

Lipschitz-Free Spaces over Hilbert Spaces

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Abstract

We develop Lipschitz-free spaces from their definition and linearization property to the theorem that the free space over a real Hilbert space contains no linearly isomorphic copy of c_0 . A common principle organizes the proofs: a summable family of scalar Lipschitz tests defines one operator into ℓ_1 , where Schur's property controls weak convergence. We use this principle to prove compact reduction for weakly null sequences and to treat compact convex subsets of Hilbert spaces through explicit square-root approximations, finite Hilbert averages, and multiplicative damping. All estimates in this part are obtained from algebraic identities and finite sums, without using derivatives or integrals. Nearest-point retractions and the Bessaga–Pełczyński theorem then give the result for the whole Hilbert space. The exposition specializes existing arguments to these conclusions and supplies the free-space background for a companion account of Lipschitz homeomorphic, linearly non-isomorphic Banach spaces.

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1. Introduction

A Lipschitz-free space turns Lipschitz maps into bounded linear operators. Starting from a metric space M with a distinguished point 0 , one forms a Banach space $\mathcal{F}(M)$ generated by an isometric image of M . Every Lipschitz map from M into a Banach space that sends 0 to 0 then has a unique bounded linear extension to $\mathcal{F}(M)$, with operator norm equal to the Lipschitz constant. This construction makes methods from Banach space theory available for studying Lipschitz maps. It also raises a structural question: what can the geometry of M tell us about the Banach space $\mathcal{F}(M)$?

We develop the basic tools needed to answer one particular instance of that question. Our final result is that, for every real Hilbert space H ,

$$\mathcal{F}(H) \text{ contains no linearly isomorphic copy of } c_0. \quad (1.1)$$

Here c_0 is the space of real sequences tending to zero, with the supremum norm. A *linearly isomorphic copy* of a Banach space E in a Banach space Y means the range of a bounded linear map $S : E \rightarrow Y$ for which there exists $b > 0$ such that

$$\|Sx\| \geq b\|x\| \quad (x \in E).$$

We also write $E \hookrightarrow Y$ when such a map exists. All spaces in this note are real.

The proof of (1.1) has two stages. We first show that weakly null sequences in a free space can be approximated on one compact set. More precisely, if M is complete and $\mu_n \rightarrow 0$ in $\mathcal{F}(M)$, then for every $\tau > 0$ there exists a compact set $K \subset M$ containing 0 such that

$$\sup_n \text{dist}(\mu_n, \mathcal{F}(K)) < \tau. \quad (1.2)$$

The notation $\mathcal{F}(K) \subset \mathcal{F}(M)$ will be justified by the extension theorem for real-valued Lipschitz functions. The compact set in (1.2) depends on the sequence and on τ , and approximates every term of the sequence simultaneously.

We then prove that $\mathcal{F}(C)$ contains no copy of c_0 when C is a compact convex subset of a Hilbert space. To pass from this result to (1.1), suppose that $S : c_0 \rightarrow \mathcal{F}(H)$ were an embedding. Apply compact reduction to the weakly null sequence (Se_n) , where (e_n) is the usual unit vector sequence of c_0 . The closed convex hull of the resulting compact set is still compact. Its nearest-point retraction induces a projection of free spaces, and the Bessaga–Pełczyński theorem then gives the contradiction. The details of this transfer are proved in Section 6.

1.1 A common method for the two compactness arguments

Both stages use families of real-valued Lipschitz functions to test vectors in a free space. The useful estimate is

$$\sum_{j \geq 1} |f_j(x) - f_j(y)| \leq Ld(x, y) \quad (x, y \in M), \quad (1.3)$$

where each f_j vanishes at 0. It says that $x \mapsto (f_j(x))_{j \geq 1}$ is an ℓ_1 -valued Lipschitz map. After linearization, the same family defines a bounded operator

$$A : \mathcal{F}(M) \rightarrow \ell_1, \quad A\mu = (\langle f_j, \mu \rangle)_{j \geq 1}.$$

Here $\delta_M(x)$ denotes the canonical image of x in $\mathcal{F}(M)$. The pairing is first defined by $\langle f, \delta_M(x) \rangle = f(x)$ and then extended linearly and continuously. Its construction is given in Section 2.1. If $\mu_n \rightharpoonup 0$, Schur's property of ℓ_1 gives $\|A\mu_n\|_1 \rightarrow 0$. Thus a family satisfying (1.3) cannot have pairings bounded away from zero along a weakly null sequence. There is a corresponding uniform tail estimate for weakly Cauchy sequences.

The two main arguments obtain (1.3) in different ways. In compact reduction, the functions are arranged to have disjoint nonzero sets. On a compact convex subset of a Hilbert space, their nonzero sets can overlap. Instead, explicit approximations and Hilbert averages control linear coefficients of their increments, with quadratic errors. Finite products make the coefficient bounds telescope, and subdivision of line segments removes the quadratic errors. This gives the same increment estimate using finite sums. The operator argument is proved once in Section 3, so that the later proofs can concentrate on constructing the appropriate functions.

1.2 Scope and sources

Section 2 develops free-space duality, linearization, and the inclusion of the free space of a subset. Section 3 proves the summable-test principle and the elementary cutoff estimates. Section 4 proves compact reduction for weakly null sequences. Section 5 treats compact convex subsets of Hilbert spaces, and Section 6 passes to the whole Hilbert space.

Our proof of compact reduction is tailored to sequences and follows Aliaga–Noûs–Petitjean–Procházka [1]. The argument for compact convex subsets uses the averaging and damping method of Kochanek–Pernecká [5], whose theorem proves weak sequential completeness in a more general setting. Here we develop the part needed to exclude c_0 . Section 5 gives a self-contained algebraic reconstruction of the approximation and segment estimates: its square-root formulas and finite-subdivision calculations are proved here, not quoted as formulas from that paper. The nearest-point retraction used in the final transfer also appears in the proof of Theorem C of Aliaga–Pernecká–Quero [2]. These are existing results and methods; the purpose of this note is to give a detailed route through the special forms needed here.

The choice of material is motivated by OpenAI's construction of Lipschitz homeomorphic separable Banach spaces that are not linearly isomorphic [8]. A companion exposition will treat that construction using the results proved here. The present note can be read independently: its goal is (1.1), and all arguments leading to that goal are developed below.

We assume standard Banach space theory, including Hahn–Banach, Schur's property of ℓ_1 , and the Bessaga–Pełczyński theorem on unconditional series. The operator consequence of the last theorem is derived explicitly in Section 6.1. We also use basic Hilbert space geometry, finite sums, and finite nets in compact metric spaces. The Lipschitz-free space constructions, the square-root approximation, and the specific compactness arguments are proved in the text.

2. Lipschitz functions and free spaces

We begin with the evaluation of scalar Lipschitz functions. The resulting norm gives both the isometric inclusion of the metric space and the linearization of Banach-valued maps. The extension theorem then identifies the free space of a subset as a closed subspace, which will be needed when we approximate vectors on compact sets.

2.1 From a Lipschitz function to a linear functional

Let $(M, d, 0)$ be a pointed metric space. The point 0 is distinguished, but M need not be a vector space. For $f : M \rightarrow \mathbb{R}$, write

$$\text{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}, \quad \text{Lip}_0(M) = \{f : M \rightarrow \mathbb{R} : f(0) = 0, \text{Lip}(f) < \infty\}.$$

The supremum over no pairs is zero. The norm on $\text{Lip}_0(M)$ is $\text{Lip}(f)$. It is a norm, rather than a seminorm, because a function of Lipschitz constant zero is constant and here must vanish at 0. For every $x \in M$, evaluation is a bounded linear functional because

$$|f(x)| = |f(x) - f(0)| \leq \text{Lip}(f)d(x, 0).$$

We denote this functional by $\delta_M(x)$, so that $\delta_M(x)(f) = f(x)$, and define

$$\mathcal{F}(M) = \overline{\text{span}}\{\delta_M(x) : x \in M\} \subset \text{Lip}_0(M)^*. \quad (2.1)$$

The closure is in the dual norm. Thus $\mathcal{F}(M)$ is a Banach space. In particular, every element can be approximated in norm by a finite sum of evaluations. The norm of such a sum is

$$\left\| \sum_{j=1}^m a_j \delta_M(x_j) \right\| = \sup_{\substack{f(0)=0 \\ \text{Lip}(f) \leq 1}} \left| \sum_{j=1}^m a_j f(x_j) \right|. \quad (2.2)$$

The evaluation map preserves distance.

Since every function in the supremum is 1-Lipschitz,

$$\|\delta_M(x) - \delta_M(y)\| = \sup_{\substack{f(0)=0 \\ \text{Lip}(f) \leq 1}} |f(x) - f(y)| \leq d(x, y).$$

For the reverse inequality, fix y and define $f_y(t) = d(t, y) - d(0, y)$. The reverse triangle inequality gives $|f_y(s) - f_y(t)| = |d(s, y) - d(t, y)| \leq d(s, t)$, and $f_y(0) = 0$. This particular function satisfies $f_y(x) - f_y(y) = d(x, y)$. Hence

$$\delta_M(0) = 0, \quad \|\delta_M(x) - \delta_M(y)\| = d(x, y). \quad (2.3)$$

For a normed-space domain, the right side is $\|x - y\|$.

Linearity takes place in the free-space variable.

For $f \in \text{Lip}_0(M)$ and a finite evaluation sum, set

$$\left\langle f, \sum_j a_j \delta_M(x_j) \right\rangle = \sum_j a_j f(x_j). \quad (2.4)$$

This is the action of the element of $\text{Lip}_0(M)^*$ on f . Formula (2.2) gives

$$\left| \sum_j a_j f(x_j) \right| \leq \text{Lip}(f) \left\| \sum_j a_j \delta_M(x_j) \right\|.$$

It therefore extends continuously to $\mathcal{F}(M)$, with $|\langle f, \mu \rangle| \leq \text{Lip}(f) \|\mu\|$. This does not assert linearity of f on M ; it asserts linearity in $\mu \in \mathcal{F}(M)$.

Conversely, given $\ell \in \mathcal{F}(M)^*$, define $f_\ell(x) = \ell(\delta_M(x))$. Then $f_\ell(0) = 0$, and

$$|f_\ell(x) - f_\ell(y)| \leq \|\ell\| \|\delta_M(x) - \delta_M(y)\| = \|\ell\| d(x, y).$$

On finite evaluation sums, $\ell(\sum a_j \delta_M(x_j)) = \sum a_j f_\ell(x_j)$. Density shows that ℓ is the functional associated with f_ℓ . The two norm inequalities just proved give

$$\mathcal{F}(M)^* = \text{Lip}_0(M) \quad \text{isometrically.} \quad (2.5)$$

These are the standard evaluation and duality descriptions of free spaces; see [4].

For a Lipschitz function not vanishing at 0, our convention is

$$\langle f, \mu \rangle = \langle f - f(0), \mu \rangle, \quad \langle f, \delta_M(x) \rangle = f(x) - f(0).$$

In particular, constants have zero pairing.

2.2 Vector-valued linearization and subsets

Proposition 2.1 (Linearization). *Let E be a real Banach space and $F : M \rightarrow E$ a Lipschitz map with $F(0) = 0$. There is a unique bounded linear operator $\hat{F} : \mathcal{F}(M) \rightarrow E$ satisfying $\hat{F}\delta_M(x) = F(x)$ for every $x \in M$, with $\|\hat{F}\| = \text{Lip}(F)$.*

Proof. The prescribed values force the formula

$$\hat{F} \left(\sum_j a_j \delta_M(x_j) \right) = \sum_j a_j F(x_j).$$

We must first check that two representations of a free-space vector give the same result. If $e^* \in E^*$ and $\|e^*\| \leq 1$, the function $e^* \circ F$ vanishes at zero and has Lipschitz constant at most $\text{Lip}(F)$. Consequently,

$$\begin{aligned} \left\| \sum_j a_j F(x_j) \right\| &= \sup_{\|e^*\| \leq 1} \left| \sum_j a_j e^*(F(x_j)) \right| \\ &\leq \text{Lip}(F) \left\| \sum_j a_j \delta_M(x_j) \right\|. \end{aligned}$$

If the input sum is zero, its proposed image is zero. Subtracting two representations proves well-definedness. The same inequality extends the linear map to the norm closure, since E is complete, and gives $\|\widehat{F}\| \leq \text{Lip}(F)$.

For any $x, y \in M$, the defining formula now gives

$$\begin{aligned} \|F(x) - F(y)\| &= \|\widehat{F}(\delta_M(x) - \delta_M(y))\| \\ &\leq \|\widehat{F}\| \|\delta_M(x) - \delta_M(y)\| = \|\widehat{F}\| d(x, y). \end{aligned}$$

Thus $\text{Lip}(F) \leq \|\widehat{F}\|$. Finally, any two bounded linear operators with the prescribed values agree on the evaluation span and therefore on its closure. $\square$

For a pointed Lipschitz map $r : M \rightarrow N$ between metric spaces, the Banach-valued map to linearize is $\delta_N \circ r : M \rightarrow \mathcal{F}(N)$. Its resulting operator is

$$\widehat{r} : \mathcal{F}(M) \rightarrow \mathcal{F}(N), \quad \widehat{r}\delta_M(x) = \delta_N(r(x)), \quad \|\widehat{r}\| = \text{Lip}(r).$$

The equality of constants follows because δ_N preserves distance. This is the version used later for a nearest-point retraction.

McShane extension and the free space of a subset.

Let $0 \in K \subset M$ and $f \in \text{Lip}_0(K)$, with $\text{Lip}(f) \leq L$. For $L > 0$, the McShane extension [7] is

$$\widetilde{f}(x) = \inf_{z \in K} (f(z) + Ld(x, z)), \quad x \in M. \quad (2.6)$$

The expression is finite. Indeed, $z = 0$ gives the upper bound $Ld(x, 0)$, while, for every $z \in K$,

$$\begin{aligned} f(z) + Ld(x, z) &\geq -Ld(z, 0) + Ld(x, z) \\ &\geq -Ld(x, 0). \end{aligned}$$

For $x \in K$, the Lipschitz inequality gives

$$f(z) + Ld(x, z) \geq f(x) \quad (z \in K),$$

and $z = x$ gives equality. Thus $\widetilde{f}|_K = f$.

For arbitrary $x, y \in M$ and $z \in K$, the triangle inequality gives

$$f(z) + Ld(x, z) \leq f(z) + Ld(y, z) + Ld(x, y).$$

Taking the infimum over z yields

$$\widetilde{f}(x) \leq \widetilde{f}(y) + Ld(x, y).$$

Interchanging x and y , we obtain

$$|\widetilde{f}(x) - \widetilde{f}(y)| \leq Ld(x, y).$$

For $L = 0$ the function is zero and its zero extension suffices.

The inclusion of evaluations from K into $\mathcal{F}(M)$ is isometric. For a finite sum

$$\nu = \sum_j a_j \delta_K(x_j), \quad x_j \in K,$$

restriction of functions gives

$$\|\nu\|_{\mathcal{F}(M)} \leq \|\nu\|_{\mathcal{F}(K)}.$$

Extension by (2.6) gives the opposite inequality:

$$\sup_{\substack{f \in \text{Lip}_0(K) \\ \text{Lip}(f) \leq 1}} \left| \sum_j a_j f(x_j) \right| \leq \sup_{\substack{g \in \text{Lip}_0(M) \\ \text{Lip}(g) \leq 1}} \left| \sum_j a_j g(x_j) \right|.$$

The resulting identification is

$$\mathcal{F}(K) = \overline{\text{span}}\{\delta_M(x) : x \in K\} \subset \mathcal{F}(M).$$

In particular, for a Lipschitz f on M ,

$$|\langle f, \nu \rangle| \leq \text{Lip}(f|_K) \|\nu\| \quad (\nu \in \mathcal{F}(K)). \quad (2.7)$$

Only the restriction to K matters in this pairing.

We shall also use the following characterization:

$$\mu \in \mathcal{F}(K) \iff \langle f, \mu \rangle = 0 \text{ for every } f \in \text{Lip}_0(M) \text{ with } f|_K = 0. \quad (2.8)$$

The forward implication follows on evaluations and then by continuity. For the reverse implication, if μ is outside the closed subspace $\mathcal{F}(K)$, Hahn–Banach gives $\ell \in \mathcal{F}(M)^*$ vanishing on that subspace but not at μ . The corresponding function $f(x) = \ell(\delta_M(x))$ vanishes on K and contradicts the right side.

Two consequences used later.

If M is separable, take a countable dense subset containing 0. By (2.3), rational finite linear combinations of its evaluations are dense in $\mathcal{F}(M)$. Thus $\mathcal{F}(M)$ is separable.

Suppose $f_n \in \text{Lip}_0(M)$, $\text{Lip}(f_n) \leq L$, and $f_n(x) \rightarrow f(x)$ at every point. Taking the limit in $|f_n(x) - f_n(y)| \leq Ld(x, y)$ shows $\text{Lip}(f) \leq L$ and $f(0) = 0$. For $\mu \in \mathcal{F}(M)$ and a finite evaluation sum $\nu = \sum_{j=1}^m a_j \delta_M(x_j)$,

$$\begin{aligned} |\langle f_n - f, \mu \rangle| &\leq |\langle f_n - f, \nu \rangle| + 2L \|\mu - \nu\| \\ &\leq \sum_{j=1}^m |a_j| |f_n(x_j) - f(x_j)| + 2L \|\mu - \nu\|. \end{aligned}$$

Choose ν close to μ first; the finite sum then tends to zero. This proves

$$\langle f_n, \mu \rangle \longrightarrow \langle f, \mu \rangle \quad (\mu \in \mathcal{F}(M)). \quad (2.9)$$

3. Summable Lipschitz tests

To exploit weak convergence we shall combine a family of scalar Lipschitz functions into one map with values in ℓ_1 . We first prove the estimate that makes this possible and its consequences for weakly null and weakly Cauchy sequences. We then construct such families by cutting off functions and separating their nonzero sets.

3.1 One operator for a family of scalar tests

The next lemma gives four equivalent ways to control a family of Lipschitz functions. The point-pair formulation is convenient for construction; the operator formulation is convenient for weak convergence.

Lemma 3.1 (Summable tests). *Let $f_j \in \text{Lip}_0(M)$ for $j \geq 1$, and let $C \geq 0$. The following conditions are equivalent.*

(i) For all $x, y \in M$,

$$\sum_{j \geq 1} |f_j(x) - f_j(y)| \leq Cd(x, y). \quad (3.1)$$

(ii) For every finite $F \subset \mathbb{N}$ and real coefficients $|a_j| \leq 1$,

$$\text{Lip} \left(\sum_{j \in F} a_j f_j \right) \leq C. \quad (3.2)$$

(iii) The formula

$$\mathbf{f} : M \rightarrow \ell_1, \quad \mathbf{f}(x) = (f_j(x))_{j \geq 1} \quad (3.3)$$

defines a map with $\text{Lip}(\mathbf{f}) \leq C$.

(iv) The formula

$$A : \mathcal{F}(M) \rightarrow \ell_1, \quad A\mu = (\langle f_j, \mu \rangle)_{j \geq 1} \quad (3.4)$$

defines a bounded linear operator with $\|A\| \leq C$.

The last two conditions include that the sequences belong to ℓ_1 .

Proof. (i) implies (ii). For every allowed finite sum,

$$\left| \sum_{j \in F} a_j f_j(x) - \sum_{j \in F} a_j f_j(y) \right| \leq \sum_{j \in F} |f_j(x) - f_j(y)| \leq Cd(x, y).$$

(ii) implies (i). For fixed x, y choose $\sigma_j \in \{-1, 1\}$ so that $\sigma_j(f_j(x) - f_j(y)) = |f_j(x) - f_j(y)|$. Then

$$\sum_{j \in F} |f_j(x) - f_j(y)| = \left| \sum_{j \in F} \sigma_j (f_j(x) - f_j(y)) \right| \leq Cd(x, y).$$

Increasing $F = \{1, \dots, N\}$ gives the infinite sum.

(i) is equivalent to (iii). Setting $y = 0$ in (i) gives

$$\sum_j |f_j(x)| \leq Cd(x, 0),$$

so $\mathbf{f}(x) \in \ell_1$. Moreover,

$$\|\mathbf{f}(x) - \mathbf{f}(y)\|_1 = \sum_j |f_j(x) - f_j(y)|.$$

This equality proves both implications.

(iii) implies (iv). Since $\mathbf{f}(0) = 0$, its linearization $\hat{\mathbf{f}}$ has norm at most C . For a finite evaluation sum,

$$\left(\hat{\mathbf{f}} \sum_k a_k \delta_M(x_k) \right)_j = \sum_k a_k f_j(x_k) = \left\langle f_j, \sum_k a_k \delta_M(x_k) \right\rangle.$$

For a fixed j , the j th coordinate functional on ℓ_1 and the functional represented by f_j on $\mathcal{F}(M)$ are continuous. If finite evaluation sums μ_m converge to μ , then

$$(\widehat{\mathbf{f}}\mu)_j = \lim_m (\widehat{\mathbf{f}}\mu_m)_j = \lim_m \langle f_j, \mu_m \rangle = \langle f_j, \mu \rangle.$$

Define $A = \widehat{\mathbf{f}}$. Its values already belong to ℓ_1 by vector-valued linearization, and the last equality proves the coordinate formula in (iv), with $\|A\| \leq C$.

(iv) implies (i). The displayed coordinate formula gives $A(\delta_M(x) - \delta_M(y)) = (f_j(x) - f_j(y))_j$. Consequently,

$$\sum_j |f_j(x) - f_j(y)| = \|A(\delta_M(x) - \delta_M(y))\|_1 \leq C \|\delta_M(x) - \delta_M(y)\| = Cd(x, y). \quad \square$$

We write $x_n \rightharpoonup x$ in a Banach space E when $e^*(x_n) \rightarrow e^*(x)$ for every $e^* \in E^*$. A sequence is *weakly Cauchy* when every scalar sequence $(e^*(x_n))_n$ is Cauchy. By (2.5), in a free space these conditions are tested against all $f \in \text{Lip}_0(M)$. Bounded linear operators preserve weak convergence and weak-Cauchy sequences. More explicitly, for a bounded linear operator $T : E \rightarrow Y$ and $y^* \in Y^*$, $y^*(Tx_n) = (T^*y^*)(x_n)$, where $T^*y^* \in E^*$. Thus the required scalar convergence, or scalar Cauchy property, follows from the corresponding assumption on (x_n) .

Corollary 3.2. *Suppose (f_j) satisfies Lemma 3.1.*

(i) *If $\mu_n \rightharpoonup 0$ in $\mathcal{F}(M)$, then*

$$\sum_j |\langle f_j, \mu_n \rangle| \longrightarrow 0. \quad (3.5)$$

(ii) *If (μ_n) is weakly Cauchy, then*

$$\lim_{J \rightarrow \infty} \sup_n \sum_{j \geq J} |\langle f_j, \mu_n \rangle| = 0. \quad (3.6)$$

In particular, $\sup_n |\langle f_j, \mu_n \rangle| \rightarrow 0$ as $j \rightarrow \infty$.

Proof. By Lemma 3.1, the operator

$$A : \mathcal{F}(M) \longrightarrow \ell_1, \quad A\mu = (\langle f_j, \mu \rangle)_{j \geq 1},$$

is bounded. Put $x_n = A\mu_n$. If $\mu_n \rightharpoonup 0$, then for each $\phi \in (\ell_1)^*$,

$$\phi(x_n) = \phi(A\mu_n) = (A^*\phi)(\mu_n) \longrightarrow 0,$$

since $A^*\phi \in \mathcal{F}(M)^*$. Hence $x_n \rightharpoonup 0$ in ℓ_1 , and Schur's property gives

$$\sum_j |\langle f_j, \mu_n \rangle| = \|x_n\|_1 \longrightarrow 0.$$

Now suppose (μ_n) is weakly Cauchy. We first show that (x_n) is norm Cauchy. Otherwise there are $\eta > 0$ and increasing pairs of indices $n_k < m_k < n_{k+1}$ such that $\|x_{n_k} - x_{m_k}\|_1 \geq \eta$. For every $\phi \in (\ell_1)^*$, the scalar sequence $(\phi(x_n))_n$ is Cauchy, so $\phi(x_{n_k} - x_{m_k}) \rightarrow 0$. Schur's property applied to these weakly null differences gives $\|x_{n_k} - x_{m_k}\|_1 \rightarrow 0$, a contradiction. Completeness yields $x_n \rightarrow x$ in ℓ_1 for some $x = (x_j)_j \in \ell_1$.

Fix $\eta > 0$. Choose n_0 so that

$$\|x_n - x\|_1 < \frac{\eta}{2} \quad (n \geq n_0).$$

Since only finitely many indices $n < n_0$ remain, choose J so that

$$\sum_{j \geq J} |x_j| < \frac{\eta}{2}, \quad \sum_{j \geq J} |(x_n)_j| < \eta \quad (1 \leq n < n_0).$$

For $n \geq n_0$,

$$\sum_{j \geq J} |(x_n)_j| \leq \|x_n - x\|_1 + \sum_{j \geq J} |x_j| < \eta.$$

This proves the uniform tail estimate for all n . If $j \geq J$, then

$$|(x_n)_j| \leq \sum_{k \geq J} |(x_n)_k|,$$

which gives the individual-coordinate estimate. $\square$

Condition (3.2) is the bounded-finite-sums description of a weakly unconditionally Cauchy series in $\text{Lip}_0(M)$. We shall use the concrete estimate rather than this terminology. It is much weaker than $\sum_j \text{Lip}(f_j) < \infty$: the functions may each have a large Lipschitz constant, provided their variations do not add without bound at the same pair of points.

3.2 Cutoffs and functions with disjoint nonzero sets

Lemma 3.3 (Cutoff products). *For $z \in M$ and $r > 0$, define*

$$\chi_{z,r}(x) = \max\{0, \min\{1, 2 - 2d(x, z)/r\}\}.$$

It takes values in $[0, 1]$, equals one when $d(x, z) \leq r/2$, equals zero when $d(x, z) \geq r$, and has Lipschitz constant at most $2/r$. If $f(z) = 0$ and $\text{Lip}(f) \leq L$, then

$$\text{Lip}(\chi_{z,r}f) \leq 3L, \quad \text{Lip}((1 - \chi_{z,r})f) \leq 3L. \quad (3.7)$$

Proof. Let

$$\theta(s) = \max\{0, \min\{1, s\}\} \quad (s \in \mathbb{R}).$$

This scalar truncation is 1-Lipschitz, so

$$|\theta(s) - \theta(t)| \leq |s - t| \quad (s, t \in \mathbb{R}).$$

The cutoff is the composition

$$\chi_{z,r}(x) = \theta\left(2 - \frac{2d(x, z)}{r}\right).$$

Applying the scalar inequality to the two inputs and then using the reverse triangle inequality gives

$$\begin{aligned} |\chi_{z,r}(x) - \chi_{z,r}(y)| &\leq \left| \left(2 - \frac{2d(x, z)}{r}\right) - \left(2 - \frac{2d(y, z)}{r}\right) \right| \\ &= \frac{2}{r} |d(x, z) - d(y, z)| \leq \frac{2}{r} d(x, y). \end{aligned}$$

The stated values follow by substituting $d(x, z) \leq r/2$ and $d(x, z) \geq r$ into its formula.

Let χ denote either $\chi_{z,r}$ or $1 - \chi_{z,r}$. If $\chi(x) = \chi(y)$, then

$$|f(x)\chi(x) - f(y)\chi(y)| = |\chi(x)| |f(x) - f(y)| \leq Ld(x, y).$$

Otherwise at least one endpoint lies in $B(z, r)$: outside that ball the cutoff, and also its complement, has a constant value. Call that endpoint x . The normalization $f(z) = 0$ implies $|f(x)| \leq Lr$. Expanding the product difference in the order that uses this endpoint gives

$$\begin{aligned} f(x)\chi(x) - f(y)\chi(y) &= \chi(y)(f(x) - f(y)) + f(x)(\chi(x) - \chi(y)), \\ |f(x)\chi(x) - f(y)\chi(y)| &\leq Ld(x, y) + Lr\frac{2}{r}d(x, y) = 3Ld(x, y). \end{aligned}$$

No global bound on f has been used, and the constant is independent of r . $\square$

Lemma 3.4 (Disjoint nonzero sets). *Let $u_j : M \rightarrow \mathbb{R}$ have pairwise disjoint nonzero sets and satisfy $\text{Lip}(u_j) \leq L$. Then*

$$\sum_j |u_j(x) - u_j(y)| \leq 2Ld(x, y) \quad (x, y \in M). \quad (3.8)$$

Consequently the normalized functions $u_j - u_j(0)$ satisfy Lemma 3.1 with constant $2L$.

Proof. At each point at most one function is nonzero. If both points have their nonzero value in the same function u_p , then

$$\sum_j |u_j(x) - u_j(y)| = |u_p(x) - u_p(y)| \leq Ld(x, y).$$

If $u_p(x) \neq 0$ and $u_q(y) \neq 0$ with $p \neq q$, disjointness gives $u_p(y) = u_q(x) = 0$. Hence

$$\begin{aligned} \sum_j |u_j(x) - u_j(y)| &= |u_p(x)| + |u_q(y)| \\ &= |u_p(x) - u_p(y)| + |u_q(x) - u_q(y)| \leq 2Ld(x, y). \end{aligned}$$

If all functions vanish at one endpoint, at most one difference remains and is at most $Ld(x, y)$. If they vanish at both endpoints, the sum is zero. These cases exhaust all possibilities. Subtracting $u_j(0)$ does not change any difference in (3.8). $\square$

The nonzero sets are considered *before* subtracting constants. Their closures need not be disjoint or a positive distance apart.

Lemma 3.5 (Multiplication on a free space). *Let $h : M \rightarrow \mathbb{R}$ be bounded and Lipschitz, and suppose $h(x) = 0$ whenever $d(x, 0) > R$. The prescription $S_h\delta_M(x) = h(x)\delta_M(x)$ extends to a bounded linear operator $S_h : \mathcal{F}(M) \rightarrow \mathcal{F}(M)$ satisfying*

$$\|S_h\| \leq \|h\|_\infty + R\text{Lip}(h). \quad (3.9)$$

If $h = 1$ on a set A containing 0, this operator fixes $\mathcal{F}(A)$. Its range lies in $\overline{\text{span}}\{\delta_M(x) : h(x) \neq 0\}$. Adding the base point to this set of evaluation points does not change the span, because $\delta_M(0) = 0$.

Proof. Define $\Phi_h : M \rightarrow \mathcal{F}(M)$ by $\Phi_h(x) = h(x)\delta_M(x)$. It fixes zero because $\delta_M(0) = 0$. If $h(x) = h(y) = 0$, its difference is zero. Otherwise interchange x, y if necessary so that $h(y) \neq 0$ and therefore $d(y, 0) \leq R$. Then

$$\begin{aligned} \Phi_h(x) - \Phi_h(y) &= h(x)(\delta_M(x) - \delta_M(y)) + (h(x) - h(y))\delta_M(y), \\ \|\Phi_h(x) - \Phi_h(y)\| &\leq |h(x)|d(x, y) + |h(x) - h(y)|d(y, 0) \\ &\leq (\|h\|_\infty + R\text{Lip}(h))d(x, y). \end{aligned}$$

Thus Φ_h is Lipschitz and vanishes at zero, which are precisely the assumptions of Proposition 2.1. Define $S_h = \widehat{\Phi_h} : \mathcal{F}(M) \rightarrow \mathcal{F}(M)$. Its values and norm are

$$S_h(\delta_M(x)) = \Phi_h(x) = h(x)\delta_M(x), \quad \|S_h\| = \text{Lip}(\Phi_h) \leq \|h\|_\infty + R \text{Lip}(h).$$

If $x \in A$ and $h(x) = 1$, then $S_h\delta_M(x) = \delta_M(x)$; continuity extends this identity to $\mathcal{F}(A)$. Every image of an evaluation is either zero or a scalar multiple of an evaluation at a point where $h \neq 0$. Finite linear combinations and then norm limits give the range assertion. $\square$

4. Compact reduction for weakly null sequences

Let M be a complete pointed metric space, and suppose $\mu_n \rightharpoonup 0$ in $\mathcal{F}(M)$. We seek a single compact set $K \ni 0$ such that all the μ_n are close to $\mathcal{F}(K)$. The proof proceeds in three stages. First we restrict attention to one bounded set. Next we obtain a finite cover at any fixed positive scale. Finally we combine these covers at successively smaller scales while keeping the accumulated approximation error small.

The first two stages use functions with disjoint nonzero sets and the uniform tail estimate of Corollary 3.2. The third uses the multipliers of Lemma 3.5 to preserve weak convergence at each finite stage. This is the weakly null sequence form of the compact-reduction theorem of [1].

4.1 First prevent escape to infinity

For a closed set $A \subset M$ containing 0, the distance to $\mathcal{F}(A)$ can be measured by Lipschitz functions vanishing on A :

$$\text{dist}(\mu, \mathcal{F}(A)) = \max \{ \langle f, \mu \rangle : f \in \text{Lip}_0(M), \text{Lip}(f) \leq 1, f|_A = 0 \}. \quad (4.1)$$

Indeed, such a function vanishes on $\mathcal{F}(A)$. For every $\nu \in \mathcal{F}(A)$ we therefore have

$$\langle f, \mu \rangle = \langle f, \mu - \nu \rangle \leq |\langle f, \mu - \nu \rangle| \leq \|\mu - \nu\|.$$

Taking the infimum over ν proves that each admissible pairing is at most the distance. If the distance is zero, $f = 0$ attains this value. Otherwise the closed-subspace form of Hahn–Banach gives a functional $\ell \in \mathcal{F}(M)^*$ of norm one, vanishing on $\mathcal{F}(A)$, such that $\ell(\mu) = \text{dist}(\mu, \mathcal{F}(A))$. By free-space duality, $f(x) = \ell(\delta_M(x))$ belongs to $\text{Lip}_0(M)$, has Lipschitz constant one, and vanishes on A . Its pairing with μ attains the displayed distance. This proves both the equality and the maximum assertion. Thus a positive distance from $\mathcal{F}(A)$ can be detected by one Lipschitz function vanishing on A .

Lemma 4.1. *If $\mu_n \rightharpoonup 0$ in $\mathcal{F}(M)$, then for every $\eta > 0$ there exists $R < \infty$ such that*

$$\sup_n \text{dist}(\mu_n, \mathcal{F}(\overline{B}(0, R))) < \eta.$$

Proof. Suppose the assertion fails. Decreasing the failed tolerance, we obtain $\eta > 0$ such that every radius leaves some term at distance greater than η . Start with $R_0 = 1$. Suppose R_{j-1} is fixed. Choose n_j so that

$$\text{dist}(\mu_{n_j}, \mathcal{F}(\overline{B}(0, 2R_{j-1}))) > \eta.$$

Equation (4.1) gives a function f_j with $\text{Lip}(f_j) \leq 1$, zero on that ball, and $\langle f_j, \mu_{n_j} \rangle > \eta$. Choose a finite evaluation sum

$$\nu_j = \sum_{k=1}^{N_j} a_{jk} \delta_M(x_{jk}), \quad \|\mu_{n_j} - \nu_j\| < \eta/24.$$

The free-space dual norm and $\text{Lip}(f_j) \leq 1$ give

$$|\langle f_j, \mu_{n_j} - \nu_j \rangle| \leq \text{Lip}(f_j) \|\mu_{n_j} - \nu_j\| < \frac{\eta}{24}.$$

Subtracting this error from the pairing greater than η , we get

$$\langle f_j, \nu_j \rangle = \langle f_j, \mu_{n_j} \rangle - \langle f_j, \mu_{n_j} - \nu_j \rangle > \eta - \eta/24 > \eta/2.$$

Choose $R_j > 2R_{j-1}$ with $d(x_{jk}, 0) \leq R_j$ for every k . Recall from Lemma 3.3 that

$$\chi_{0,2R_j}(x) = \max\{0, \min\{1, 2 - d(x, 0)/R_j\}\}.$$

It equals one on $\overline{B}(0, R_j)$, vanishes outside $B(0, 2R_j)$, and has Lipschitz constant at most $1/R_j$. Put $u_j = \chi_{0,2R_j} f_j$. Since $f_j(0) = 0$ and $\text{Lip}(f_j) \leq 1$, Lemma 3.3 gives $\text{Lip}(u_j) \leq 3$; also $u_j(0) = 0$. The cutoff equals one at all the evaluation points of ν_j , so $u_j = f_j$ there. Consequently,

$$\begin{aligned} \langle u_j, \mu_{n_j} \rangle &\geq \langle u_j, \nu_j \rangle - 3 \|\mu_{n_j} - \nu_j\| \\ &= \langle f_j, \nu_j \rangle - 3 \|\mu_{n_j} - \nu_j\| > \eta/2 - 3\eta/24 > \eta/4. \end{aligned}$$

The nonzero set of u_j lies in $\{x : 2R_{j-1} < d(x, 0) < 2R_j\}$. These annuli are disjoint as j varies. Lemma 3.4, applied to these functions with $L = 3$, gives

$$\sum_j |u_j(x) - u_j(y)| \leq 6d(x, y) \quad (x, y \in M).$$

Each u_j vanishes at 0, so the last increment estimate is condition (i) of Lemma 3.1 with constant 6. It defines the bounded linear operator

$$A : \mathcal{F}(M) \rightarrow \ell_1, \quad A\mu = (\langle u_j, \mu \rangle)_{j \geq 1}, \quad \|A\| \leq 6.$$

A weakly null sequence is weakly Cauchy. Corollary 3.2(ii) therefore gives, for these particular functions,

$$\sup_n |\langle u_j, \mu_n \rangle| \longrightarrow 0 \quad (j \rightarrow \infty).$$

This contradicts the lower bound at n_j . The indices n_j need not increase, since this conclusion is uniform over all n . $\square$

4.2 Then prevent escape in infinitely many directions

A bounded ball need not be compact. The next lemma says that, at any one positive scale, finitely many balls suffice for uniform approximation of the sequence. Completeness is still not needed at this stage. For a nonempty set A , write

$$d(x, A) = \inf_{a \in A} d(x, a).$$

Taking the infimum over a in $d(x, a) \leq d(x, y) + d(y, a)$, and then interchanging x, y , gives

$$|d(x, A) - d(y, A)| \leq d(x, y). \quad (4.2)$$

In particular, each set $\{x : d(x, A) \leq \delta\}$ is closed.

Lemma 4.2. *Suppose N is a pointed metric space with $d(x, 0) \leq R$ for all $x \in N$, where $R > 0$, and $\mu_n \rightarrow 0$ in $\mathcal{F}(N)$. For every $\eta, \delta > 0$ there is a finite $F \subset N$ containing 0 such that*

$$\sup_n \text{dist}(\mu_n, \mathcal{F}([F]_\delta)) < \eta, \quad [F]_\delta = \{x \in N : d(x, F) \leq \delta\}.$$

Proof. Suppose that the conclusion fails for some $\eta_0, \delta > 0$, and set $\eta = \eta_0/2$. For every finite $F \subset N$ containing 0, the failure of the asserted bound means that

$$\sup_n \text{dist}(\mu_n, \mathcal{F}([F]_\delta)) \geq \eta_0 > \eta.$$

Consequently, for every such F there exists an index n for which $\text{dist}(\mu_n, \mathcal{F}([F]_\delta)) > \eta$. The smaller threshold ensures this strict inequality even when the supremum at η_0 is not attained. We shall construct functions with disjoint nonzero sets that retain a positive pairing with selected terms of the sequence.

Set $L = \max\{1, R/\delta\}$ and $F_0 = \{0\}$. Suppose that the finite set F_{j-1} has been chosen and contains 0 and every evaluation point selected at earlier stages. The failure statement supplies n_j such that

$$\text{dist}(\mu_{n_j}, \mathcal{F}([F_{j-1}]_\delta)) > \eta.$$

The set $[F_{j-1}]_\delta$ is closed by (4.2) and contains 0. Applying the distance formula (4.1) in $\mathcal{F}(N)$ therefore gives $f_j \in \text{Lip}_0(N)$ with

$$\text{Lip}(f_j) \leq 1, \quad f_j|_{[F_{j-1}]_\delta} = 0, \quad \langle f_j, \mu_{n_j} \rangle > \eta.$$

Choose a finite evaluation sum and its set of evaluation points as follows:

$$\nu_j = \sum_{k=1}^{N_j} a_{jk} \delta_N(x_{jk}), \quad \|\mu_{n_j} - \nu_j\| < \frac{\eta}{16L}, \quad A_j = \{0, x_{j1}, \dots, x_{jN_j}\}.$$

By free-space duality, the error in the pairing is at most $\text{Lip}(f_j) \|\mu_{n_j} - \nu_j\|$. Thus

$$\langle f_j, \nu_j \rangle \geq \langle f_j, \mu_{n_j} \rangle - \text{Lip}(f_j) \|\mu_{n_j} - \nu_j\| > \eta - \frac{\eta}{16L} \geq \frac{15\eta}{16} > \frac{\eta}{2}.$$

We first retain a nonnegative part of f_j with a large pairing. For a real number t , write $t^+ = \max\{t, 0\}$ and $t^- = \max\{-t, 0\}$. If $s \leq t$, then $0 \leq t^+ - s^+ \leq t - s$: when s, t have the same sign this follows from the defining formula, and when $s \leq 0 \leq t$ it is the inequality $t \leq t - s$. Replacing s, t by $-t, -s$ gives the corresponding bound for the negative parts. Define the pointwise parts by $f_j^\pm(x) = (f_j(x))^\pm$. Then, for every $x, y \in N$,

$$|f_j^\pm(x) - f_j^\pm(y)| \leq |f_j(x) - f_j(y)| \leq d(x, y).$$

Both f_j^+ and f_j^- therefore have Lipschitz constant at most one and vanish on $[F_{j-1}]_\delta$. Since $f_j = f_j^+ - f_j^-$,

$$\frac{\eta}{2} < \langle f_j, \nu_j \rangle = \langle f_j^+, \nu_j \rangle - \langle f_j^-, \nu_j \rangle \leq |\langle f_j^+, \nu_j \rangle| + |\langle f_j^-, \nu_j \rangle|.$$

One of these functions, denoted by g_j , satisfies $|\langle g_j, \nu_j \rangle| > \eta/4$. The preceding difference estimate gives $\text{Lip}(g_j) \leq 1$. Also $g_j \geq 0$ and $g_j(0) = 0$. Using the hypothesis $d(x, 0) \leq R$ for every $x \in N$, we obtain

$$0 \leq g_j(x) = |g_j(x) - g_j(0)| \leq d(x, 0) \leq R \quad (x \in N).$$

To preserve its values on A_j while restricting its nonzero set, define $u_j : N \rightarrow \mathbb{R}$ by

$$u_j(x) = \max \left\{ 0, \max_{y \in A_j} (g_j(y) - Ld(x, y)) \right\}. \quad (4.3)$$

For $x, x' \in N$ and $y \in A_j$, the triangle inequality gives

$$g_j(y) - Ld(x, y) \leq g_j(y) - Ld(x', y) + Ld(x, x') \leq u_j(x') + Ld(x, x').$$

The zero term in (4.3) is also at most the last quantity. Taking the maximum and then interchanging x, x' gives $|u_j(x) - u_j(x')| \leq Ld(x, x')$. Thus $\text{Lip}(u_j) \leq L$. Furthermore, $L \geq 1$ and $\text{Lip}(g_j) \leq 1$ imply

$$g_j(y) - Ld(x, y) \leq g_j(x) + (1 - L)d(x, y) \leq g_j(x).$$

Together with $g_j \geq 0$, this shows that $0 \leq u_j \leq g_j$. For $x \in A_j$, the term $y = x$ gives the reverse inequality, so $u_j(x) = g_j(x)$ there. In particular, $u_j(0) = 0$, and the defining formula for the pairing gives

$$\langle u_j, \nu_j \rangle = \sum_{k=1}^{N_j} a_{jk} u_j(x_{jk}) = \sum_{k=1}^{N_j} a_{jk} g_j(x_{jk}) = \langle g_j, \nu_j \rangle.$$

The approximation of μ_{n_j} and the bound $\text{Lip}(u_j) \leq L$ now yield

$$\begin{aligned} |\langle u_j, \mu_{n_j} \rangle| &\geq |\langle g_j, \nu_j \rangle| - |\langle u_j, \mu_{n_j} - \nu_j \rangle| \\ &\geq |\langle g_j, \nu_j \rangle| - L \|\mu_{n_j} - \nu_j\| > \frac{\eta}{4} - L \frac{\eta}{16L} = \frac{3\eta}{16}. \end{aligned}$$

Set $F_j = F_{j-1} \cup A_j$ to complete the induction step.

If $u_j(x) > 0$, the finite maximum in (4.3) provides $y \in A_j$ with

$$Ld(x, y) < g_j(y) \leq R, \quad d(x, A_j) \leq d(x, y) < R/L \leq \delta.$$

For $k > j$, the construction gives $A_j \subset F_{k-1}$. Hence such an x belongs to $[F_{k-1}]_\delta$, where $g_k = 0$, and $0 \leq u_k(x) \leq g_k(x) = 0$. The functions u_j consequently have pairwise disjoint nonzero sets. Lemma 3.4, with their common Lipschitz bound L , gives

$$\sum_j |u_j(x) - u_j(y)| \leq 2L d(x, y) \quad (x, y \in N).$$

As each u_j vanishes at zero, Lemma 3.1 applies and defines the bounded linear operator

$$A : \mathcal{F}(N) \rightarrow \ell_1, \quad A\mu = (\langle u_j, \mu \rangle)_{j \geq 1}, \quad \|A\| \leq 2L.$$

The weakly null sequence (μ_n) is weakly Cauchy, so Corollary 3.2(ii) gives

$$\sup_n |\langle u_j, \mu_n \rangle| \longrightarrow 0 \quad (j \rightarrow \infty).$$

This contradicts $|\langle u_j, \mu_{n_j} \rangle| > 3\eta/16$. No increase of the indices n_j is needed, because the convergence is uniform over all n . $\square$

The finite maximum in (4.3) has three separate purposes: it preserves the values used in the selected pairing, keeps one Lipschitz bound, and leaves the previous neighborhoods available as zero sets for later functions. All three are needed for the disjointness argument.

4.3 Constructing one compact set at all scales

Theorem 4.3 (Compact reduction for a sequence). *Let M be complete and let $\mu_n \rightharpoonup 0$ in $\mathcal{F}(M)$. For every $\tau > 0$ there is a compact $K \subset M$ containing 0 with*

$$\sup_n \text{dist}(\mu_n, \mathcal{F}(K)) < \tau. \quad (4.4)$$

Proof. The two preceding lemmas control one radius and one scale. We now combine successively smaller scales without changing choices already made. The approximations are obtained by bounded linear multipliers; thus at each finite step they still form a weakly null sequence.

1. Put all approximations in one bounded set.

Apply Lemma 4.1 with tolerance $\tau/16$, and enlarge the resulting radius if necessary to obtain $R \geq 1$ with

$$\sup_n \text{dist}(\mu_n, \mathcal{F}(\overline{B}(0, R))) < \tau/16.$$

Set $K_0 = \overline{B}(0, 2R)$ and, recalling Lemma 3.3, define

$$h_0(x) = \chi_{0,2R}(x) = \max\{0, \min\{1, 2 - d(x, 0)/R\}\}.$$

This function equals one on $\overline{B}(0, R)$, vanishes when $d(x, 0) \geq 2R$, and has Lipschitz constant at most $1/R$. Lemma 3.5 therefore defines $T_0 = S_{h_0} : \mathcal{F}(M) \rightarrow \mathcal{F}(K_0)$ and gives

$$\|T_0\| \leq \|h_0\|_\infty + 2R \text{Lip}(h_0) \leq 1 + (2R)(1/R) = 3.$$

The same lemma shows that T_0 fixes $\mathcal{F}(\overline{B}(0, R))$. For each n , choose

$$\zeta_n \in \mathcal{F}(\overline{B}(0, R))$$

with $\|\mu_n - \zeta_n\| < \tau/16$. Since $T_0\zeta_n = \zeta_n$,

$$\|T_0\mu_n - \mu_n\| = \|(T_0 - I)(\mu_n - \zeta_n)\| \leq 4\|\mu_n - \zeta_n\| < \tau/4.$$

All these free spaces are identified with their isometric subspaces of $\mathcal{F}(M)$. In particular, the difference $T_0 - I$ in this estimate is taken in $\mathcal{F}(M)$.

2. Keep the sets nested while reducing their scale.

For $m \geq 1$, put $\delta_m = 2^{-m}$ and $e_m = \tau 2^{-m-2}$. Suppose a closed set $K_{m-1} \subset K_0$ containing zero and a bounded linear operator $T_{m-1} : \mathcal{F}(M) \rightarrow \mathcal{F}(K_{m-1})$ have been constructed. The sequence $T_{m-1}\mu_n$ is weakly null in $\mathcal{F}(K_{m-1})$: for every $f \in \text{Lip}_0(K_{m-1})$, the composite $\mu \mapsto \langle f, T_{m-1}\mu \rangle$ is a bounded linear functional on $\mathcal{F}(M)$ and hence tends to zero at μ_n . All points of K_{m-1} are at distance at most $2R$ from 0. Apply Lemma 4.2 on the pointed metric space K_{m-1} , with radius bound $2R$, scale δ_m , and tolerance $e_m/(2 + 2R/\delta_m)$. It gives a finite set $F_m \subset K_{m-1}$ containing 0 and, for every n , a vector $\xi_{m,n} \in \mathcal{F}([F_m]_{\delta_m})$ such that

$$\|T_{m-1}\mu_n - \xi_{m,n}\| < \frac{e_m}{2 + 2R/\delta_m} \quad (n \geq 1).$$

Here the neighborhood is taken inside K_{m-1} . On that same metric space define

$$h_m(x) = \max\{0, \min\{1, 2 - d(x, F_m)/\delta_m\}\}, \quad K_m = \{x \in K_{m-1} : d(x, F_m) \leq 2\delta_m\}.$$

Recall the scalar truncation from Lemma 3.3:

$$\theta(s) = \max\{0, \min\{1, s\}\}, \quad |\theta(s) - \theta(t)| \leq |s - t|.$$

Here the defining formula is

$$h_m(x) = \theta \left(2 - \frac{d(x, F_m)}{\delta_m} \right).$$

For $x, y \in K_{m-1}$, apply the scalar inequality and then (4.2) to obtain

$$\begin{aligned} |h_m(x) - h_m(y)| &\leq \left| \left(2 - \frac{d(x, F_m)}{\delta_m} \right) - \left(2 - \frac{d(y, F_m)}{\delta_m} \right) \right| \\ &= \frac{1}{\delta_m} |d(x, F_m) - d(y, F_m)| \\ &\leq \frac{1}{\delta_m} d(x, y). \end{aligned}$$

Thus $\text{Lip}(h_m) \leq 1/\delta_m$. Every point of K_{m-1} has distance at most $2R$ from the base point. Lemma 3.5 therefore gives

$$\|S_{h_m}\| \leq \|h_m\|_\infty + 2R \text{Lip}(h_m) \leq 1 + \frac{2R}{\delta_m}.$$

On $[F_m]_{\delta_m}$ we have $h_m = 1$, so $S_{h_m}\xi_{m,n} = \xi_{m,n}$. The function is zero whenever $d(x, F_m) \geq 2\delta_m$, so its multiplier has range in $\mathcal{F}(K_m)$. We may therefore regard the multiplier as a bounded linear map $S_{h_m} : \mathcal{F}(K_{m-1}) \rightarrow \mathcal{F}(K_m)$ and define $T_m = S_{h_m} \circ T_{m-1} : \mathcal{F}(M) \rightarrow \mathcal{F}(K_m)$. To compare it with T_{m-1} , we may subtract $\xi_{m,n}$ inside $S_{h_m} - I$ because that vector is fixed. Thus

$$\begin{aligned} \|T_m\mu_n - T_{m-1}\mu_n\| &= \|(S_{h_m} - I)(T_{m-1}\mu_n - \xi_{m,n})\| \\ &\leq (2 + 2R/\delta_m) \|T_{m-1}\mu_n - \xi_{m,n}\| < \left(2 + \frac{2R}{\delta_m} \right) \frac{e_m}{2 + 2R/\delta_m} = e_m. \end{aligned} \quad (4.5)$$

The final substitution explains the denominator in the chosen tolerance. The set

$$K_m = \{x \in K_{m-1} : d(x, F_m) \leq 2\delta_m\}$$

is closed in M : the set K_{m-1} is closed and the function $x \mapsto d(x, F_m)$ is continuous by (4.2). Also $K_m \subset K_{m-1}$, and $0 \in K_m$ because $0 \in F_m$. Since F_m is finite, the distance to F_m is attained, so

$$K_m \subset \bigcup_{y \in F_m} \overline{B}(y, 2\delta_m).$$

This is the required finite cover at scale $2\delta_m$.

For $p > m$, telescoping and (4.5) give

$$\begin{aligned} \sup_n \|T_p\mu_n - T_m\mu_n\| &\leq \sum_{k=m+1}^p \sup_n \|T_k\mu_n - T_{k-1}\mu_n\| \\ &\leq \sum_{k=m+1}^p e_k \leq \tau \sum_{k=m+1}^{\infty} 2^{-k-2} = \tau 2^{-m-2}. \end{aligned}$$

Completeness of $\mathcal{F}(M)$ therefore gives limits ν_n satisfying

$$\sup_n \|\nu_n - T_m\mu_n\| \leq \tau 2^{-m-2}.$$

In particular, the convergence is uniform in n . Since

$$\sum_{m \geq 1} e_m = \frac{\tau}{4},$$

these limits satisfy

$$\sup_n \|\nu_n - \mu_n\| \leq \sup_n \|T_0\mu_n - \mu_n\| + \sum_{m \geq 1} e_m \leq \tau/2 < \tau. \quad (4.6)$$

For fixed m , all $T_p\mu_n$ with $p \geq m$ lie in the closed subspace $\mathcal{F}(K_m)$, so $\nu_n \in \bigcap_m \mathcal{F}(K_m)$.

3. Understand the intersection of the supporting sets.

Set $K = \bigcap_{m \geq 0} K_m$. It contains zero and is closed in the complete space M . For every m it has a finite cover by balls of radius $2\delta_m$, and $\delta_m \rightarrow 0$. Hence it is totally bounded and therefore compact. We also need the stronger fact

$$\sup_{x \in K_m} d(x, K) \longrightarrow 0. \quad (4.7)$$

Suppose instead that $m_j \rightarrow \infty$ and $x_j \in K_{m_j}$ satisfy $d(x_j, K) \geq a > 0$. Given any scale $\rho > 0$, choose m with $2\delta_m < \rho/2$. All sufficiently late x_j lie in K_m , so the range of this tail has a finite cover by balls of radius less than $\rho/2$. Adding a ball for each of the finitely many earlier points shows that the whole sequence has totally bounded range. For each $k \geq 1$, cover this range by finitely many balls of radius 2^{-k-2} . At least one ball contains infinitely many of the indices retained at the preceding step; retain those indices. Choose $j_1 < j_2 < \dots$ with j_k in the infinite set retained at step k . For $\ell > k$, both selected points lie in the ball retained at step k , and hence

$$d(x_{j_k}, x_{j_\ell}) < 2 \cdot 2^{-k-2} < 2^{-k}.$$

Thus (x_{j_k}) is Cauchy. Completeness of M gives a limit $x_{j_k} \rightarrow x \in M$. For each fixed m , eventually $m_{j_k} \geq m$, so $x_{j_k} \in K_{m_{j_k}} \subset K_m$. Closedness of K_m gives $x \in K_m$. This holds for every m , so $x \in K$. But then

$$a \leq d(x_{j_k}, K) \leq d(x_{j_k}, x) \longrightarrow 0,$$

a contradiction.

4. Pass from the intersection of free spaces to $\mathcal{F}(K)$.

This last step is not automatic from set inclusion. Let $\nu \in \bigcap_m \mathcal{F}(K_m)$ and let $f \in \text{Lip}_0(M)$ vanish on K . Put $L = \text{Lip}(f)$. Since $f(y) = 0$ for every $y \in K$,

$$|f(x)| = |f(x) - f(y)| \leq Ld(x, y) \quad (y \in K), \quad |f(x)| \leq Ld(x, K).$$

Equation (4.7) consequently gives

$$b_m := \sup_{x \in K_m} |f(x)| \leq L \sup_{x \in K_m} d(x, K) \longrightarrow 0.$$

Define a scalar function and its composition by

$$\vartheta_b(t) = \begin{cases} t - b, & t > b, \\ 0, & |t| \leq b, \\ t + b, & t < -b, \end{cases} \quad f_m = \vartheta_{b_m} \circ f.$$

The slopes of ϑ_b on its three linear pieces are 1, 0, 1. Splitting an interval at $-b$ and b when necessary shows that $0 \leq \vartheta_b(t) - \vartheta_b(s) \leq t - s$ for $s \leq t$. Its defining formula also gives $\vartheta_b(0) = 0$ and $|\vartheta_b(t) - t| \leq b$. Therefore

$$|f_m(x) - f_m(y)| \leq |f(x) - f(y)| \leq Ld(x, y), \quad |f_m(x) - f(x)| \leq b_m.$$

Thus $\text{Lip}(f_m) \leq L$, $f_m(0) = 0$, and $f_m \rightarrow f$ pointwise. On K_m we have $|f| \leq b_m$, so $f_m = 0$ there. Since $\nu \in \mathcal{F}(K_m)$, this gives $\langle f_m, \nu \rangle = 0$. The hypotheses of the bounded-pointwise pairing result (2.9) are now satisfied, and it yields

$$\langle f, \nu \rangle = \lim_m \langle f_m, \nu \rangle = 0.$$

This holds for every $f \in \text{Lip}_0(M)$ vanishing on K . The annihilator characterization (2.8), with this closed set K , therefore gives $\nu \in \mathcal{F}(K)$. In particular,

$$\nu_n \in \mathcal{F}(K) \quad (n \geq 1).$$

Equation (4.6) proves the required distance bound. $\square$

This is the sequence version required here, not the full assertion of [1, Theorem 2.3]. The intersection argument was proved for the particular nested sets we constructed; no general theorem about supports of free-space elements has been assumed.

5. Free spaces over compact convex subsets of Hilbert spaces

We now prove that $\mathcal{F}(C)$ contains no copy of c_0 when C is a compact convex subset of a real Hilbert space H and $0 \in C$. The result follows from the stronger theorem of Kochanek–Pernecká [5]. The proof below uses the Hilbert averaging and multiplicative damping from that argument, but obtains all estimates from square-root identities and finite sums.

The final contradiction has the same form as in Section 3. A putative copy of c_0 supplies a bounded weakly Cauchy sequence (ν_m) in $\mathcal{F}(C)$. We shall construct functions $g_k \in \text{Lip}_0(C)$ such that

$$\sum_{k \geq 1} |g_k(x) - g_k(y)| \leq 2 \|x - y\| \quad (x, y \in C), \quad \langle g_k, \nu_{m_k} \rangle > \frac{b}{4} \quad (k \geq 1) \quad (5.1)$$

for a fixed $b > 0$ and suitable indices m_k . The first estimate gives an operator into ℓ_1 ; Corollary 3.2(ii) then makes the second estimate impossible.

The preparation has three parts. Explicit approximations give a vector coefficient for each small increment of a function, with a quadratic error. Hilbert averages make these coefficients small at finitely many points. Finally, differences along subdivided line segments telescope; the quadratic errors add up to a quantity tending to zero. No measure on the segments or integration formula is needed.

5.1 Approximation by square-root formulas

The elementary upper approximants $f(a) + \|x - a\|$ suggest the construction. We replace the distance by $\sqrt{\|x - a\|^2 + \delta^2}$, whose increment has an explicit linear term, and then combine finitely many such functions. The next lemma provides every estimate needed later.

Lemma 5.1 (Approximation with an explicit linear term). *Let $C \subset H$ be compact, with $0 \in C$, and let $f : C \rightarrow \mathbb{R}$ satisfy $f(0) = 0$ and $\text{Lip}(f) \leq 1$. For every $\varepsilon > 0$ there are a function $g : H \rightarrow \mathbb{R}$, a map $\Gamma_g : H \rightarrow H$, and finite constants $K_g, M_g \geq 0$ such that*

$$g(0) = 0, \quad \text{Lip}(g) \leq 1, \quad \sup_{x \in C} |g(x) - f(x)| < \varepsilon, \quad (5.2)$$

and, for all $x, y \in H$,

$$\|\Gamma_g(x)\| \leq 1, \quad \|\Gamma_g(y) - \Gamma_g(x)\| \leq K_g \|y - x\|, \quad (5.3)$$

$$|g(y) - g(x) - \langle \Gamma_g(x), y - x \rangle| \leq M_g \|y - x\|^2. \quad (5.4)$$

Thus $\Gamma_g(x)$ is the coefficient of a linear approximation to the increment at x . Both g and Γ_g will be given by finite formulas.

Proof. **1. A single square root.**

Fix $a \in H$ and $\delta > 0$, and define

$$r(x) = \sqrt{\|x - a\|^2 + \delta^2}, \quad \Gamma_r(x) = \frac{x - a}{r(x)}.$$

These expressions are defined everywhere because $r(x) \geq \delta$. In the Hilbert space $H \oplus_2 \mathbb{R}$, the reverse triangle inequality gives

$$|r(y) - r(x)| = \left| \|(y - a, \delta)\| - \|(x - a, \delta)\| \right| \leq \|y - x\|.$$

Also $\|\Gamma_r(x)\| \leq 1$. Subtracting the two quotients gives

$$\begin{aligned} \|\Gamma_r(y) - \Gamma_r(x)\| &= \left\| \frac{y - x}{r(y)} + (x - a) \left(\frac{1}{r(y)} - \frac{1}{r(x)} \right) \right\| \\ &\leq \frac{\|y - x\|}{r(y)} + \frac{\|x - a\|}{r(x)} \frac{|r(x) - r(y)|}{r(y)} \\ &\leq \frac{2}{\delta} \|y - x\|. \end{aligned}$$

To compute the increment, put $h = y - x$. Expanding the two squares gives

$$\begin{aligned} (r(y) - r(x))(r(y) + r(x)) &= r(y)^2 - r(x)^2 \\ &= \|x - a + h\|^2 - \|x - a\|^2 \\ &= 2\langle x - a, h \rangle + \|h\|^2. \end{aligned}$$

Since $x - a = r(x)\Gamma_r(x)$, subtracting the linear term yields

$$\begin{aligned} r(y) - r(x) - \langle \Gamma_r(x), h \rangle &= \frac{2r(x)\langle \Gamma_r(x), h \rangle + \|h\|^2 - (r(y) + r(x))\langle \Gamma_r(x), h \rangle}{r(y) + r(x)} \\ &= \frac{\|h\|^2 - \langle \Gamma_r(x), h \rangle(r(y) - r(x))}{r(y) + r(x)}. \end{aligned} \tag{5.5}$$

Consequently,

$$\begin{aligned} |r(y) - r(x) - \langle \Gamma_r(x), h \rangle| &\leq \frac{\|h\|^2 + \|h\| |r(y) - r(x)|}{2\delta} \\ &\leq \frac{\|h\|^2}{\delta}. \end{aligned} \tag{5.6}$$

Thus this function has the required bounds with $K_r = 2/\delta$ and $M_r = 1/\delta$. Adding a constant to r changes none of these bounds.

2. Replace the minimum of two functions by a square-root formula.

For $\tau > 0$ and $s, t \in \mathbb{R}$, define

$$m_\tau(s, t) = \frac{s + t - \sqrt{(s - t)^2 + \tau^2}}{2}.$$

The formula

$$\min\{s, t\} = \frac{s + t - |s - t|}{2}$$

shows how this operation approximates the minimum:

$$\begin{aligned} m_\tau(s, t) &= \frac{s + t - |s - t|}{2} - \frac{\sqrt{(s - t)^2 + \tau^2} - |s - t|}{2} \\ &= \min\{s, t\} - \frac{\sqrt{(s - t)^2 + \tau^2} - |s - t|}{2}. \end{aligned}$$

Since

$$0 \leq \sqrt{(s-t)^2 + \tau^2} - |s-t| \leq \tau,$$

we obtain

$$\min\{s, t\} - \frac{\tau}{2} \leq m_\tau(s, t) \leq \min\{s, t\}. \quad (5.7)$$

The function m_τ is increasing in each argument. Indeed, for $c \geq 0$ the reverse triangle inequality in $\mathbb{R}^2$ gives

$$-c \leq \sqrt{(s+c-t)^2 + \tau^2} - \sqrt{(s-t)^2 + \tau^2} \leq c,$$

and hence

$$0 \leq m_\tau(s+c, t) - m_\tau(s, t) \leq c.$$

Symmetry gives the same statement for the second argument, and

$$m_\tau(s+c, t+c) = m_\tau(s, t) + c.$$

If $u, v : H \rightarrow \mathbb{R}$ are 1-Lipschitz, then

$$\begin{aligned} m_\tau(u(x), v(x)) &\leq m_\tau(u(y) + \|x-y\|, v(y) + \|x-y\|) \\ &= m_\tau(u(y), v(y)) + \|x-y\|. \end{aligned}$$

Interchanging x and y proves $\text{Lip}(m_\tau(u, v)) \leq 1$.

Suppose that u and v already have coefficient maps and error bounds as in (5.3)–(5.4). Put

$$h = y - x, \quad d(x) = u(x) - v(x), \quad R_\tau(s) = \sqrt{s^2 + \tau^2},$$

and write

$$w(x) = m_\tau(u(x), v(x)) = \frac{u(x) + v(x) - R_\tau(d(x))}{2}.$$

Define

$$\alpha(x) = \frac{1}{2} \left(1 - \frac{d(x)}{R_\tau(d(x))} \right), \quad 1 - \alpha(x) = \frac{1}{2} \left(1 + \frac{d(x)}{R_\tau(d(x))} \right).$$

Then $0 \leq \alpha(x) \leq 1$. Set

$$\Gamma_w(x) = \alpha(x)\Gamma_u(x) + (1 - \alpha(x))\Gamma_v(x). \quad (5.8)$$

This is a convex combination, so $\|\Gamma_w(x)\| \leq 1$.

We next estimate the variation of the weight. Since R_τ is 1-Lipschitz, $R_\tau(s) \geq \tau$, and $|s| \leq R_\tau(s)$,

$$\frac{t}{R_\tau(t)} - \frac{s}{R_\tau(s)} = \frac{t-s}{R_\tau(t)} + \frac{s}{R_\tau(s)} \frac{R_\tau(s) - R_\tau(t)}{R_\tau(t)},$$

whence

$$\begin{aligned} \left| \frac{t}{R_\tau(t)} - \frac{s}{R_\tau(s)} \right| &\leq \frac{|t-s|}{R_\tau(t)} + \frac{|s|}{R_\tau(s)} \frac{|R_\tau(s) - R_\tau(t)|}{R_\tau(t)} \\ &\leq \frac{2}{\tau} |t-s|. \end{aligned}$$

Taking $s = d(x)$ and $t = d(y)$ gives

$$\begin{aligned} |\alpha(y) - \alpha(x)| &= \frac{1}{2} \left| \frac{d(y)}{R_\tau(d(y))} - \frac{d(x)}{R_\tau(d(x))} \right| \\ &\leq \frac{1}{\tau} |d(y) - d(x)| \\ &= \frac{1}{\tau} |(u(y) - u(x)) - (v(y) - v(x))| \\ &\leq \frac{1}{\tau} (|u(y) - u(x)| + |v(y) - v(x)|) \\ &\leq \frac{2}{\tau} \|h\|. \end{aligned}$$

To estimate the coefficient map, expand the two products separately:

$$\begin{aligned}\alpha(y)\Gamma_u(y) - \alpha(x)\Gamma_u(x) &= \alpha(y)(\Gamma_u(y) - \Gamma_u(x)) \\ &\quad + (\alpha(y) - \alpha(x))\Gamma_u(x),\end{aligned}$$

and

$$\begin{aligned}(1 - \alpha(y))\Gamma_v(y) - (1 - \alpha(x))\Gamma_v(x) \\ = (1 - \alpha(y))(\Gamma_v(y) - \Gamma_v(x)) \\ - (\alpha(y) - \alpha(x))\Gamma_v(x).\end{aligned}$$

Adding these identities yields

$$\begin{aligned}\Gamma_w(y) - \Gamma_w(x) &= \alpha(y)(\Gamma_u(y) - \Gamma_u(x)) \\ &\quad + (1 - \alpha(y))(\Gamma_v(y) - \Gamma_v(x)) \\ &\quad + (\alpha(y) - \alpha(x))(\Gamma_u(x) - \Gamma_v(x)).\end{aligned}$$

Therefore,

$$\begin{aligned}\|\Gamma_w(y) - \Gamma_w(x)\| &\leq \alpha(y) \|\Gamma_u(y) - \Gamma_u(x)\| \\ &\quad + (1 - \alpha(y)) \|\Gamma_v(y) - \Gamma_v(x)\| \\ &\quad + |\alpha(y) - \alpha(x)| (\|\Gamma_u(x)\| + \|\Gamma_v(x)\|) \\ &\leq (\alpha(y)K_u + (1 - \alpha(y))K_v) \|h\| + 2|\alpha(y) - \alpha(x)| \\ &\leq \left(\max\{K_u, K_v\} + \frac{4}{\tau} \right) \|h\|.\end{aligned}$$

It remains to estimate the quadratic error. First,

$$\begin{aligned}(u(y) - u(x)) - (v(y) - v(x)) \\ = (u(y) - v(y)) - (u(x) - v(x)) \\ = d(y) - d(x).\end{aligned}$$

Substituting the formulas for $\alpha(x)$ and $1 - \alpha(x)$ gives

$$\begin{aligned}\alpha(x)(u(y) - u(x)) + (1 - \alpha(x))(v(y) - v(x)) \\ = \frac{1}{2} \left(1 - \frac{d(x)}{R_\tau(d(x))} \right) (u(y) - u(x)) \\ + \frac{1}{2} \left(1 + \frac{d(x)}{R_\tau(d(x))} \right) (v(y) - v(x)) \\ = \frac{1}{2} (u(y) - u(x) + v(y) - v(x)) \\ - \frac{d(x)}{2R_\tau(d(x))} ((u(y) - u(x)) - (v(y) - v(x))) \\ = \frac{1}{2} \left[u(y) - u(x) + v(y) - v(x) - \frac{d(x)}{R_\tau(d(x))} (d(y) - d(x)) \right].\end{aligned}$$

On the other hand,

$$w(y) - w(x) = \frac{1}{2} [u(y) - u(x) + v(y) - v(x) - (R_\tau(d(y)) - R_\tau(d(x)))].$$

Subtracting the preceding weighted increment, we obtain

$$\begin{aligned}w(y) - w(x) - \alpha(x)(u(y) - u(x)) - (1 - \alpha(x))(v(y) - v(x)) \\ = -\frac{1}{2} \left[R_\tau(d(y)) - R_\tau(d(x)) - \frac{d(x)}{R_\tau(d(x))} (d(y) - d(x)) \right].\end{aligned}$$

Moreover, by (5.8),

$$\langle \Gamma_w(x), h \rangle = \alpha(x) \langle \Gamma_u(x), h \rangle + (1 - \alpha(x)) \langle \Gamma_v(x), h \rangle.$$

Adding and subtracting the weighted increments now gives

$$\begin{aligned} w(y) - w(x) - \langle \Gamma_w(x), h \rangle &= \alpha(x)(u(y) - u(x) - \langle \Gamma_u(x), h \rangle) \\ &\quad + (1 - \alpha(x))(v(y) - v(x) - \langle \Gamma_v(x), h \rangle) \\ &\quad - \frac{1}{2} \left[R_\tau(d(y)) - R_\tau(d(x)) - \frac{d(x)}{R_\tau(d(x))} (d(y) - d(x)) \right]. \end{aligned}$$

Apply (5.6) on $\mathbb{R}$ with

$$a = 0, \quad \delta = \tau, \quad x = d(x), \quad y = d(y).$$

This gives

$$\left| R_\tau(d(y)) - R_\tau(d(x)) - \frac{d(x)}{R_\tau(d(x))} (d(y) - d(x)) \right| \leq \frac{|d(y) - d(x)|^2}{\tau}.$$

Since u and v are 1-Lipschitz,

$$\begin{aligned} |d(y) - d(x)| &= |(u(y) - u(x)) - (v(y) - v(x))| \\ &\leq |u(y) - u(x)| + |v(y) - v(x)| \\ &\leq 2 \|h\|. \end{aligned}$$

Substituting these estimates into the preceding decomposition gives

$$\begin{aligned} |w(y) - w(x) - \langle \Gamma_w(x), h \rangle| &\leq \alpha(x) M_u \|h\|^2 + (1 - \alpha(x)) M_v \|h\|^2 + \frac{|d(y) - d(x)|^2}{2\tau} \\ &\leq \max\{M_u, M_v\} \|h\|^2 + \frac{(2 \|h\|)^2}{2\tau} \\ &= \left(\max\{M_u, M_v\} + \frac{2}{\tau} \right) \|h\|^2. \end{aligned}$$

Thus the operation can be repeated finitely many times. The constants K_w, M_w may increase, but remain finite; the bounds $\text{Lip}(w) \leq 1$ and $\|\Gamma_w(x)\| \leq 1$ do not increase.

3. Use the values of f on a finite net.

Choose a finite $\varepsilon/16$ -net $\{a_1, \dots, a_N\} \subset C$, including 0, and put

$$u_i(x) = f(a_i) + \sqrt{\|x - a_i\|^2 + (\varepsilon/8)^2} \quad (x \in H, 1 \leq i \leq N).$$

For every $x \in C$ and every i , the Lipschitz bound for f gives

$$f(x) \leq f(a_i) + \|x - a_i\| \leq u_i(x).$$

Choose a_i with $\|x - a_i\| < \varepsilon/16$. Then

$$\begin{aligned} u_i(x) &\leq f(a_i) + \|x - a_i\| + \frac{\varepsilon}{8} \\ &\leq f(x) + 2 \|x - a_i\| + \frac{\varepsilon}{8} < f(x) + \frac{\varepsilon}{4}. \end{aligned}$$

It follows that

$$f(x) \leq \min_{1 \leq i \leq N} u_i(x) < f(x) + \frac{\varepsilon}{4} \quad (x \in C).$$

Set

$$\tau = \frac{\varepsilon}{4 \max\{1, N-1\}}, \quad U_1 = u_1, \quad U_i = m_\tau(U_{i-1}, u_i) \quad (2 \leq i \leq N).$$

Repeated use of (5.7) gives

$$\min_{1 \leq j \leq i} u_j - \frac{(i-1)\tau}{2} \leq U_i \leq \min_{1 \leq j \leq i} u_j.$$

Indeed, these bounds are equalities for $i = 1$. For $i \geq 2$, the lower induction step is

$$\begin{aligned} U_i &\geq \min\{U_{i-1}, u_i\} - \frac{\tau}{2} \\ &\geq \min\left\{\min_{j < i} u_j - \frac{(i-2)\tau}{2}, u_i\right\} - \frac{\tau}{2} \\ &\geq \min_{j \leq i} u_j - \frac{(i-1)\tau}{2}; \end{aligned}$$

the upper step is $U_i \leq \min\{U_{i-1}, u_i\} \leq \min_{j \leq i} u_j$. Since $(N-1)\tau/2 \leq \varepsilon/8$, these estimates give

$$f(x) - \frac{\varepsilon}{8} \leq U_N(x) < f(x) + \frac{\varepsilon}{4} \quad (x \in C).$$

Finally let $g = U_N - U_N(0)$ and $\Gamma_g = \Gamma_{U_N}$. Steps 1 and 2 give the required coefficient and quadratic bounds. A constant shift changes neither increments nor Lipschitz constants, and, using $0 \in C$ and $f(0) = 0$, we obtain

$$\begin{aligned} g(0) &= 0, \\ \sup_C |g - f| &\leq \sup_C |U_N - f| + |U_N(0) - f(0)| \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

This proves the lemma, including the case $N = 1$, when no minimum operation is used. $\square$

The vector coefficient contains first-order approximation information. We have established that information by algebra, rather than by using a differentiability or smoothing theorem. We shall use only the displayed increment and Lipschitz estimates.

Finite linear combinations preserve the estimates.

Suppose $f_1, \dots, f_N$ have these coefficient maps and bounds, and $\sum_i |c_i| \leq 1$. For

$$z = \sum_{i=1}^N c_i f_i, \quad \Gamma_z = \sum_{i=1}^N c_i \Gamma_{f_i}, \quad K = \max_i K_{f_i}, \quad M = \max_i M_{f_i},$$

we have

$$\begin{aligned} |z(y) - z(x)| &\leq \sum_i |c_i| |f_i(y) - f_i(x)| \leq \|y - x\|, \\ \|\Gamma_z(x)\| &\leq \sum_i |c_i| \|\Gamma_{f_i}(x)\| \leq 1, \\ \|\Gamma_z(y) - \Gamma_z(x)\| &\leq \sum_i |c_i| K_{f_i} \|y - x\| \leq K \|y - x\|, \\ |z(y) - z(x) - \langle \Gamma_z(x), y - x \rangle| &\leq \sum_i |c_i| M_{f_i} \|y - x\|^2 \leq M \|y - x\|^2. \end{aligned} \tag{5.9}$$

If all $f_i(0) = 0$, then $z(0) = 0$. In particular, these estimates apply to half the difference of two averages, since the absolute values of its coefficients sum to one.

5.2 Hilbert averages

The next lemma finds two close block averages. Its input may lie in any Hilbert space, including a finite direct sum of copies of H .

Lemma 5.2 (Two ordered blocks with close averages). *Let $v_1, \dots, v_{2^m}$ lie in a Hilbert ball of radius B , where $m \geq 1$. There are two adjacent dyadic intervals $I, J \subset \{1, \dots, 2^m\}$ of equal length, with $\max I < \min J$, such that*

$$\left\| \frac{1}{|I|} \sum_{i \in I} v_i - \frac{1}{|J|} \sum_{i \in J} v_i \right\| \leq \frac{2B}{\sqrt{m}}. \quad (5.10)$$

Proof. Translating the center of the ball to zero does not change differences of averages, so we may assume $\|v_i\| \leq B$. For $0 \leq r \leq m$ and $1 \leq s \leq 2^r$, let

$$I_{r,s} = \{(s-1)2^{m-r} + 1, \dots, s2^{m-r}\}, \quad a_{r,s} = \frac{1}{|I_{r,s}|} \sum_{i \in I_{r,s}} v_i.$$

Thus $I_{r,s}$ is the disjoint union of its two consecutive halves $I_{r+1,2s-1}$ and $I_{r+1,2s}$, and

$$a_{r,s} = \frac{a_{r+1,2s-1} + a_{r+1,2s}}{2}.$$

Each average has norm at most B . Define

$$E_r = \frac{1}{2^r} \sum_{s=1}^{2^r} \|a_{r,s}\|^2.$$

The parallelogram identity, in the form

$$\frac{\|a\|^2 + \|b\|^2}{2} - \left\| \frac{a+b}{2} \right\|^2 = \frac{1}{4} \|a-b\|^2,$$

therefore gives

$$\begin{aligned} E_{r+1} - E_r &= 2^{-r} \sum_{s=1}^{2^r} \left(\frac{\|a_{r+1,2s-1}\|^2 + \|a_{r+1,2s}\|^2}{2} - \|a_{r,s}\|^2 \right) \\ &= \frac{1}{4 \cdot 2^r} \sum_{s=1}^{2^r} \|a_{r+1,2s-1} - a_{r+1,2s}\|^2 \geq 0. \end{aligned}$$

Since $0 \leq E_0$ and $E_m \leq B^2$, telescoping these nonnegative differences gives

$$\sum_{r=0}^{m-1} (E_{r+1} - E_r) = E_m - E_0 \leq B^2.$$

Choose $r \in \{0, \dots, m-1\}$ at which these finitely many increments have their minimum. The minimum is at most the average, so

$$E_{r+1} - E_r \leq \frac{1}{m} \sum_{q=0}^{m-1} (E_{q+1} - E_q) \leq \frac{B^2}{m}.$$

For this r , the formula for $E_{r+1} - E_r$ gives

$$\frac{1}{2^r} \sum_{s=1}^{2^r} \|a_{r+1,2s-1} - a_{r+1,2s}\|^2 \leq \frac{4B^2}{m}.$$

Choose $s \in \{1, \dots, 2^r\}$ attaining the minimum:

$$\|a_{r+1,2s-1} - a_{r+1,2s}\|^2 = \min_{1 \leq q \leq 2^r} \|a_{r+1,2q-1} - a_{r+1,2q}\|^2.$$

Again the minimum is at most the average, and therefore

$$\|a_{r+1,2s-1} - a_{r+1,2s}\|^2 \leq \frac{1}{2^r} \sum_{q=1}^{2^r} \|a_{r+1,2q-1} - a_{r+1,2q}\|^2 \leq \frac{4B^2}{m}.$$

Take $I = I_{r+1,2s-1}$ and $J = I_{r+1,2s}$. Their lengths agree and every index of I precedes every index of J . Taking the square root of the last bound proves (5.10). $\square$

5.3 Finite representations and subdivisions

To estimate a pairing, we first express its vector as a finite sum of endpoint differences. We then subdivide each segment and add the function increments. The following representation has exactly the free-space norm as its total weight.

Lemma 5.3 (A finite representation of minimum total weight). *For every finite evaluation sum $\mu \in \mathcal{F}(C)$ there are finitely many $a_s \in \mathbb{R}$ and $p_s, q_s \in C$, with $p_s \neq q_s$, such that*

$$\mu = \sum_{s=1}^r a_s (\delta_C(p_s) - \delta_C(q_s)), \quad \sum_{s=1}^r |a_s| \|p_s - q_s\| = \|\mu\|.$$

For $\mu = 0$ the empty representation is allowed.

Proof. For $\mu = 0$, use the empty sum. Suppose $\mu \neq 0$, and let $A \subset C$ be a finite set containing 0 and all the evaluation points in one finite-sum representation of μ .

In the finite-dimensional space $\mathcal{F}(A)$, consider

$$\mathcal{M}_A = \left\{ \frac{\delta_A(p) - \delta_A(q)}{\|p - q\|} : p, q \in A, p \neq q \right\}, \quad B = \text{conv } \mathcal{M}_A.$$

The set $\mathcal{M}_A$ is finite. Interchanging p and q changes the sign of a molecule, so B is symmetric. It is therefore a compact balanced convex set. Moreover, for each $p \in A \setminus \{0\}$,

$$\delta_A(p) = \|p\| \frac{\delta_A(p) - \delta_A(0)}{\|p\|},$$

so B spans $\mathcal{F}(A)$. Each molecule has norm one by (2.3); hence

$$B \subset \{v \in \mathcal{F}(A) : \|v\| \leq 1\}.$$

For the reverse inclusion, free-space duality (2.5) identifies a functional with a function $f \in \text{Lip}_0(A)$, and

$$\begin{aligned} \sup_{w \in B} |\langle f, w \rangle| &= \max_{\substack{p, q \in A \\ p \neq q}} \left| \left\langle f, \frac{\delta_A(p) - \delta_A(q)}{\|p - q\|} \right\rangle \right| \\ &= \max_{\substack{p, q \in A \\ p \neq q}} \frac{|f(p) - f(q)|}{\|p - q\|} = \text{Lip}(f). \end{aligned}$$

If $\|v\| \leq 1$ but $v \notin B$, finite-dimensional Hahn–Banach separation gives a functional strictly separating v from B . Symmetry and spanning of B allow us to normalize it so that

$$\sup_{w \in B} |\langle f, w \rangle| = 1, \quad \langle f, v \rangle > 1.$$

The displayed maximum then gives $\text{Lip}(f) = 1$, whereas the free-space norm gives

$$\langle f, v \rangle \leq |\langle f, v \rangle| \leq \text{Lip}(f) \|v\| \leq 1,$$

a contradiction. Thus B is exactly the unit ball of $\mathcal{F}(A)$.

The inclusion $\mathcal{F}(A) \subset \mathcal{F}(C)$ is isometric by the McShane argument in Section 2. In particular, μ has the same norm in the two spaces. Since B is the convex hull of the finite set $\mathcal{M}_A$ and

$$\frac{\mu}{\|\mu\|} \in B,$$

the definition of a convex hull gives an ordinary finite convex combination

$$\frac{\mu}{\|\mu\|} = \sum_{s=1}^r \lambda_s \frac{\delta_C(p_s) - \delta_C(q_s)}{\|p_s - q_s\|}, \quad \lambda_s \geq 0, \quad \sum_{s=1}^r \lambda_s = 1.$$

Put

$$a_s = \frac{\|\mu\| \lambda_s}{\|p_s - q_s\|} \quad (1 \leq s \leq r).$$

Then

$$\begin{aligned} \mu &= \sum_{s=1}^r a_s (\delta_C(p_s) - \delta_C(q_s)), \\ \sum_{s=1}^r |a_s| \|p_s - q_s\| &= \|\mu\| \sum_{s=1}^r \lambda_s = \|\mu\|. \end{aligned} \tag{5.11}$$

This is the required representation. $\square$

For each finite evaluation sum, choose one representation as in Lemma 5.3 and keep it fixed. The finite averages below depend on this choice, not just on the vector μ . Set $D = \text{diam}(C)$ and, for this representation, write

$$w_s = |a_s| \|p_s - q_s\|, \quad \sum_{s=1}^r w_s = \|\mu\|.$$

For an integer $q \geq 1$, subdivide every segment into q equal parts:

$$x_{s,\ell}^{(q)} = q_s + \frac{\ell}{q}(p_s - q_s) \quad (0 \leq \ell \leq q), \quad \|x_{s,\ell+1}^{(q)} - x_{s,\ell}^{(q)}\| = \frac{\|p_s - q_s\|}{q}. \tag{5.12}$$

Convexity puts all these points in C . For a function $\varphi : C \rightarrow \mathbb{R}$, define the finite weighted sum

$$E_{\mu,q}(\varphi) = \sum_{s=1}^r \frac{w_s}{q} \sum_{\ell=0}^{q-1} \varphi(x_{s,\ell}^{(q)}). \tag{5.13}$$

The endpoint $x_{s,q}^{(q)} = p_s$ is not included in this average: each term samples the left endpoint of one small segment. For the empty representation all these sums are zero. The definition immediately gives

$$\begin{aligned} E_{\mu,q}(\varphi + \psi) &= E_{\mu,q}(\varphi) + E_{\mu,q}(\psi), \\ E_{\mu,q}(c\mathbf{1}) &= c \|\mu\| \quad (c \in \mathbb{R}), \\ \varphi \leq \psi &\implies E_{\mu,q}(\varphi) \leq E_{\mu,q}(\psi). \end{aligned} \tag{5.14}$$

For example, the constant-function formula is

$$E_{\mu,q}(c\mathbf{1}) = \sum_s \frac{w_s}{q} \sum_{\ell=0}^{q-1} c = \sum_s \frac{w_s}{q} qc = c \|\mu\|.$$

Finite-sum Cauchy–Schwarz gives one further useful estimate:

$$\begin{aligned} |E_{\mu,q}(\varphi)| &= \left| \sum_{s,\ell} \sqrt{\frac{w_s}{q}} \left(\sqrt{\frac{w_s}{q}} \varphi(x_{s,\ell}^{(q)}) \right) \right| \\ &\leq \left(\sum_{s,\ell} \frac{w_s}{q} \right)^{1/2} \left(\sum_{s,\ell} \frac{w_s}{q} |\varphi(x_{s,\ell}^{(q)})|^2 \right)^{1/2} \\ &= \|\mu\|^{1/2} E_{\mu,q}(|\varphi|^2)^{1/2}. \end{aligned} \tag{5.15}$$

Here and below $\sum_{s,\ell}$ means $\sum_{s=1}^r \sum_{\ell=0}^{q-1}$ for the indicated subdivision.

We need to compare different subdivision sizes, and to estimate a pairing using one such finite sum. Both facts follow by adding finitely many inequalities.

Lemma 5.4 (Two subdivision estimates). *Fix the representation of μ above.*

(i) *If $\varphi : C \rightarrow \mathbb{R}$ is K -Lipschitz, then, for integers $q, q' \geq 1$,*

$$|E_{\mu,q}(\varphi) - E_{\mu,q'}(\varphi)| \leq \|\mu\| KD \left(\frac{1}{q} + \frac{1}{q'} \right). \tag{5.16}$$

(ii) *Suppose $h \in \text{Lip}_0(C)$, $\varphi : C \rightarrow [0, \infty)$, and $M \geq 0$ satisfy*

$$|h(y) - h(x)| \leq \varphi(x) \|y - x\| + M \|y - x\|^2 \quad (x, y \in C).$$

Then, for every integer $q \geq 1$,

$$|\langle h, \mu \rangle| \leq E_{\mu,q}(\varphi) + \frac{MD \|\mu\|}{q}. \tag{5.17}$$

Proof. For (i), first compare q with the common refinement qq' . Writing an index of the finer grid as $\ell q' + j$ gives

$$\begin{aligned} E_{\mu,qq'}(\varphi) &= \sum_s \frac{w_s}{q} \sum_{\ell=0}^{q-1} \frac{1}{q'} \sum_{j=0}^{q'-1} \varphi \left(q_s + \left(\frac{\ell}{q} + \frac{j}{qq'} \right) (p_s - q_s) \right), \\ E_{\mu,q}(\varphi) &= \sum_s \frac{w_s}{q} \sum_{\ell=0}^{q-1} \frac{1}{q'} \sum_{j=0}^{q'-1} \varphi \left(q_s + \frac{\ell}{q} (p_s - q_s) \right). \end{aligned}$$

The corresponding arguments differ by

$$\left\| \frac{j}{qq'} (p_s - q_s) \right\| \leq \frac{\|p_s - q_s\|}{q} \leq \frac{D}{q}.$$

Subtracting the two sums and using the Lipschitz bound for φ yields

$$\begin{aligned} |E_{\mu,qq'}(\varphi) - E_{\mu,q}(\varphi)| &\leq \sum_s \frac{w_s}{q} \sum_{\ell=0}^{q-1} \frac{1}{q'} \sum_{j=0}^{q'-1} \frac{KD}{q} \\ &= \sum_s \frac{w_s}{q} q \frac{1}{q'} q' \frac{KD}{q} = \frac{\|\mu\| KD}{q}. \end{aligned}$$

Interchanging q and q' gives

$$|E_{\mu,qq'}(\varphi) - E_{\mu,q'}(\varphi)| \leq \frac{\|\mu\| KD}{q'}.$$

Combining the two bounds gives

$$\begin{aligned} |E_{\mu,q}(\varphi) - E_{\mu,q'}(\varphi)| &\leq |E_{\mu,q}(\varphi) - E_{\mu,qq'}(\varphi)| + |E_{\mu,qq'}(\varphi) - E_{\mu,q'}(\varphi)| \\ &\leq \frac{\|\mu\| KD}{q} + \frac{\|\mu\| KD}{q'} = \|\mu\| KD \left(\frac{1}{q} + \frac{1}{q'} \right). \end{aligned}$$

For (ii), the endpoint difference is the telescoping sum

$$h(p_s) - h(q_s) = \sum_{\ell=0}^{q-1} (h(x_{s,\ell+1}^{(q)}) - h(x_{s,\ell}^{(q)})).$$

Insert this identity into the pairing and apply the assumed increment estimate to each small segment:

$$\begin{aligned} |\langle h, \mu \rangle| &= \left| \sum_s a_s (h(p_s) - h(q_s)) \right| \\ &\leq \sum_s |a_s| \sum_{\ell=0}^{q-1} |h(x_{s,\ell+1}^{(q)}) - h(x_{s,\ell}^{(q)})| \\ &\leq \sum_s |a_s| \sum_{\ell=0}^{q-1} \left(\varphi(x_{s,\ell}^{(q)}) \frac{\|p_s - q_s\|}{q} + M \frac{\|p_s - q_s\|^2}{q^2} \right) \\ &= E_{\mu,q}(\varphi) + \frac{M}{q} \sum_s w_s \|p_s - q_s\| \\ &\leq E_{\mu,q}(\varphi) + \frac{MD \|\mu\|}{q}. \end{aligned} \quad \square$$

We shall obtain estimates valid for every q beyond a specified threshold. Later, with μ and h fixed, only the explicit error $MD \|\mu\| / q$ will be sent to zero. There is no need to introduce a limit of the finite averages as a new object.

5.4 The triangular data supplied by a copy of c_0

Suppose there is an embedding $S : c_0 \rightarrow \mathcal{F}(C)$. Rescaling S , we may assume that, for some $0 < b \leq 1$,

$$b \|a\|_\infty \leq \|Sa\| \leq \|a\|_\infty \quad (a \in c_0).$$

Define $\nu_m = S(e_1 + \cdots + e_m)$. Then $\|\nu_m\| \leq 1$. For every $f \in \mathcal{F}(C)^*$, the functional S^*f belongs to ℓ_1 , and

$$\langle f, \nu_m \rangle = \sum_{j=1}^m (S^*f)_j.$$

These scalar partial sums converge. Hence (ν_m) is weakly Cauchy.

For each $i \geq 1$, define a functional on the subspace $S(c_0)$ by

$$\ell_i(Sa) = ba_i \quad (a \in c_0).$$

This is well defined because S is injective. The lower bound for S gives

$$|\ell_i(Sa)| = b|a_i| \leq b\|a\|_\infty \leq \|Sa\|,$$

so $\|\ell_i\| \leq 1$. Hahn–Banach extends it to

$$\tilde{\ell}_i \in \mathcal{F}(C)^*, \quad \tilde{\ell}_i|_{S(c_0)} = \ell_i, \quad \|\tilde{\ell}_i\| = \|\ell_i\| \leq 1.$$

Free-space duality (2.5) represents this extended functional by the Lipschitz function

$$f_i(x) = \tilde{\ell}_i(\delta_C(x)) \quad (x \in C), \quad f_i(0) = 0, \quad \text{Lip}(f_i) = \|\tilde{\ell}_i\| \leq 1.$$

The correspondence for a general free-space vector is

$$\langle f_i, \mu \rangle = \tilde{\ell}_i(\mu) \quad (\mu \in \mathcal{F}(C)).$$

In particular, $\nu_j = S(e_1 + \cdots + e_j)$ lies in $S(c_0)$, where the extension agrees with ℓ_i . Thus

$$\begin{aligned} \langle f_i, \nu_j \rangle &= \tilde{\ell}_i(\nu_j) \\ &= \ell_i(S(e_1 + \cdots + e_j)) \\ &= b(e_1 + \cdots + e_j)_i \\ &= \begin{cases} b, & i \leq j, \\ 0, & i > j. \end{cases} \end{aligned} \tag{5.18}$$

Thus earlier functions detect a later partial sum, and later functions vanish on an earlier partial sum. The ordering in this array will allow us to average functions without destroying detection.

Approximate the vectors and the functions.

Put

$$\tau = \frac{b}{64},$$

and choose, for every j , a finite evaluation sum μ_j with $\|\mu_j - \nu_j\| < \tau$. Since $\|\nu_j\| \leq 1$ and $b \leq 1$,

$$\|\mu_j\| \leq \|\nu_j\| + \|\mu_j - \nu_j\| < 1 + \frac{b}{64} \leq 1 + \frac{1}{64} < 2.$$

Fix one representation from Lemma 5.3 for each μ_j . All later expressions $E_{\mu_j, q}$ use that same representation.

For each integer N , apply Lemma 5.1 to $f_1, \dots, f_N$ on C . This is a separate finite construction for each N ; extension of these functions to H is not needed first. Using the same index convention as in Section 4, write

$$\mu_j = \sum_{k=1}^{N_j} a_{jk} \delta_C(x_{jk}).$$

Put

$$M_N = \max \left\{ 1, \max_{1 \leq j \leq N} \sum_{k=1}^{N_j} |a_{jk}| \right\}.$$

Choose the approximations $f_i^{(N)} : H \rightarrow \mathbb{R}$ so that

$$\sup_C |f_i^{(N)} - f_i| < \frac{b}{64M_N}, \quad f_i^{(N)}(0) = 0, \quad \text{Lip}(f_i^{(N)}) \leq 1.$$

Each approximation comes with its fixed map $\Gamma_{f_i^{(N)}}$ and finite constants in (5.3)–(5.4). For $1 \leq i, j \leq N$,

$$\begin{aligned} |\langle f_i^{(N)}, \mu_j \rangle - \langle f_i, \nu_j \rangle| &\leq |\langle f_i^{(N)} - f_i, \mu_j \rangle| + |\langle f_i, \mu_j - \nu_j \rangle| \\ &\leq \sum_{k=1}^{N_j} |a_{jk}| |f_i^{(N)}(x_{jk}) - f_i(x_{jk})| + \text{Lip}(f_i) \|\mu_j - \nu_j\| \\ &< \frac{b}{64M_N} \sum_{k=1}^{N_j} |a_{jk}| + \frac{b}{64} \\ &\leq \frac{b}{64} + \frac{b}{64} = \frac{b}{32} < \frac{b}{8}. \end{aligned}$$

Together with (5.18), this gives

$$\left| \langle f_i^{(N)}, \mu_j \rangle - b \mathbf{1}_{\{i \leq j\}} \right| \leq \frac{b}{8} \quad (1 \leq i, j \leq N). \quad (5.19)$$

For every $N \geq 1$ we therefore have the finite indexed family

$$((f_i^{(N)}, \mu_i))_{i=1}^N.$$

The functions may depend on N , but the vector carrying the index i is always the same μ_i . If $B_N \subset \{1, \dots, N\}$, then the subfamily

$$((f_i^{(N)}, \mu_i))_{i \in B_N}$$

is listed in increasing order of the original index i , and (5.19) continues to hold for all $i, j \in B_N$.

5.5 Selecting averages from finite indexed families

The next lemma chooses one difference of two averages from a finite family. It also identifies infinitely many later finite families and, in each of them, many indices on which the coefficient of the chosen function is small. All indices remain the original indices of the vectors μ_i .

Lemma 5.5 (Selection by finite Hilbert averages). *Let $(\mu_j)_{j \geq 1}$ be finite evaluation sums in $\mathcal{F}(C)$, with $\|\mu_j\| \leq 2$ and fixed representations from Lemma 5.3. Let $\mathcal{M} \subset \mathbb{N}$ be infinite. For each $n \in \mathcal{M}$, suppose that a nonempty set $B_n \subset \{1, \dots, n\}$ and the finite family*

$$((f_i^{(n)}, \mu_i))_{i \in B_n}$$

is given in increasing order of its indices, with

$$|B_n| \longrightarrow \infty \quad (n \rightarrow \infty, n \in \mathcal{M}).$$

Each $f_i^{(n)} : H \rightarrow \mathbb{R}$ vanishes at 0, is 1-Lipschitz, and has a fixed coefficient map satisfying (5.3)–(5.4). Assume that $b > 0$ satisfies

$$\left| \langle f_i^{(n)}, \mu_j \rangle - b \mathbf{1}_{\{i \leq j\}} \right| \leq \frac{b}{8} \quad (n \in \mathcal{M}, i, j \in B_n). \quad (5.20)$$

For every $t \geq 1$ and $\eta > 0$, one can choose $n_0 \in \mathcal{M}$, two nonempty equal-size subsets $I, J \subset B_{n_0}$ with $\max I < \min J$, and a function

$$z = \frac{1}{2} \left(\frac{1}{|I|} \sum_{i \in I} f_i^{(n_0)} - \frac{1}{|J|} \sum_{i \in J} f_i^{(n_0)} \right) \quad (5.21)$$

such that

$$z(0) = 0, \quad \text{Lip}(z) \leq 1, \quad \langle z, \mu_{\max I} \rangle \geq \frac{3b}{8}, \quad t \sup_C |z| \leq \frac{\eta}{2}. \quad (5.22)$$

The map Γ_z is the same linear combination of the input maps as in (5.21); it satisfies (5.3)–(5.4) with finite constants K_z, M_z . There are also an integer $q_0 \geq 1$, an infinite set $\mathcal{N} \subset \{n \in \mathcal{M} : n > n_0\}$, and nonempty subsets $A_n \subset B_n$ such that

$$|A_n| \longrightarrow \infty \quad (n \rightarrow \infty, n \in \mathcal{N}), \quad (5.23)$$

$$E_{\mu_j, q}(\|\Gamma_z\|) \leq \eta \quad (n \in \mathcal{N}, j \in A_n, q \geq q_0). \quad (5.24)$$

Here $\|\Gamma_z\|$ denotes the scalar function $x \mapsto \|\Gamma_z(x)\|$ on C . For each $n \in \mathcal{N}$, the selected indices are the elements of A_n .

Proof. 1. Fix one finite family, then choose a sufficiently fine subdivision.

Let

$$R = \max \left\{ 1, \sup_{x \in C} \|x\| \right\}, \quad D = \text{diam}(C),$$

and choose a nonempty finite $\eta/(4t)$ -net $\mathcal{S} \subset C$. Define B_* and choose $m \geq 1$ with

$$B_*^2 = 2 + t^2 |\mathcal{S}| R^2, \quad \frac{2B_*}{\sqrt{m}} \leq \frac{\eta}{2}.$$

Since $|B_n| \rightarrow \infty$ along $n \in \mathcal{M}$, choose $n_0 \in \mathcal{M}$ with $|B_{n_0}| \geq 2^m$, and list its first 2^m indices as

$$i_1 < i_2 < \cdots < i_{2^m}.$$

Only the functions at these indices will be averaged. Put

$$K = \max_{1 \leq a \leq 2^m} K_{f_{i_a}^{(n_0)}}, \quad M = \max_{1 \leq a \leq 2^m} M_{f_{i_a}^{(n_0)}},$$

and choose an integer $q_0 \geq 1$ such that

$$q_0 \geq \max \left\{ 1, \frac{8KD}{\eta} \right\}. \quad (5.25)$$

The index n_0 and its finite family are fixed before K and q_0 are chosen. No bound on these constants, uniform in n , is required.

Split the ordered indices into consecutive halves as in Lemma 5.2. Explicitly, let

$$I_{r,s} = \{i_{(s-1)2^{m-r}+1}, \dots, i_{s2^{m-r}}\} \quad (0 \leq r \leq m, 1 \leq s \leq 2^r),$$

and collect the pairs of halves in the finite set

$$\mathcal{P}_m = \{(I_{r+1,2s-1}, I_{r+1,2s}) : 0 \leq r < m, 1 \leq s \leq 2^r\}.$$

Every $P = (I, J) \in \mathcal{P}_m$ satisfies $|I| = |J|$ and $\max I < \min J$, and

$$|\mathcal{P}_m| = \sum_{r=0}^{m-1} 2^r = 2^m - 1.$$

For each such pair define $z_{I,J}$ by the right-hand side of (5.21), and define $\Gamma_{z_{I,J}}$ by the same linear combination. The sum of the absolute coefficients is

$$\sum_{i \in I} \frac{1}{2|I|} + \sum_{i \in J} \frac{1}{2|J|} = \frac{1}{2} + \frac{1}{2} = 1.$$

Thus (5.9) gives, for every candidate,

$$\begin{aligned} z_{I,J}(0) &= 0, \quad \text{Lip}(z_{I,J}) \leq 1, \quad \|\Gamma_{z_{I,J}}(x)\| \leq 1, \quad \text{Lip}(\Gamma_{z_{I,J}}) \leq K, \\ |z_{I,J}(y) - z_{I,J}(x) - \langle \Gamma_{z_{I,J}}(x), y - x \rangle| &\leq M \|y - x\|^2. \end{aligned} \quad (5.26)$$

2. Fix $n > n_0$ and $j \in B_n$, and find one usable pair.

Fix $n \in \mathcal{M}$ with $n > n_0$, and fix $j \in B_n$. Use the fixed representation of μ_j from Section 5.3; in this paragraph denote its data by a_s, p_s, q_s, w_s and its subdivision points by $x_{s,\ell}^{(q_0)}$. We encode simultaneously the coefficient values at these subdivision points and the function values on $\mathcal{S}$. For this purpose, form the finite Hilbert direct sum

$$\mathcal{H}_j = \left(\bigoplus_{s=1}^r \bigoplus_{\ell=0}^{q_0-1} H \right)_2 \oplus_2 \ell_2(\mathcal{S}).$$

Its first component has $r q_0$ copies of H ; H itself need not be finite-dimensional. For each of the selected indices i , put

$$V_i^j = \left(\left(\sqrt{\frac{w_s}{q_0}} \Gamma_{f_i^{(n_0)}}(x_{s,\ell}^{(q_0)}) \right)_{s,\ell}, (t f_i^{(n_0)}(p))_{p \in \mathcal{S}} \right).$$

The zero representation causes no difficulty: its first component is the zero space. The bound on the second component follows from

$$|f_i^{(n_0)}(p)| = |f_i^{(n_0)}(p) - f_i^{(n_0)}(0)| \leq \|p\| \leq R.$$

For the whole vector, the definition of the direct-sum norm gives

$$\begin{aligned} \|V_i^j\|_{\mathcal{H}_j}^2 &= \sum_s \frac{w_s}{q_0} \sum_{\ell=0}^{q_0-1} \left\| \Gamma_{f_i^{(n_0)}}(x_{s,\ell}^{(q_0)}) \right\|^2 + t^2 \sum_{p \in \mathcal{S}} |f_i^{(n_0)}(p)|^2 \\ &\leq \sum_s \frac{w_s}{q_0} q_0 + t^2 |\mathcal{S}| R^2 \\ &= \|\mu_j\| + t^2 |\mathcal{S}| R^2 \leq 2 + t^2 |\mathcal{S}| R^2 = B_*^2. \end{aligned}$$

For any candidate pair $P = (I, J)$, the definitions of $z_{I,J}$ and its coefficient map give

$$\begin{aligned} &\frac{1}{|I|} \sum_{i \in I} V_i^j - \frac{1}{|J|} \sum_{i \in J} V_i^j \\ &= \left(\left(2 \sqrt{\frac{w_s}{q_0}} \Gamma_{z_{I,J}}(x_{s,\ell}^{(q_0)}) \right)_{s,\ell}, (2t z_{I,J}(p))_{p \in \mathcal{S}} \right). \end{aligned} \tag{5.27}$$

For example, at a fixed sample point the first component is

$$\sqrt{\frac{w_s}{q_0}} \left(\frac{1}{|I|} \sum_{i \in I} \Gamma_{f_i^{(n_0)}}(x_{s,\ell}^{(q_0)}) - \frac{1}{|J|} \sum_{i \in J} \Gamma_{f_i^{(n_0)}}(x_{s,\ell}^{(q_0)}) \right) = 2 \sqrt{\frac{w_s}{q_0}} \Gamma_{z_{I,J}}(x_{s,\ell}^{(q_0)}).$$

Apply Lemma 5.2 to $V_{i_1}^j, \dots, V_{i_{2m}}^j$ in their listed order. Its two blocks correspond to a pair $(I, J) \in \mathcal{P}_m$, and

$$\left\| \frac{1}{|I|} \sum_{i \in I} V_i^j - \frac{1}{|J|} \sum_{i \in J} V_i^j \right\|_{\mathcal{H}_j} \leq \frac{2B_*}{\sqrt{m}} \leq \frac{\eta}{2}. \tag{5.28}$$

For this fixed n and j , the pair may depend on j .

3. Fix one pair on index sets whose cardinalities tend to infinity.

For $P = (I, J) \in \mathcal{P}_m$ and $n \in \mathcal{M}$, $n > n_0$, define

$$A_P(n) = \left\{ j \in B_n : \left\| \frac{1}{|I|} \sum_{i \in I} V_i^j - \frac{1}{|J|} \sum_{i \in J} V_i^j \right\|_{\mathcal{H}_j} \leq \frac{\eta}{2} \right\}.$$

To see that these sets cover B_n , take any $j \in B_n$. Apply Step 2 to this particular μ_j . It supplies a pair P with

$$\left\| \frac{1}{|I|} \sum_{i \in I} V_i^j - \frac{1}{|J|} \sum_{i \in J} V_i^j \right\|_{\mathcal{H}_j} \leq \frac{\eta}{2} \implies j \in A_P(n) \subset \bigcup_{Q \in \mathcal{P}_m} A_Q(n).$$

Thus $B_n \subset \bigcup_P A_P(n)$. Conversely each $A_P(n) \subset B_n$ by definition. Consequently,

$$\begin{aligned} |B_n| &= \left| \bigcup_{P \in \mathcal{P}_m} A_P(n) \right| \\ &\leq \sum_{P \in \mathcal{P}_m} |A_P(n)| \\ &\leq (2^m - 1) \max_{P \in \mathcal{P}_m} |A_P(n)|. \end{aligned} \tag{5.29}$$

The first inequality allows overlap: a shared index is counted once on the left and more than once in the sum. Choose a maximizing pair P_n , so that

$$|A_{P_n}(n)| = \max_{P \in \mathcal{P}_m} |A_P(n)| \geq \frac{|B_n|}{2^m - 1}.$$

There are finitely many possible pairs. Since

$$\{n \in \mathcal{M} : n > n_0\} = \bigcup_{P \in \mathcal{P}_m} \{n \in \mathcal{M} : n > n_0, P_n = P\}$$

is infinite, at least one set on the right is infinite. Fix its pair $P = (I, J)$, and denote that infinite set by $\mathcal{N}$. For $n \in \mathcal{N}$, set $A_n = A_P(n)$. Then

$$|A_n| = |A_{P_n}(n)| \geq \frac{|B_n|}{2^m - 1} \longrightarrow \infty \quad (n \rightarrow \infty, n \in \mathcal{N}).$$

In particular, $A_n \neq \emptyset$. For each $n \in \mathcal{N}$, the indices in A_n refer to the original functions $f_i^{(n)}$ and the original vectors μ_i .

4. Compute both estimates for the chosen function.

Set $z = z_{I,J}$ for the fixed pair from Step 3. For $n \in \mathcal{N}$ and $j \in A_n = A_P(n)$, the defining inequality and (5.27) give the following calculation:

$$\begin{aligned} \frac{\eta^2}{4} &\geq \left\| \frac{1}{|I|} \sum_{i \in I} V_i^j - \frac{1}{|J|} \sum_{i \in J} V_i^j \right\|_{\mathcal{H}_j}^2 \\ &= \left\| \left(\left(2\sqrt{\frac{w_s}{q_0}} \Gamma_z(x_{s,\ell}^{(q_0)}) \right)_{s,\ell}, (2tz(p))_{p \in \mathcal{S}} \right) \right\|_{\mathcal{H}_j}^2 \\ &= \sum_s \frac{4w_s}{q_0} \sum_{\ell=0}^{q_0-1} \left\| \Gamma_z(x_{s,\ell}^{(q_0)}) \right\|^2 + 4t^2 \sum_{p \in \mathcal{S}} |z(p)|^2 \\ &= 4E_{\mu_j, q_0}(\|\Gamma_z\|^2) + 4t^2 \sum_{p \in \mathcal{S}} |z(p)|^2. \end{aligned} \tag{5.30}$$

Both summands are nonnegative. Dividing by four gives

$$E_{\mu_j, q_0}(\|\Gamma_z\|^2) \leq \frac{\eta^2}{16}, \quad t^2 \sum_{p \in \mathcal{S}} |z(p)|^2 \leq \frac{\eta^2}{16}.$$

Finite-sum Cauchy–Schwarz (5.15) now gives

$$E_{\mu_j, q_0}(\|\Gamma_z\|) \leq \|\mu_j\|^{1/2} E_{\mu_j, q_0}(\|\Gamma_z\|^2)^{1/2} \leq \frac{\sqrt{2}\eta}{4}.$$

This estimate initially holds only at q_0 . To obtain every finer subdivision, first note from (5.26) that

$$|\|\Gamma_z(x)\| - \|\Gamma_z(y)\|| \leq \|\Gamma_z(x) - \Gamma_z(y)\| \leq K \|x - y\|.$$

Therefore (5.16), $\|\mu_j\| \leq 2$, and the choice (5.25) imply, for every $q \geq q_0$,

$$\begin{aligned} E_{\mu_j, q}(\|\Gamma_z\|) &\leq E_{\mu_j, q_0}(\|\Gamma_z\|) + 2KD \left(\frac{1}{q_0} + \frac{1}{q} \right) \\ &\leq \frac{\sqrt{2}\eta}{4} + \frac{4KD}{q_0} \\ &\leq \frac{\sqrt{2}\eta}{4} + \frac{\eta}{2} = \frac{\sqrt{2}+2}{4}\eta < \eta. \end{aligned} \tag{5.31}$$

This proves (5.24) uniformly for all selected indices and all $q \geq q_0$.

To obtain the value estimate, choose any $n \in \mathcal{N}$ and any $j \in A_n$; such a pair exists because $\mathcal{N}$ is infinite and every A_n is nonempty. The second summand of (5.30) gives, for every $p \in \mathcal{S}$,

$$t|z(p)| \leq t \left(\sum_{u \in \mathcal{S}} |z(u)|^2 \right)^{1/2} \leq \frac{\eta}{4}.$$

For any $x \in C$, choose $p \in \mathcal{S}$ with $\|x - p\| < \eta/(4t)$. Using $\text{Lip}(z) \leq 1$ from (5.26), we get

$$\begin{aligned} t|z(x)| &\leq t|z(p)| + t|z(x) - z(p)| \\ &\leq \frac{\eta}{4} + t\|x - p\| < \frac{\eta}{4} + t \frac{\eta}{4t} = \frac{\eta}{2}. \end{aligned}$$

Taking the supremum proves $t \sup_C |z| \leq \eta/2$.

5. Compute the pairing in the selected finite family.

The chosen vector is the original $\mu_{\max I}$. The ordering of the two blocks gives

$$i \in I \implies i \leq \max I, \quad i \in J \implies i > \max I.$$

Apply (5.20) with $n = n_0$ and $j = \max I$:

$$\begin{aligned} i \in I &\implies |\langle f_i^{(n_0)}, \mu_{\max I} \rangle - b| \leq \frac{b}{8} \implies \langle f_i^{(n_0)}, \mu_{\max I} \rangle \geq \frac{7b}{8}, \\ i \in J &\implies |\langle f_i^{(n_0)}, \mu_{\max I} \rangle| \leq \frac{b}{8} \implies \langle f_i^{(n_0)}, \mu_{\max I} \rangle \leq \frac{b}{8}. \end{aligned}$$

Consequently,

$$\begin{aligned} \langle z, \mu_{\max I} \rangle &= \frac{1}{2} \left(\frac{1}{|I|} \sum_{i \in I} \langle f_i^{(n_0)}, \mu_{\max I} \rangle - \frac{1}{|J|} \sum_{i \in J} \langle f_i^{(n_0)}, \mu_{\max I} \rangle \right) \\ &\geq \frac{1}{2} \left(\frac{7b}{8} - \frac{b}{8} \right) = \frac{3b}{8}. \end{aligned}$$

The other asserted properties of z and Γ_z were established in (5.26). $\square$

5.6 Finite products and the final contradiction

Theorem 5.6 (Compact convex subsets of Hilbert spaces). *If C is a compact convex subset of a real Hilbert space and $0 \in C$, then $\mathcal{F}(C)$ contains no linear copy of c_0 .*

Proof. If $C = \{0\}$, then $\mathcal{F}(C) = \{0\}$. Otherwise suppose there is an embedding $S : c_0 \rightarrow \mathcal{F}(C)$, normalized by

$$b \|a\|_\infty \leq \|Sa\| \leq \|a\|_\infty \quad (a \in c_0), \quad 0 < b \leq 1.$$

Section 5.4 gives a weakly Cauchy sequence, its finite approximations, and the finite indexed families of functions:

$$\begin{aligned} \nu_j &= S(e_1 + \cdots + e_j), \quad \|\nu_j\| \leq 1, \quad \|\mu_j - \nu_j\| < \frac{b}{64}, \quad \|\mu_j\| < 2, \\ f_i^{(n)}(0) &= 0, \quad \text{Lip}(f_i^{(n)}) \leq 1, \\ \left| \langle f_i^{(n)}, \mu_j \rangle - b \mathbf{1}_{\{i \leq j\}} \right| &\leq \frac{b}{8} \quad (1 \leq i, j \leq n). \end{aligned} \tag{5.32}$$

Each function has its fixed coefficient map and quadratic-error constant from Lemma 5.1. Every μ_j has one fixed representation for $E_{\mu_j, q}$. Neither the vector nor its representation will change.

The functions we seek will be products

$$g_k = \beta_k z_k, \quad \beta_k(x) = \prod_{\ell < k} (1 - a_\ell(x)), \quad a_\ell(x) = \|\Gamma_{z_\ell}(x)\| \in [0, 1].$$

The reason for this product is the exact identity

$$\beta_k(x) a_k(x) = \beta_k(x) - \beta_{k+1}(x).$$

Thus the coefficients in the increment bounds will telescope. For each k , Lemma 5.5 will choose z_k and an original vector μ_{m_k} so that $\langle z_k, \mu_{m_k} \rangle$ is large, while all earlier coefficient functions $a_1, \dots, a_{k-1}$ have small finite averages on μ_{m_k} .

1. Choose z_k and μ_{m_k} by induction.

Set

$$\eta_k = b 2^{-k-8} \quad (k \geq 1), \quad \sum_{k \geq 1} \eta_k = \frac{b}{256}.$$

We shall choose, for every $k \geq 1$, a function z_k , an original index m_k , and an integer $q_k \geq 1$. Put

$$a_k(x) = \|\Gamma_{z_k}(x)\|.$$

The required conditions are

$$\begin{aligned} z_k(0) &= 0, \quad \text{Lip}(z_k) \leq 1, \quad \|\Gamma_{z_k}(x)\| \leq 1, \\ \langle z_k, \mu_{m_k} \rangle &\geq \frac{3b}{8}, \quad t_k \sup_C |z_k| \leq \frac{\eta_k}{2}, \end{aligned} \tag{5.33}$$

where

$$t_k = \max \left\{ 1, \sum_{\ell < k} K_{z_\ell} \right\},$$

and, for every $\ell < k$,

$$E_{\mu_{m_k}, q}(a_\ell) \leq \eta_\ell \quad (q \geq q_\ell). \tag{5.34}$$

Each z_k also has finite constants K_{z_k}, M_{z_k} satisfying (5.3)–(5.4).

The admissible original indices are defined recursively. Start with

$$\Lambda_0 = \mathbb{N}.$$

After z_k and q_k have been chosen, set

$$\Lambda_k = \Lambda_{k-1} \cap \{i \in \mathbb{N} : E_{\mu_i, q}(a_k) \leq \eta_k \text{ for every integer } q \geq q_k\}. \quad (5.35)$$

The induction will show that every Λ_k is infinite. Thus choosing $m_k \in \Lambda_{k-1}$ automatically gives all estimates in (5.34).

Assume that $z_1, \dots, z_{k-1}$, $m_1, \dots, m_{k-1}$, and $q_1, \dots, q_{k-1}$ have been chosen, and that Λ_{k-1} is infinite. Define

$$\beta_k(x) = \prod_{\ell < k} (1 - a_\ell(x)), \quad t_k = \max \left\{ 1, \sum_{\ell < k} K_{z_\ell} \right\}.$$

For $k = 1$ these are the empty product and empty sum, so $\beta_1 = 1$ and $t_1 = 1$.

For every $\ell < k$,

$$\begin{aligned} |a_\ell(y) - a_\ell(x)| &\leq \|\Gamma_{z_\ell}(y) - \Gamma_{z_\ell}(x)\| \\ &\leq K_{z_\ell} \|y - x\|. \end{aligned}$$

Changing the factors of the product one at a time gives

$$\beta_k(y) - \beta_k(x) = \sum_{\ell < k} (a_\ell(x) - a_\ell(y)) \prod_{r < \ell} (1 - a_r(y)) \prod_{\ell < r < k} (1 - a_r(x)).$$

Since every factor belongs to $[0, 1]$,

$$\begin{aligned} |\beta_k(y) - \beta_k(x)| &\leq \sum_{\ell < k} |a_\ell(x) - a_\ell(y)| \\ &\leq \left(\sum_{\ell < k} K_{z_\ell} \right) \|y - x\| \\ &\leq t_k \|y - x\|. \end{aligned} \quad (5.36)$$

Thus $\text{Lip}(\beta_k) \leq t_k$ before z_k is chosen.

Let

$$n_* = \min \Lambda_{k-1}, \quad B_n = \Lambda_{k-1} \cap \{1, \dots, n\} \quad (n \geq n_*).$$

Every B_n is nonempty. Moreover, $|B_n| \rightarrow \infty$: indeed, given $r \geq 1$, choose

$$i_1 < i_2 < \dots < i_r, \quad i_1, \dots, i_r \in \Lambda_{k-1}.$$

Then $n \geq i_r$ implies $\{i_1, \dots, i_r\} \subset B_n$, and hence $|B_n| \geq r$.

Apply Lemma 5.5 with the explicit data

$$\mathcal{M} = \{n \in \mathbb{N} : n \geq n_*\}, \quad B_n = \Lambda_{k-1} \cap \{1, \dots, n\}, \quad t = t_k, \quad \eta = \eta_k.$$

For each n , use the previously constructed pairs $(f_i^{(n)}, \mu_i)$ with $i \in B_n$. Since $B_n \subset \{1, \dots, n\}$, the input estimate is precisely

$$\left| \langle f_i^{(n)}, \mu_j \rangle - b \mathbf{1}_{\{i \leq j\}} \right| \leq \frac{b}{8} \quad (i, j \in B_n),$$

which follows from (5.32). The first two lines of that display also give

$$\|\mu_i\| < 2, \quad f_i^{(n)}(0) = 0, \quad \text{Lip}(f_i^{(n)}) \leq 1 \quad (i \in B_n),$$

and Lemma 5.1 supplies the fixed coefficient maps and quadratic-error bounds. The representation of every μ_i was fixed in Section 5.4. Thus all hypotheses of Lemma 5.5 have been verified.

The lemma gives n_0 , two equal-size sets $I, J \subset B_{n_0}$ with $\max I < \min J$, an integer $q_k \geq 1$, an infinite set $\mathcal{N}$, and sets $A_n \subset B_n$ for $n \in \mathcal{N}$. Define

$$z_k = \frac{1}{2} \left(\frac{1}{|I|} \sum_{i \in I} f_i^{(n_0)} - \frac{1}{|J|} \sum_{i \in J} f_i^{(n_0)} \right), \quad m_k = \max I.$$

With these choices, (5.22) gives (5.33), and the lemma supplies finite constants K_{z_k}, M_{z_k} . Because

$$m_k = \max I \in I \subset B_{n_0} \subset \Lambda_{k-1},$$

the definition of Λ_{k-1} gives (5.34).

Now define $a_k(x) = \|\Gamma_{z_k}(x)\|$ and define Λ_k by (5.35). The lemma also gives

$$|A_n| \longrightarrow \infty \quad (n \rightarrow \infty, n \in \mathcal{N})$$

and

$$E_{\mu_i, q}(a_k) \leq \eta_k \quad (n \in \mathcal{N}, i \in A_n, q \geq q_k).$$

For such n and i ,

$$i \in A_n \subset B_n \subset \Lambda_{k-1},$$

so the last estimate and (5.35) give $i \in \Lambda_k$. Hence

$$A_n \subset \Lambda_k \quad (n \in \mathcal{N}).$$

If Λ_k were finite, then $|A_n| \leq |\Lambda_k|$ for every $n \in \mathcal{N}$, contradicting $|A_n| \rightarrow \infty$. Therefore Λ_k is infinite, and the induction continues.

2. Estimate the increments of $g_k = \beta_k z_k$.

On C set $g_k = \beta_k z_k$. In particular $g_k(0) = 0$. The product difference is the elementary identity

$$g_k(y) - g_k(x) = \beta_k(x)(z_k(y) - z_k(x)) + z_k(y)(\beta_k(y) - \beta_k(x)). \quad (5.37)$$

Since $0 \leq \beta_k \leq 1$, (5.36) and (5.33) first give the ordinary Lipschitz estimate

$$\begin{aligned} |g_k(y) - g_k(x)| &\leq |z_k(y) - z_k(x)| + \sup_C |z_k| t_k \|y - x\| \\ &\leq \left(1 + \frac{\eta_k}{2}\right) \|y - x\|. \end{aligned} \quad (5.38)$$

Thus $g_k \in \text{Lip}_0(C)$ and, since $\eta_k = b 2^{-k-8} \leq b/512$,

$$\text{Lip}(g_k) \leq 1 + \frac{\eta_k}{2} \leq 1 + \frac{b}{1024} < 2.$$

For the sum over k , use the more precise quadratic estimate for z_k . Subtracting its linear term from (5.37) gives

$$\begin{aligned} g_k(y) - g_k(x) - \beta_k(x) \langle \Gamma_{z_k}(x), y - x \rangle \\ = \beta_k(x)(z_k(y) - z_k(x) - \langle \Gamma_{z_k}(x), y - x \rangle) \\ + z_k(y)(\beta_k(y) - \beta_k(x)). \end{aligned}$$

By (5.4), (5.36), and (5.33), its absolute value is at most

$$M_{z_k} \|y - x\|^2 + \frac{\eta_k}{2} \|y - x\|.$$

Using $a_k(x) = \|\Gamma_{z_k}(x)\|$ in the remaining inner product therefore yields

$$|g_k(y) - g_k(x)| \leq \left(\beta_k(x) a_k(x) + \frac{\eta_k}{2} \right) \|y - x\| + M_{z_k} \|y - x\|^2. \quad (5.39)$$

3. Telescope the coefficients and then subdivide one segment.

From the definitions of the finite products,

$$\beta_{k+1}(x) = \beta_k(x)(1 - a_k(x)), \quad \beta_k(x)a_k(x) = \beta_k(x) - \beta_{k+1}(x).$$

Hence, for every finite N ,

$$\begin{aligned} \sum_{k=1}^N \beta_k(x) a_k(x) &= \sum_{k=1}^N (\beta_k(x) - \beta_{k+1}(x)) \\ &= \beta_1(x) - \beta_{N+1}(x) = 1 - \beta_{N+1}(x) \leq 1. \end{aligned} \quad (5.40)$$

Fix this N and $x, y \in C$. For any integer $q \geq 1$, let

$$x_\ell = x + \frac{\ell}{q}(y - x) \quad (0 \leq \ell \leq q).$$

These points lie in C , and

$$\|x_{\ell+1} - x_\ell\| = \frac{\|y - x\|}{q}.$$

Apply (5.39) at each pair of consecutive points, then use (5.40):

$$\begin{aligned} \sum_{k=1}^N |g_k(y) - g_k(x)| &= \sum_{k=1}^N \left| \sum_{\ell=0}^{q-1} (g_k(x_{\ell+1}) - g_k(x_\ell)) \right| \\ &\leq \sum_{\ell=0}^{q-1} \sum_{k=1}^N |g_k(x_{\ell+1}) - g_k(x_\ell)| \\ &\leq \sum_{\ell=0}^{q-1} \left(\sum_{k=1}^N \beta_k(x_\ell) a_k(x_\ell) + \frac{1}{2} \sum_{k=1}^N \eta_k \right) \frac{\|y - x\|}{q} \\ &\quad + \sum_{\ell=0}^{q-1} \sum_{k=1}^N M_{z_k} \frac{\|y - x\|^2}{q^2} \\ &\leq \left(1 + \frac{b}{512} \right) \|y - x\| + \frac{\|y - x\|^2}{q} \sum_{k=1}^N M_{z_k}. \end{aligned} \quad (5.41)$$

Keep N fixed and let $q \rightarrow \infty$. The sum of the M_{z_k} is finite, so the last term tends to zero. Thus

$$\sum_{k=1}^N |g_k(y) - g_k(x)| \leq \left(1 + \frac{b}{512} \right) \|y - x\|.$$

Now let $N \rightarrow \infty$. The nonnegative partial sums increase, giving

$$\sum_{k \geq 1} |g_k(y) - g_k(x)| \leq \left(1 + \frac{b}{512} \right) \|y - x\| \leq 2 \|y - x\|. \quad (5.42)$$

There was no bound on $\sum_{k \geq 1} M_{z_k}$, nor was one needed: the subdivision limit was taken first for each finite set of functions.

4. The large pairings survive multiplication.

The same telescoping identity, now stopped at $k - 1$, gives

$$1 - \beta_k(x) = \sum_{\ell < k} \beta_\ell(x) a_\ell(x) \leq \sum_{\ell < k} a_\ell(x). \quad (5.43)$$

Here even the quadratic estimate is unnecessary. Since z_k is 1-Lipschitz, the exact product difference gives

$$\begin{aligned} & (g_k - z_k)(y) - (g_k - z_k)(x) \\ &= (\beta_k(x) - 1)(z_k(y) - z_k(x)) + z_k(y)(\beta_k(y) - \beta_k(x)), \\ & |(g_k - z_k)(y) - (g_k - z_k)(x)| \\ &\leq (1 - \beta_k(x)) \|y - x\| + \sup_C |z_k| t_k \|y - x\| \\ &\leq \left(\sum_{\ell < k} a_\ell(x) + \frac{\eta_k}{2} \right) \|y - x\|. \end{aligned}$$

The function $h = g_k - z_k$ vanishes at zero and is Lipschitz. The preceding estimate has the form required by Lemma 5.4(ii), with

$$\varphi(x) = \sum_{\ell < k} a_\ell(x) + \frac{\eta_k}{2}, \quad M = 0, \quad \mu = \mu_{m_k}.$$

Choose one integer

$$q \geq \max\{1, q_1, \dots, q_{k-1}\};$$

for $k = 1$, this means only $q \geq 1$. The lemma gives

$$\begin{aligned} |\langle g_k - z_k, \mu_{m_k} \rangle| &\leq E_{\mu_{m_k}, q} \left(\sum_{\ell < k} a_\ell + \frac{\eta_k}{2} \mathbf{1}_C \right) + \frac{0 \cdot D \|\mu_{m_k}\|}{q} \\ &= E_{\mu_{m_k}, q} \left(\sum_{\ell < k} a_\ell + \frac{\eta_k}{2} \mathbf{1}_C \right). \end{aligned}$$

To evaluate this finite sum, write the fixed representation of μ_{m_k} as

$$\mu_{m_k} = \sum_{s=1}^r c_s (\delta_C(p_s) - \delta_C(q_s)), \quad w_s = |c_s| \|p_s - q_s\|, \quad \sum_{s=1}^r w_s = \|\mu_{m_k}\|.$$

Then

$$\begin{aligned}
& E_{\mu_{m_k}, q} \left(\sum_{\ell < k} a_\ell + \frac{\eta_k}{2} \mathbf{1}_C \right) \\
&= \sum_{s=1}^r \frac{w_s}{q} \sum_{j=0}^{q-1} \left(\sum_{\ell < k} a_\ell(x_{s,j}^{(q)}) + \frac{\eta_k}{2} \right) \\
&= \sum_{\ell < k} E_{\mu_{m_k}, q}(a_\ell) + \frac{\eta_k}{2} \sum_{s=1}^r \frac{w_s}{q} \sum_{j=0}^{q-1} 1 \\
&= \sum_{\ell < k} E_{\mu_{m_k}, q}(a_\ell) + \frac{\eta_k}{2} \sum_{s=1}^r w_s \\
&= \sum_{\ell < k} E_{\mu_{m_k}, q}(a_\ell) + \frac{\eta_k}{2} \|\mu_{m_k}\| \\
&\leq \sum_{\ell < k} \eta_\ell + \frac{\eta_k}{2} \cdot 2 \\
&= \sum_{\ell=1}^k \eta_\ell \leq \sum_{\ell=1}^{\infty} \frac{b}{2^{\ell+8}} = \frac{b}{256}.
\end{aligned} \tag{5.44}$$

The inequality before the last line uses $m_k \in \Lambda_{k-1}$, so that

$$E_{\mu_{m_k}, q}(a_\ell) \leq \eta_\ell \quad (\ell < k, \quad q \geq q_\ell),$$

and also $\|\mu_{m_k}\| < 2$. This estimate is obtained at a finite subdivision; no further limit is used here. The remaining approximation error, by (5.38) and (5.32), is

$$\begin{aligned}
|\langle g_k, \nu_{m_k} - \mu_{m_k} \rangle| &\leq \text{Lip}(g_k) \|\nu_{m_k} - \mu_{m_k}\| \\
&< 2 \cdot \frac{b}{64} = \frac{b}{32}.
\end{aligned}$$

Together with the selected pairing in (5.33), this yields

$$\begin{aligned}
\langle g_k, \nu_{m_k} \rangle &\geq \langle z_k, \mu_{m_k} \rangle - |\langle g_k - z_k, \mu_{m_k} \rangle| - |\langle g_k, \nu_{m_k} - \mu_{m_k} \rangle| \\
&> \frac{3b}{8} - \frac{b}{256} - \frac{b}{32} = \frac{(96 - 1 - 8)b}{256} = \frac{87b}{256} > \frac{b}{4}.
\end{aligned}$$

On the other hand, (5.42) and Corollary 3.2(ii), applied to the weakly Cauchy sequence (ν_m) , give

$$\sup_m |\langle g_k, \nu_m \rangle| \longrightarrow 0.$$

This contradicts

$$\sup_m |\langle g_k, \nu_m \rangle| \geq |\langle g_k, \nu_{m_k} \rangle| > \frac{b}{4} \quad (k \geq 1). \quad \square$$

The proof used the explicit linear coefficients only in the finite Hilbert averages and in the quadratic increment bounds. The final test functions $g_k = \beta_k z_k$ were estimated directly from product differences; no differentiability of these products was needed.

6. Free spaces over Hilbert spaces

Compact reduction now allows us to use the preceding theorem for an arbitrary real Hilbert space. A hypothetical embedding of c_0 will give a bounded operator into the free space of a compact convex set. We first derive the consequence of Bessaga–Pełczyński that makes this operator compact, and then show why compactness is impossible.

6.1 Bessaga–Pełczyński and operators on c_0

A series $\sum_n x_n$ in a Banach space E is weakly unconditionally Cauchy if

$$\sum_n |e^*(x_n)| < \infty \quad (e^* \in E^*).$$

This is one of the standard equivalent formulations; see [6, Proposition 4.3.9]. We use the following theorem of Bessaga and Pełczyński [3]; see also [6, Theorem 4.3.12]: if E contains no linear copy of c_0 , every weakly unconditionally Cauchy series in E converges unconditionally in norm.

Here is its operator consequence, including the passage from a series to an operator-norm estimate:

$$c_0 \not\hookrightarrow E \implies \text{every bounded linear } T : c_0 \rightarrow E \text{ is compact.} \quad (6.1)$$

The notation $\not\hookrightarrow$ excludes a bounded linear embedding with a positive lower bound.

To see (6.1), let E contain no linear copy of c_0 , and let $T : c_0 \rightarrow E$ be bounded and linear. Let (e_n) be the standard unit vectors of c_0 . Since $T^*e^* \in c_0^* = \ell_1$,

$$\sum_n |e^*(Te_n)| = \|T^*e^*\|_1 < \infty.$$

Thus the theorem gives unconditional convergence of $\sum_n Te_n$. For $\eta > 0$, choose N so that every finite sum over indices larger than N has norm less than $\eta/2$. Splitting a signed sum into its positive and negative parts gives

$$\left\| \sum_{n \in F} \sigma_n Te_n \right\| < \eta \quad (F \subset \{N+1, N+2, \dots\} \text{ finite, } \sigma_n = \pm 1).$$

For real coefficients $|a_n| \leq 1$, write their vector as a convex combination of sign vectors:

$$(a_n)_{n \in F} = \sum_{\sigma \in \{-1, 1\}^F} t_\sigma \sigma, \quad t_\sigma \geq 0, \quad \sum_\sigma t_\sigma = 1.$$

This is possible because $[-1, 1]^F$ is the convex hull of its vertices. Substituting this identity into the finite linear combination yields

$$\left\| \sum_{n \in F} a_n Te_n \right\| = \left\| \sum_\sigma t_\sigma \sum_{n \in F} \sigma_n Te_n \right\| \leq \sum_\sigma t_\sigma \left\| \sum_{n \in F} \sigma_n Te_n \right\| < \eta.$$

Let $R_N : c_0 \rightarrow c_0$ retain its first N coordinates. For $a = (a_n) \in B_{c_0}$, the finite tails converge in c_0 . Continuity of T and the preceding estimate give

$$T(I - R_N)a = \lim_{m \rightarrow \infty} \sum_{n=N+1}^m a_n Te_n, \quad \|T(I - R_N)a\| \leq \eta.$$

The bound holds for every a in the unit ball. Hence

$$\|T - TR_N\| = \|T(I - R_N)\| \longrightarrow 0.$$

Each TR_N has finite rank, proving (6.1).

6.2 The transfer to the whole Hilbert space

Theorem 6.1. *For every real Hilbert space H , the free space $\mathcal{F}(H)$ contains no linear copy of c_0 .*

Proof. Suppose that $S : c_0 \rightarrow \mathcal{F}(H)$ is a bounded linear embedding. Choose $b > 0$ such that

$$\|Sa\| \geq b \|a\|_\infty \quad (a \in c_0).$$

For every $\ell \in \mathcal{F}(H)^*$, the sequence $S^*\ell$ belongs to $c_0^* = \ell_1$, and hence

$$\ell(Se_n) = (S^*\ell)_n \rightarrow 0.$$

Thus $Se_n \rightarrow 0$. Apply Theorem 4.3 with $M = H$ and $\tau = b/4$ to obtain a compact set $K \ni 0$ such that

$$\sup_n \text{dist}(Se_n, \mathcal{F}(K)) < b/4.$$

We will project onto the free space of a compact convex set containing K , while keeping the norms of these projected vectors bounded away from zero.

1. The closed convex hull is still compact.

Let $C = \overline{\text{conv}} K$. We will construct finite nets at every scale to show that C is totally bounded. For $\delta > 0$ choose a finite δ -net $F \subset K$. Given a convex combination

$$\sum_j a_j x_j, \quad x_j \in K, \quad a_j \geq 0, \quad \sum_j a_j = 1,$$

choose $y_j \in F$ with $\|x_j - y_j\| < \delta$. Then

$$\left\| \sum_j a_j x_j - \sum_j a_j y_j \right\| \leq \sum_j a_j \|x_j - y_j\| < \delta.$$

The convex hull of finite F is the continuous image of a finite-dimensional compact simplex, so it is compact. Choose a finite δ -net G in $\text{conv } F$. The preceding approximation and this net give

$$d(x, G) \leq d(x, \text{conv } F) + \delta < 2\delta \quad (x \in \text{conv } K).$$

By continuity of the distance to G , $d(x, G) \leq 2\delta < 3\delta$ for $x \in C = \overline{\text{conv}} K$. Thus G is a finite 3δ -net for C . Thus C is totally bounded and closed in complete H , hence compact.

2. The nearest-point projection is nonexpansive.

Let $r_C(x)$ be the nearest point in C to x . Existence follows from compactness. To see uniqueness, suppose u, v both minimize the distance to x , with value d . Their midpoint lies in C , and

$$\left\| x - \frac{u+v}{2} \right\|^2 = \frac{\|x-u\|^2 + \|x-v\|^2}{2} - \frac{\|u-v\|^2}{4} = d^2 - \frac{\|u-v\|^2}{4}.$$

Minimality of d forces $u = v$.

Let $x \in H$, $w \in C$, and $0 < t \leq 1$. Convexity gives

$$r_C(x) + t(w - r_C(x)) \in C.$$

The nearest-point property therefore yields

$$\begin{aligned} 0 &\leq \|x - r_C(x) - t(w - r_C(x))\|^2 - \|x - r_C(x)\|^2 \\ &= -2t \langle x - r_C(x), w - r_C(x) \rangle + t^2 \|w - r_C(x)\|^2. \end{aligned}$$

Dividing by $2t > 0$ gives

$$\langle x - r_C(x), w - r_C(x) \rangle \leq \frac{t}{2} \|w - r_C(x)\|^2.$$

Letting $t \downarrow 0$, we obtain the variational inequality

$$\langle x - r_C(x), w - r_C(x) \rangle \leq 0 \quad (w \in C). \quad (6.2)$$

Apply (6.2) first with $w = r_C(y)$:

$$\langle x - r_C(x), r_C(y) - r_C(x) \rangle \leq 0,$$

so

$$\langle x - r_C(x), r_C(x) - r_C(y) \rangle \geq 0.$$

Apply it again with base point y and $w = r_C(x)$:

$$\langle y - r_C(y), r_C(x) - r_C(y) \rangle \leq 0.$$

Subtracting the second inner product from the first gives

$$\begin{aligned} 0 &\leq \langle x - r_C(x), r_C(x) - r_C(y) \rangle \\ &\quad - \langle y - r_C(y), r_C(x) - r_C(y) \rangle \\ &= \langle x - y, r_C(x) - r_C(y) \rangle - \|r_C(x) - r_C(y)\|^2. \end{aligned}$$

Consequently, by Cauchy–Schwarz,

$$\begin{aligned} \|r_C(x) - r_C(y)\|^2 &\leq \langle x - y, r_C(x) - r_C(y) \rangle \\ &\leq \|x - y\| \|r_C(x) - r_C(y)\|. \end{aligned}$$

If $r_C(x) = r_C(y)$, the required inequality is immediate. Otherwise, dividing by $\|r_C(x) - r_C(y)\|$ gives

$$\|r_C(x) - r_C(y)\| \leq \|x - y\|.$$

Thus r_C is nonexpansive. It fixes every point of C , including zero.

3. Linearize the retraction and compare with the identity.

The map $\delta_C \circ r_C : H \rightarrow \mathcal{F}(C)$ is 1-Lipschitz and fixes zero. Proposition 2.1 therefore gives its linearization

$$P_C : \mathcal{F}(H) \rightarrow \mathcal{F}(C), \quad P_C \delta_H(x) = \delta_C(r_C(x)), \quad \|P_C\| \leq 1.$$

Under the standard inclusion $\mathcal{F}(C) \subset \mathcal{F}(H)$, it fixes each $\delta_H(x)$ for $x \in C$ and therefore is a projection onto $\mathcal{F}(C)$. Regard $I - P_C$ as an operator on $\mathcal{F}(H)$. Since $\mathcal{F}(K) \subset \mathcal{F}(C)$, every $\nu \in \mathcal{F}(K)$ satisfies

$$P_C \nu = \nu, \quad (I - P_C)\mu = (I - P_C)(\mu - \nu) \quad (\mu \in \mathcal{F}(H)).$$

Consequently,

$$\begin{aligned} \|\mu - P_C \mu\| &= \|(I - P_C)(\mu - \nu)\| \\ &\leq (1 + \|P_C\|) \|\mu - \nu\| \leq 2 \|\mu - \nu\|. \end{aligned}$$

This holds for every $\nu \in \mathcal{F}(K)$. Taking the infimum gives

$$\|\mu - P_C \mu\| \leq 2 \operatorname{dist}(\mu, \mathcal{F}(K)). \quad (6.3)$$

For $\mu = Se_n$, the lower bound for S and the choice of K therefore give

$$\begin{aligned} \|P_C Se_n\| &\geq \|Se_n\| - \|Se_n - P_C Se_n\| \\ &\geq b - 2 \operatorname{dist}(Se_n, \mathcal{F}(K)) > b - 2(b/4) = b/2. \end{aligned}$$

4. Obtain the compact-operator contradiction.

By Theorem 5.6, $\mathcal{F}(C)$ contains no c_0 . The Bessaga–Pełczyński consequence (6.1) therefore makes the bounded operator $P_C S : c_0 \rightarrow \mathcal{F}(C)$ compact. Since (e_n) is bounded, some subsequence $(P_C S e_{n_j})_j$ converges in norm, say to $v \in \mathcal{F}(C)$. For every $\ell \in \mathcal{F}(C)^*$, however,

$$\ell(v) = \lim_j \ell(P_C S e_{n_j}) = \lim_j ((P_C S)^* \ell)_{n_j} = 0,$$

because $(P_C S)^* \ell \in \ell_1$. Thus $v = 0$ by Hahn–Banach, contrary to $\|P_C S e_{n_j}\| > b/2$ for every j . $\square$

The nearest-point retraction also appears in the Hilbert-space transfer in [2, proof of Theorem C]. In this argument we needed only the distance form of compact reduction for $(S e_n)$, not an approximation of the whole image of the unit ball.

Use of ChatGPT

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References

- [1] R. J. Aliaga, C. Noûs, C. Petitjean, and A. Procházka, *Compact reduction in Lipschitz-free spaces*, *Studia Math.* **260** (2021), no. 3, 341–359. doi:10.4064/sm200925-18-1. The theorem numbering cited here is that of arXiv:2004.14250v3.
- [2] R. J. Aliaga, E. Pernecká, and A. Quero, *Pełczyński’s property (V^*) in Lipschitz-free spaces*, *Results Math.* **80** (2025), Article 197. doi:10.1007/s00025-025-02510-6. The version used for the Hilbert-space transfer is arXiv:2411.15107v3, proof of Theorem C.
- [3] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, *Studia Math.* **17** (1958), 151–164. doi:10.4064/sm-17-2-151-164.
- [4] G. Godefroy and N. J. Kalton, *Lipschitz-free Banach spaces*, *Studia Math.* **159** (2003), no. 1, 121–141. doi:10.4064/sm159-1-6.
- [5] T. Kochanek and E. Pernecká, *Lipschitz-free spaces over compact subsets of superreflexive spaces are weakly sequentially complete*, *Bull. Lond. Math. Soc.* **50** (2018), no. 4, 680–696. doi:10.1112/blms.12179. The theorem numbering cited here is that of arXiv:1703.07896v2.
- [6] R. E. Megginson, *An Introduction to Banach Space Theory*, Graduate Texts in Mathematics, vol. 183, Springer-Verlag, New York, 1998. doi:10.1007/978-1-4612-0603-3.
- [7] E. J. McShane, *Extension of range of functions*, *Bull. Amer. Math. Soc.* **40** (1934), no. 12, 837–842. doi:10.1090/S0002-9904-1934-05978-0.
- [8] OpenAI, *Lipschitz Equivalent Separable Banach Spaces Need Not Be Linearly Isomorphic*, preprint dated 24 September 2026, OpenAI Math Release. The source used is the 23-page manuscript at repository commit 1581d4cbdd69a0faf34b2089614fe2a814316a6a. Official preprint at the pinned commit.