T_n)h = \text{NF}(ph) \quad (8)$$

for every polynomial p . The constant term uses (4). In particular, the reduction rule itself gives

$$T_i^N h = \text{NF}(X_i^N h) = \text{NF}(q_i h) = q_i(T_1, \dots, T_n)h. \quad (9)$$

For a one-variable illustration, take $N = 3$ and $q(X) = a + bX + cX^2$. In the ordered basis $1, X, X^2$ of E , multiplication by X followed by reduction gives

$$T(1) = X, \quad T(X) = X^2, \quad T(X^2) = a + bX + cX^2,$$

so its matrix and relation are

$$T = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}, \quad T^3 = aI + bT + cT^2.$$

This is the companion matrix of $X^3 - cX^2 - bX - a$. Every real eigenvalue λ of T therefore satisfies $\lambda^3 = a + b\lambda + c\lambda^2$. Our construction extends this mechanism to several multiplication operators; their commutativity is what lets us use one common eigenvector for all the equations.

Extracting the real solution

By Lemma 2, the commuting operators on the odd-dimensional space E have a common real eigenvector:

$$0 \neq v \in E, \quad T_i v = \lambda_i v, \quad \lambda_i \in \mathbb{R}.$$

For a monomial and hence for a polynomial, this implies

$$T_1^{\alpha_1} \dots T_n^{\alpha_n} v = \lambda^{\alpha} v, \quad q_i(T_1, \dots, T_n) v = q_i(\lambda_1, \dots, \lambda_n) v.$$

Applying (9) to v therefore gives

$$(\lambda_i^N - q_i(\lambda_1, \dots, \lambda_n))v = 0.$$

Since $v \neq 0$, the scalar coefficient must vanish. Thus $\lambda = (\lambda_1, \dots, \lambda_n)$ solves the system, proving Lemma 3. $\square$

What has been proved here. The polynomial relations are imposed by a terminating, consistent reduction, not by an assumed solution. The resulting operators are constructed before their eigenvalues are found. No fixed-point theorem has been used. This is a concrete instance of the ideal-projector and real-zero machinery in Shekhtman [2, Theorems 1.2 and 4.5]; that paper attributes the projection identity to de Boer.

4. Polynomial approximation and a growth estimate

From now on, let $n \geq 1$, let $C = [-1, 1]^n$, and use the maximum norm $\|x\|_\infty = \max_i |x_i|$. A polynomial approximation on C can grow arbitrarily outside C . This matters because Lemma 3 provides a real solution, but does not tell us where the solution lies. We will choose a high odd power whose growth controls this difficulty. The needed exponent will depend on both the degrees and the coefficients of the polynomials.

An elementary inequality for powers

For a positive integer N , an integer $0 \leq k \leq N$, and real $t \geq 1$,

$$t^k - 1 \leq \frac{k}{N}(t^N - 1). \quad (10)$$

The case $k = 0$ is immediate. For $k \geq 1$, the numbers $1, t, \dots, t^{N-1}$ are nondecreasing, so the average of the first k is at most the average of all N :

$$\frac{1 + t + \dots + t^{k-1}}{k} \leq \frac{1 + t + \dots + t^{N-1}}{N}.$$

Multiplying by $k(t - 1)$ proves (10), including when $t = 1$.

Control along rays from the cube

A polynomial map $Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$ means a map $Q = (Q_1, \dots, Q_n)$ whose coordinate functions are real polynomials in the n input variables. The subscript i labels an output coordinate; each Q_i may still mix all of $X_1, \dots, X_n$. Using the multi-index notation above, each coordinate has a unique finite expansion

$$Q_i(X) = \sum_{\alpha \in \mathbb{N}_0^n} c_{i,\alpha} X^\alpha, \quad c_{i,\alpha} \in \mathbb{R},$$

with only finitely many nonzero coefficients. We use this monomial expansion because, along the ray $x = tu$,

$$(tu)^\alpha = t^{|\alpha|} u^\alpha,$$

so the one-variable estimate (10) can be applied with $k = |\alpha|$. Define

$$B = \max_{1 \leq i \leq n} \sum_{\alpha \in \mathbb{N}_0^n} |\alpha| |c_{i,\alpha}|. \quad (11)$$

Choose an integer $d \geq 0$ bounding the total degrees of all Q_i ; zero polynomials are allowed. If $N > d$, $u \in C$, and $t \geq 1$, then

$$\begin{aligned} Q_i(tu) - Q_i(u) &= \sum_{\alpha \in \mathbb{N}_0^n} c_{i,\alpha} (t^{|\alpha|} - 1) u^\alpha, \\ |Q_i(tu) - Q_i(u)| &\leq \sum_{\alpha \in \mathbb{N}_0^n} |c_{i,\alpha}| (t^{|\alpha|} - 1) |u^\alpha| \\ &\leq \frac{B}{N} (t^N - 1). \end{aligned} \quad (12)$$

Indeed, $|u^\alpha| \leq 1$, and (10) applies to each $k = |\alpha|$. The constant terms cancel because $t^0 - 1 = 0$. The estimate allows arbitrary signs and mixed monomials. It controls growth immediately outside the cube as well as far away from it.

5. From an approximate fixed point to an exact one

We now prove Theorem 1 on $C = [-1, 1]^n$. Fix $\varepsilon > 0$ and set

$$g = f - \text{id} = (g_1, \dots, g_n), \quad g_i(x) = f_i(x) - x_i.$$

Thus g takes n real coordinates as input and returns an n -component vector. Its i th output coordinate $g_i : C \rightarrow \mathbb{R}$ is a continuous scalar-valued function and may depend on all n inputs. Apply polynomial approximation separately to these finitely many coordinate functions, choosing $Q_i \in \mathbb{R}[X_1, \dots, X_n]$ such that

$$\sup_{u \in C} |Q_i(u) - g_i(u)| < \varepsilon \quad (1 \leq i \leq n).$$

Set $Q = (Q_1, \dots, Q_n)$. Since we use the maximum norm, the coordinatewise estimates are exactly

$$\sup_{u \in C} \|Q(u) - g(u)\|_\infty < \varepsilon. \quad (13)$$

For these polynomials choose B and d as above, and then choose a positive odd integer N satisfying

$$\boxed{N > d, \quad \frac{B}{N} \leq \varepsilon.} \quad (14)$$

Arbitrarily large odd integers are available, so this is possible. The order is important: first ε , then Q , then B, d , and finally N . The degree condition alone would not give the confinement estimate below.¹

Define $q_i = Q_i/\varepsilon$. Its total degree is less than N , and Lemma 3 gives a real solution $x \in \mathbb{R}^n$ of

$$\varepsilon x_i^N = Q_i(x) \quad (1 \leq i \leq n). \quad (15)$$

There is no restriction on the size of the coefficients in that lemma. We first locate this solution and only then estimate its residual.

The boundary inequality

If $u \in C$ and $|u_i| = 1$, then $f_i(u) \in [-1, 1]$ gives

$$\begin{aligned} u_i g_i(u) &= u_i f_i(u) - 1 \leq 0, \\ u_i Q_i(u) &= u_i g_i(u) + u_i (Q_i(u) - g_i(u)) < \varepsilon. \end{aligned} \quad (16)$$

The strict inequality follows from (13). This is the precise sense in which the displacement points inward at the boundary.

First locate the solution

Suppose a real solution of (15) satisfies $t = \|x\|_\infty \geq 1$. Set $u = x/t \in C$ and choose a coordinate i with $|u_i| = 1$. Since $x = tu$ and N is odd,

$$u_i x_i^N = u_i (tu_i)^N = t^N u_i^{N+1} = t^N.$$

Using (16), (12), and (14), we obtain

$$\begin{aligned} \varepsilon t^N &= u_i Q_i(x) \\ &= u_i Q_i(tu) \\ &= u_i Q_i(u) + u_i (Q_i(tu) - Q_i(u)) \\ &\leq u_i Q_i(u) + |Q_i(tu) - Q_i(u)| \\ &< \varepsilon + \frac{B}{N} (t^N - 1) \\ &\leq \varepsilon t^N, \end{aligned} \quad (17)$$

¹For $f(x) = x^2$ on $[-1, 1]$, take the exact residual polynomial $Q(x) = x^2 - x$. With $\varepsilon = 1/4$ and $N = 3$, the equation $\varepsilon x^N = Q(x)$ is $x(x-2)^2 = 0$, so $x = 2$ is an outside root even though $N > \deg Q$. Here $B = 3$, so $B/N \leq \varepsilon$ fails.

which is a contradiction, also when $t = 1$. Therefore every real solution of (15) satisfies

$$\boxed{\|x\|_\infty < 1.} \quad (18)$$

The approximation estimate was used at $u \in C$, before we knew the location of x . The function f has never been evaluated outside C .

Then estimate the error

Now $x \in C$, so we may use (13) at x itself. Equation (15) gives $\|Q(x)\|_\infty = \varepsilon \|x\|_\infty^N$, and hence

$$\begin{aligned} \|f(x) - x\|_\infty &= \|g(x)\|_\infty \\ &\leq \|g(x) - Q(x)\|_\infty + \|Q(x)\|_\infty \\ &< \varepsilon + \varepsilon \|x\|_\infty^N < 2\varepsilon. \end{aligned} \quad (19)$$

Finally make the residual zero

The function

$$r : C \rightarrow [0, \infty), \quad r(x) = \|x - f(x)\|_\infty$$

is continuous, so it attains a minimum at some $x^* \in C$. For every $\varepsilon > 0$, the construction above gives $x \in C$ satisfying (19). Consequently

$$0 \leq r(x^*) = \min_{z \in C} r(z) < 2\varepsilon \quad \text{for every } \varepsilon > 0.$$

It follows that $r(x^*) = 0$, which is precisely $f(x^*) = x^*$.

The polynomials, the odd integer N , and the finite-dimensional operators can all depend on ε . No continuous choice of roots or eigenvectors is required. Although the perturbed solutions lie in the open cube, the minimizing fixed point x^* may lie on its boundary.

To obtain the result on $[0, 1]^n$, use the coordinatewise affine homeomorphism

$$\phi : [-1, 1]^n \rightarrow [0, 1]^n, \quad \phi(u)_i = \frac{u_i + 1}{2}.$$

For a continuous $F : [0, 1]^n \rightarrow [0, 1]^n$, a fixed point u^* of $\phi^{-1} \circ F \circ \phi$ gives the fixed point $\phi(u^*)$ of F . This completes the proof of Theorem 1. $\square$

6. From a cube to Schauder

We next construct approximate fixed points in a real normed space. A finite net supplies nearby points and continuous convex weights. We apply the cube theorem to those weights, rather than applying a fixed-point theorem to their convex hull.

Theorem 4 (Schauder: compact-image form). *Let X be a real normed space and let $K \subset X$ be nonempty, closed, and convex. Let $f : K \rightarrow K$ be continuous. If*

$$H = \overline{f(K)}^X$$

is compact in X , then f has a fixed point.

The closure is taken in X , not in a completion of X . Neither completeness of X nor boundedness of K is assumed. Since K is closed and $f(K) \subset K$, we have $H \subset K$. The compact set H need not be convex. It will supply both the finite net and the convergent subsequence used at the end.

A finite net and continuous convex weights

Fix $\delta > 0$. Compactness of the nonempty set H gives points $a_1, \dots, a_m \in H$, with $m \geq 1$, such that

$$H \subset \bigcup_{j=1}^m B(a_j, \delta), \quad B(a_j, \delta) = \{y \in X : \|y - a_j\| < \delta\}.$$

Thus every point of H lies at distance strictly less than δ from at least one a_j . For a real number r , write $r_+ = \max\{r, 0\}$. For $y \in H$, set

$$w_j(y) = (\delta - \|y - a_j\|)_+, \quad \lambda_j(y) = \frac{w_j(y)}{\sum_{k=1}^m w_k(y)}. \quad (20)$$

At least one $w_j(y)$ is positive because the balls are open and cover H . The denominator is therefore positive at every $y \in H$. Each λ_j is continuous, and

$$\lambda_j(y) \geq 0, \quad \sum_{j=1}^m \lambda_j(y) = 1, \quad \lambda_j(y) > 0 \implies \|y - a_j\| < \delta.$$

Define

$$L_\delta = \text{conv}\{a_1, \dots, a_m\} \subset K, \quad P_\delta(y) = \sum_{j=1}^m \lambda_j(y) a_j. \quad (21)$$

The inclusion $L_\delta \subset K$ follows from convexity of K , and $P_\delta : H \rightarrow L_\delta$ is continuous. Its error is controlled by the same weights:

$$\begin{aligned} \|P_\delta(y) - y\| &= \left\| \sum_{j=1}^m \lambda_j(y) (a_j - y) \right\| \\ &\leq \sum_{j=1}^m \lambda_j(y) \|a_j - y\| < \delta \quad (y \in H). \end{aligned} \quad (22)$$

For the strict inequality, discard the zero-weight terms. Each remaining distance is strictly less than δ , and the remaining weights sum to one.

The set L_δ lies in the span of finitely many points, but we do not need a fixed-point theorem on L_δ . We also do not need $L_\delta \subset H$: the map P_δ is defined on H and can take values outside H .

Extend convex combinations to a whole cube

To use Theorem 1, we need a map on a cube. The natural coefficients of a convex combination satisfy $\sum_j t_j = 1$, whereas a general $t \in [0, 1]^m$ does not. We therefore extend the convex combination formula as follows:

$$S(t) = \sum_{j=1}^m t_j, \quad A(t) = \frac{\sum_{j=1}^m t_j a_j + (1 - S(t))_+ a_1}{\max\{1, S(t)\}} \quad (t \in [0, 1]^m). \quad (23)$$

Here a_1 is just one of the net points, chosen to absorb any missing weight. To check the formula, consider its two cases:

$$A(t) = \begin{cases} \sum_{j=1}^m t_j a_j + (1 - S(t)) a_1, & S(t) \leq 1, \\ \sum_{j=1}^m \frac{t_j}{S(t)} a_j, & S(t) \geq 1. \end{cases}$$

In the first case we add the missing weight $1 - S(t)$ to a_1 ; in the second we divide all weights by their sum. In both cases the coefficients are nonnegative and sum to one, so $A(t) \in L_\delta$. The formulas agree when $S(t) = 1$, and then

$$A(t) = \sum_{j=1}^m t_j a_j.$$

The single formula (23) also proves continuity: its denominator is at least one. In particular, it has no singularity at $t = 0$, where $A(0) = a_1$. No linear or affine independence of the a_j is needed.

Apply Brouwer to the coefficient cube

Define

$$F : [0, 1]^m \rightarrow [0, 1]^m, \quad F_j(t) = \lambda_j(f(A(t))) \quad (1 \leq j \leq m). \quad (24)$$

This composition is well-defined: $A(t) \in L_\delta \subset K$ and $f(A(t)) \in f(K) \subset H$, the domain of the weights. It is continuous, and its coordinates are nonnegative and sum to one, so its values do indeed belong to $[0, 1]^m$.

By Theorem 1, there is a fixed point $t^\delta \in [0, 1]^m$. Combining $t^\delta = F(t^\delta)$ with the normalization $\sum_j \lambda_j(y) = 1$ from (20), we obtain

$$\begin{aligned} t_j^\delta &= \lambda_j(f(A(t^\delta))) & (1 \leq j \leq m), \\ S(t^\delta) &= \sum_{j=1}^m t_j^\delta = \sum_{j=1}^m \lambda_j(f(A(t^\delta))) = 1. \end{aligned} \quad (25)$$

Set $x_\delta = A(t^\delta) \in L_\delta \subset K$. Because the coefficients now sum to one, the extension agrees with the original convex combination. Thus

$$x_\delta = \sum_{j=1}^m t_j^\delta a_j = \sum_{j=1}^m \lambda_j(f(x_\delta)) a_j = P_\delta(f(x_\delta)).$$

Apply (22) to $y = f(x_\delta) \in H$. We have obtained

$$\boxed{\|x_\delta - f(x_\delta)\| < \delta.} \quad (26)$$

The number of net points and the cube dimension may depend on δ . Only the approximate fixed point and its residual estimate are needed for the next step.

Use compactness of the image to take the limit

Choose $\delta_k \downarrow 0$, and for each k choose a point $x_k = x_{\delta_k} \in K$ as above. We cannot assume that K is compact. Instead, $f(x_k) \in H$, so compactness of H gives a convergent subsequence. Passing to it and relabeling x_k and δ_k , write

$$f(x_k) \longrightarrow x^* \in H \subset K.$$

The residual estimate makes the corresponding points converge to the same limit:

$$\begin{aligned} \|x_k - x^*\| &\leq \|x_k - f(x_k)\| + \|f(x_k) - x^*\| \\ &< \delta_k + \|f(x_k) - x^*\| \longrightarrow 0. \end{aligned} \quad (27)$$

Since $x^* \in K$, continuity of f now gives

$$f(x^*) = \lim_{k \rightarrow \infty} f(x_k) = x^*.$$

This proves Theorem 4. □

What compactness is required. The theorem assumes that the closure of the *whole image* $f(K)$ is compact. On an unbounded domain, it is not enough for f merely to map bounded sets into sets with compact closure. For example, $f(x) = x + 1$ on $K = X = \mathbb{R}$ has the latter property but has no fixed point. Nor do we minimize the residual over K : a continuous function on a noncompact set need not attain its infimum. The convergent subsequence comes specifically from H .

Why this argument uses only convex combinations. The maps P_δ are generally nonlinear. We have not assumed linear projections, a basis for X , or a Banach-space approximation property. We have not assumed that P_δ fixes points of L_δ ; indeed, L_δ need not be contained in its domain H . The map A makes it possible to work on the coefficient cube, where the cube theorem applies directly. No Euclidean projection onto K , compactness theorem for closed convex hulls, or fixed-point theorem on a general finite-dimensional convex set is needed in this proof.

7. Two familiar consequences

Corollary 5 (Schauder: compact-domain form). *Let K be a nonempty compact convex subset of a real normed space X . Every continuous map $f : K \rightarrow K$ has a fixed point.*

Proof. A compact subset of a normed space is closed. The continuous image $f(K)$ is compact and hence closed in X , so $H = \overline{f(K)}^X = f(K)$ is compact. Theorem 4 applies. Completeness of X is still unnecessary. □

Corollary 6 (Brouwer: compact convex-set form). *Let K be a nonempty compact convex subset of a finite-dimensional real normed space. Every continuous map $f : K \rightarrow K$ has a fixed point.*

Proof. This is the finite-dimensional special case of Corollary 5. There is no requirement that K have nonempty interior. If its ambient space has dimension zero, K is a singleton and the conclusion is immediate. □

The order of deduction is worth emphasizing: the algebra proves the cube theorem; the cube theorem on coefficient vectors proves Schauder; and the general compact convex-set form of Brouwer is then a corollary. There is no circular appeal to that last form in the proof of Schauder.

Perspective and sources

The proof separates algebraic existence from the location estimate. Odd-dimensional linear algebra supplies a real solution of the perturbed polynomial system. The boundary inequality for $f - \text{id}$ and the polynomial growth estimate then force every such solution into the cube, where its fixed-point residual is small. The main estimate is (17); the main algebraic identity is (5). The coefficient bound can require a large N and hence a large operator space of dimension N^n ; this is an existence argument, not a claim of computational efficiency.

The passage to Schauder uses only the finite-net approximation (22), the coefficient-cube map (24), and compactness of the image. The residual estimate transfers convergence from the images to the approximate fixed points themselves.

Derksen’s Lemmas 3–4 give the kernel–image induction and the common-real-eigenvector statement for commuting endomorphisms on an odd-dimensional real vector space [1, Lemmas 3–4]. Shekhtman’s Theorem 1.2 records the ideal-projector identity $P(fg) = P(fP(g))$, while Theorem 4.5 obtains a real zero of an odd-codimensional real ideal from a common eigenvector of its commuting multiplication operators [2, Theorems 1.2 and 4.5]. The reduction used here is an explicit special case of that framework. Kuga’s proof [3] uses a different construction: polynomial approximation on a simplex, induction on dimension, and a continuous function formed from the real and imaginary parts of the roots of an auxiliary polynomial equation, together with a compactness argument. We do not use that construction.

Schauder’s 1930 article is the historical source cited here [4]; the exact compact-image normed-space formulation proved in this note is not attributed verbatim to that article. Cobzaş’s Lemma 2.1 gives continuous normalized finite-net weights and the associated convex approximation map in a locally convex setting [5, Lemma 2.1]. The coefficient-cube extension and compact-image limiting step above are part of the present exposition. The proof is arranged so that the only finite-dimensional fixed-point theorem used is the cube form of Brouwer. No novelty claim is made.

The initial motivation for this note was the fixed-point background in OpenAI’s paper on Tingley’s problem [6]. At the fixed repository revision cited below, Appendix A first proves the simplex case of Brouwer from Sperner’s lemma, then the compact convex Schauder theorem, and then uses that theorem in the proof of Darbo’s condensing-map theorem. The present note gives an operator-based route to the same compact convex fixed-point theorem. It can be read independently of Tingley’s problem; no result about sphere isometries is assumed.

This is an expository note, and no historical priority is claimed for the arrangement.

Attribution and scope

Brouwer’s and Schauder’s fixed-point theorems are classical. The common-eigenvector, ideal-projector, polynomial-approximation, and Schauder-mapping sources used for context are cited above. This note organizes those ingredients into the detailed operator-and-polynomial route given here, without claiming authorship of the classical theorems or historical priority for the arrangement.

Use of ChatGPT

The entire manuscript was drafted and revised using ChatGPT.

References

- [1] H. Derksen, The Fundamental Theorem of Algebra and Linear Algebra, *American Mathematical Monthly* **110** (2003), 620–623. doi:10.1080/00029890.2003.11920001.
- [2] B. Shekhtman, Some tidbits on ideal projectors, commuting matrices and their applications, *Electronic Transactions on Numerical Analysis* **36** (2009), 17–26. Official ETNA article record.
- [3] K. Kuga, Brouwer’s Fixed-Point Theorem: An Alternative Proof, *SIAM Journal on Mathematical Analysis* **5** (1974), 893–897. doi:10.1137/0505085.
- [4] J. Schauder, Der Fixpunktsatz in Funktionalräumen, *Studia Mathematica* **2** (1930), 171–180. doi:10.4064/sm-2-1-171-180.
- [5] S. Cobzas, Fixed point theorems in locally convex spaces—the Schauder mapping method, *Fixed Point Theory and Applications* (2006), Article 57950, 1–13. doi:10.1155/FPTA/2006/57950.
- [6] OpenAI, *A positive solution to Tingley’s problem*, preprint, 23 September 2026. Appendix A, pp. 9–11. Source PDF (fixed repository revision).