

# Reading OpenAI's Construction of Lipschitz Homeomorphic but Linearly Nonisomorphic Separable Banach Spaces

Version 1

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## Abstract

We construct separable real Banach spaces that are Lipschitz homeomorphic but not linearly isomorphic, using the weighted graph spaces and the free-space linearization from OpenAI's counterexample. The Hilbert-space map is constructed here in the reverse direction. Unit vectors are chosen from the preceding output blocks and the norm of the remaining tail; this makes the inverse equation solvable by an explicit coordinate recursion. Direct two-point estimates give the distance bounds, and separated coordinate supports give complete continuity of the nonlinear correction's linearization. The Hilbert-space construction uses no differentiation, integration, or inverse function theorem. The necessary facts about Lipschitz-free spaces are stated in the forms used here, with proofs in a companion note. For every  $\varepsilon > 0$ , the resulting bijection and its inverse both have Lipschitz constant at most  $1 + \varepsilon$ ; the two Banach spaces depend on  $\varepsilon$ .

## Contents

|          |                                                         |           |
|----------|---------------------------------------------------------|-----------|
| <b>1</b> | <b>The theorem and the construction</b>                 | <b>2</b>  |
| 1.1      | The Hilbert map that the proof requires . . . . .       | 3         |
| 1.2      | How the Hilbert map is constructed . . . . .            | 3         |
| <b>2</b> | <b>The inputs from Lipschitz-free spaces</b>            | <b>4</b>  |
| 2.1      | Evaluations, duality, and linearization . . . . .       | 4         |
| 2.2      | Summable functions and cutoffs . . . . .                | 5         |
| 2.3      | Compact reduction and the Hilbert-space input . . . . . | 5         |
| <b>3</b> | <b>Weighted graph spaces: the reduction criterion</b>   | <b>6</b>  |
| 3.1      | Weighted sequences and the graph norm . . . . .         | 8         |
| 3.2      | The linear obstruction . . . . .                        | 9         |
| 3.3      | The nonlinear change of variables . . . . .             | 11        |
| <b>4</b> | <b>Local orthogonality and complete continuity</b>      | <b>15</b> |
| 4.1      | The condition on increments . . . . .                   | 15        |

|          |                                                                    |           |
|----------|--------------------------------------------------------------------|-----------|
| 4.2      | Tests confined to shrinking balls . . . . .                        | 16        |
| 4.3      | Local estimates on one compact set . . . . .                       | 18        |
| 4.4      | Proof: a finite exceptional set and a small remainder . . . . .    | 19        |
| <b>5</b> | <b>The Hilbert-space construction</b>                              | <b>25</b> |
| 5.1      | A rearrangement with one fixed direction in each block . . . . .   | 26        |
| 5.2      | Changing the fixed directions . . . . .                            | 27        |
| 5.3      | Unit directions from finitely many coordinates . . . . .           | 29        |
| 5.4      | Tail norms and a weighted estimate . . . . .                       | 31        |
| 5.5      | A scalar parameter and separated transition intervals . . . . .    | 33        |
| 5.6      | Interpolating the previously constructed directions . . . . .      | 34        |
| 5.7      | The inverse map and its two-point estimate . . . . .               | 38        |
| 5.8      | Surjectivity by a coordinate recursion . . . . .                   | 40        |
| 5.9      | Orthogonal coordinate supports on distinct neighborhoods . . . . . | 42        |
| <b>6</b> | <b>The final spaces and the choice of parameters</b>               | <b>47</b> |
|          | <b>Attribution</b>                                                 | <b>49</b> |
|          | <b>Use of ChatGPT</b>                                              | <b>49</b> |

## 1. The theorem and the construction

A surjective isometry between real normed spaces is affine by the Mazur–Ulam theorem [8]. If it fixes the origin, it is therefore a linear isometry. A bi-Lipschitz bijection need not have this consequence, even when its distortion is arbitrarily close to one. We explain the construction in [9], keeping its parameters variable and replacing its Hilbert-space map by a direct construction of the inverse.

All spaces are real. A *linear copy* of a Banach space  $E$  in a Banach space  $Y$  means the range of a bounded linear map  $T : E \rightarrow Y$  for which there is  $a > 0$  such that  $\|Tx\| \geq a\|x\|$  for every  $x \in E$ . For a Banach space  $Y$ , we use

$$c_0(Y) = \{(y_i)_{i \geq 1} : y_i \in Y, \|y_i\| \rightarrow 0\}, \quad \|(y_i)\|_{c_0(Y)} = \sup_i \|y_i\|.$$

Thus  $c_0 = c_0(\mathbb{R})$ , whereas each coordinate of  $c_0(\ell_2)$  is itself an infinite-dimensional Hilbert space.

**Theorem 1.1.** *For every  $\varepsilon > 0$  there are separable real Banach spaces  $X_\varepsilon, Y_\varepsilon$  and a bijection  $\Psi_\varepsilon : X_\varepsilon \rightarrow Y_\varepsilon$  fixing the origin such that*

$$\frac{1}{1+\varepsilon} \|s - t\|_{X_\varepsilon} \leq \|\Psi_\varepsilon(s) - \Psi_\varepsilon(t)\|_{Y_\varepsilon} \leq (1+\varepsilon) \|s - t\|_{X_\varepsilon} \quad (s, t \in X_\varepsilon). \quad (1.1)$$

*The space  $X_\varepsilon$  contains  $c_0(\ell_2)$  linearly isometrically, whereas  $Y_\varepsilon$  contains no linear copy of  $c_0(\ell_2)$ . In particular, the two spaces are not linearly isomorphic.*

The spaces may depend on  $\varepsilon$ . We make no assertion about a single fixed pair for every positive tolerance. At  $\varepsilon = 0$ , the displayed inequalities would make  $\Psi_\varepsilon$  an isometry, and Mazur–Ulam

would force linear isometry. The distinction is therefore between zero error and every positive multiplicative error, not between small and large additive errors.

The counterexample is due to OpenAI. The fixed source states its main theorem with lower and upper distance bounds  $4/21$  and  $76/25$ , respectively [9, Theorem 1.1]. The parameter-dependent bounds in Theorem 1.1 are established below through a reconstruction of that proof mechanism; they are not quoted as the literal statement of the source theorem. The quantitative form above retains its reduction to a Hilbert-space map and chooses smaller positive weights in the final norms. Section 5 gives a different realization of that map, with the same properties needed by the reduction. This is an exposition and reconstruction of the argument, not an independent solution or a claim of priority for the quantitative form.

### 1.1 The Hilbert map that the proof requires

Let  $\mathcal{H}$  be a separable infinite-dimensional Hilbert space. The main construction produces a bijection

$$h : \mathcal{H} \oplus_2 \ell_2 \longrightarrow \mathcal{H}, \quad h(0, 0) = 0,$$

with small distortion. These two Hilbert spaces are already linearly isometric; the essential additional requirement concerns the difference between  $h$  and projection onto the first coordinate. Write  $D(u, v) = h(u, v) - u$  and let

$$Q = \widehat{D} : \mathcal{F}(\mathcal{H} \oplus_2 \ell_2) \longrightarrow \mathcal{H}$$

be its linearization, so  $Q\delta(u, v) = h(u, v) - u$ . We require  $Q$  to be *completely continuous*: if  $\mu_n \rightharpoonup 0$  in its domain, then  $\|Q\mu_n\| \rightarrow 0$ . This is not a synonym for compactness.

The companion note [1] supplies the other linear fact  $c_0 \not\hookrightarrow \mathcal{F}(\mathcal{H} \oplus_2 \ell_2)$ , because  $\mathcal{H} \oplus_2 \ell_2$  is a real Hilbert space. We do not repeat its proof. Section 2 states precisely the companion results used here, including the formulas needed to apply them.

Section 3 constructs two weighted sequence spaces from  $Q$ . Their norms contain a weighted sum of the norms of the entries and a supremum of the norms of their images under  $Q$ . Complete continuity makes the second term too small on suitable Hilbert directions to permit a copy of  $c_0(\ell_2)$  in one space. Adding an  $\ell_2$  coordinate to each entry puts such a copy into the other. The identity

$$Q(\mu + \delta(Q\mu, v)) = h(Q\mu, v)$$

then produces a nonlinear bijection between the two spaces. We calculate its inverse, verify membership in both sequence spaces, and estimate both Lipschitz constants before constructing  $h$ .

### 1.2 How the Hilbert map is constructed

Write  $\mathcal{H} = (\bigoplus_{n \geq 1} H_n)_2$  with finite-dimensional blocks. For a fixed unit direction in each block, one can keep the perpendicular components in place, move the old scalar components to the even blocks, and insert the added  $\ell_2$  components in the odd blocks. This is an onto linear isometry, with an explicit inverse.

Instead of first constructing  $h$ , we construct a map in the opposite direction:

$$\Phi : \mathcal{H} \longrightarrow \mathcal{H} \oplus_2 \ell_2, \quad \Phi(y) = L_{w(y)}^{-1} y.$$

Here  $L_\xi$  is the fixed-direction isometry just described. The direction  $w_n(y)$  in block  $n$  is determined by the first  $n - 1$  blocks of  $y$  and by the tail norm  $r_n(y) = (\sum_{j \geq n} \|y_j\|^2)^{1/2}$ . For a

proposed preimage of  $(u, v)$ , norm preservation determines that radius from already computed blocks through  $r_n(y)^2 = \|u\|^2 + \|v\|^2 - \sum_{j < n} \|y_j\|^2$ . Subsection 5.8 proves that the expression is nonnegative throughout the recursion, constructs  $y_n$ , and verifies  $\Phi(y) = (u, v)$ . Thus it proves surjectivity without first producing a local inverse.

The two-point estimate for  $\Phi$  compares its values with a single fixed-direction isometry, chosen separately for each pair of inputs. A finite-coordinate construction separates distinct neighborhoods into orthogonal coordinate supports in all sufficiently late blocks. The resulting inverse  $h = \Phi^{-1}$  therefore satisfies both the metric bounds and the locality condition required by Section 4. Section 6 applies the graph-space criterion to this  $h$  and gives the final spaces and constants.

## 2. The inputs from Lipschitz-free spaces

We use the companion note [1] for the proofs in this section. The statements and conventions are recalled here so that no proof below requires a search for a definition in the companion. The underlying free-space results are attributed to [5, 2, 6, 3] as indicated at the relevant statements.

### 2.1 Evaluations, duality, and linearization

For a pointed metric space  $(M, d, 0)$ , let  $\text{Lip}_0(M)$  be the real Lipschitz functions vanishing at 0, with norm  $\text{Lip}(f)$ . For  $x \in M$ , evaluation at  $x$  is the functional  $\delta_M(x) : f \mapsto f(x)$ , and

$$\mathcal{F}(M) = \overline{\text{span}}\{\delta_M(x) : x \in M\} \subset \text{Lip}_0(M)^*.$$

The facts needed below are

$$\mathcal{F}(M)^* = \text{Lip}_0(M), \quad \langle f, \delta_M(x) \rangle = f(x), \quad \|\delta_M(x) - \delta_M(y)\| = d(x, y). \quad (2.1)$$

The first identification is isometric. A separable  $M$  has separable  $\mathcal{F}(M)$ : if  $x_j \rightarrow x$ , then the last identity gives  $\delta_M(x_j) \rightarrow \delta_M(x)$ , so evaluations on a countable dense subset have dense linear span.

**Proposition 2.1** (Linearization; [1, Proposition 2.1]). *Let  $Y$  be a Banach space and let  $F : M \rightarrow Y$  be Lipschitz with  $F(0) = 0$ . There is a unique bounded linear operator  $\widehat{F} : \mathcal{F}(M) \rightarrow Y$  such that*

$$\widehat{F}\delta_M(x) = F(x) \quad (x \in M), \quad \|\widehat{F}\| = \text{Lip}(F).$$

If  $K \subset M$  contains 0, we identify  $\mathcal{F}(K)$  isometrically with its natural subspace of  $\mathcal{F}(M)$ . In particular,

$$|\langle f, \nu \rangle| \leq \text{Lip}(f|_K) \|\nu\| \quad (\nu \in \mathcal{F}(K)). \quad (2.2)$$

For a Lipschitz function not vanishing at 0, our convention is

$$\langle f, \mu \rangle := \langle f - f(0), \mu \rangle, \quad \left\langle f, \sum_{j=1}^N a_j \delta_M(x_j) \right\rangle = \sum_{j=1}^N a_j (f(x_j) - f(0)).$$

Thus constants pair to zero. In particular, (2.2) also holds under this convention. These facts are proved in [1, Section 2]; the isometric inclusion uses the scalar extension formula of McShane [7].

## 2.2 Summable functions and cutoffs

**Lemma 2.2** (Summable tests; [1, Lemma 3.1 and Corollary 3.2]). *Let  $f_j : M \rightarrow \mathbb{R}$  be Lipschitz functions and suppose*

$$\sum_{j \geq 1} |f_j(x) - f_j(y)| \leq Cd(x, y) \quad (x, y \in M).$$

*Then*

$$A : \mathcal{F}(M) \rightarrow \ell_1, \quad A\mu = (\langle f_j, \mu \rangle)_{j \geq 1}$$

*is bounded and linear, with  $\|A\| \leq C$ . If  $\mu_n \rightarrow 0$  in  $\mathcal{F}(M)$ , then  $\|A\mu_n\|_1 \rightarrow 0$ .*

The last assertion uses Schur's property of  $\ell_1$ . The convention above allows the functions  $f_j$  to have nonzero values at the base point: subtracting those values does not change their increments.

**Lemma 2.3** (Cutoff products; [1, Lemma 3.3]). *For  $z \in M$  and  $r > 0$ , set*

$$\chi_{z,r}(x) = \max \left\{ 0, \min \left\{ 1, 2 - \frac{2d(x, z)}{r} \right\} \right\}.$$

*Then  $0 \leq \chi_{z,r} \leq 1$ , and*

$$\chi_{z,r} = 1 \text{ on } \overline{B}(z, r/2), \quad \chi_{z,r} = 0 \text{ on } M \setminus B(z, r), \quad \text{Lip}(\chi_{z,r}) \leq \frac{2}{r}.$$

*If  $f(z) = 0$  and  $\text{Lip}(f) \leq L$ , then*

$$\text{Lip}(\chi_{z,r}f) \leq 3L, \quad \text{Lip}((1 - \chi_{z,r})f) \leq 3L.$$

**Lemma 2.4** (Disjoint nonzero sets; [1, Lemma 3.4]). *Suppose  $\text{Lip}(u_j) \leq L$  and the sets  $\{x : u_j(x) \neq 0\}$  are pairwise disjoint. Then*

$$\sum_j |u_j(x) - u_j(y)| \leq 2Ld(x, y) \quad (x, y \in M).$$

Disjointness in this statement concerns the functions before constants are subtracted. The increment estimate survives that subtraction.

## 2.3 Compact reduction and the Hilbert-space input

**Theorem 2.5** (Compact reduction; [1, Theorem 4.3]). *Let  $M$  be complete and pointed, and let  $\mu_n \rightarrow 0$  in  $\mathcal{F}(M)$ . For every  $\tau > 0$  there is a compact set  $K \subset M$  containing 0 such that*

$$\sup_n \text{dist}(\mu_n, \mathcal{F}(K)) < \tau.$$

*Consequently one may choose  $\nu_n \in \mathcal{F}(K)$  with*

$$\|\mu_n - \nu_n\| < \tau, \quad \|\nu_n\| \leq \sup_j \|\mu_j\| + \tau.$$

This is the sequence form of the compact-reduction method of Aliaga–Noûs–Petitjean–Procházka [2]. The same compact set works for the entire sequence. We use precisely that uniformity in Section 4.

**Theorem 2.6** (Free spaces over Hilbert spaces; [1, Theorem 6.1]). *For every real Hilbert space  $H$ , the space  $\mathcal{F}(H)$  contains no linear copy of  $c_0$ .*

The companion proves the compact-convex case using the averaging and product argument of Kochanek–Pernecká [6], rewritten in terms of square-root formulas and finite sums. It then passes to the whole Hilbert space by compact reduction and the nearest-point retraction; compare [3, proof of Theorem C]. None of that preparation is reproved here.

We also use the following consequence of Bessaga–Pełczyński [4], with its derivation in [1, Section 6.1, equation (6.1)]:

$$c_0 \not\hookrightarrow E \implies \text{every bounded linear } T : c_0 \rightarrow E \text{ is compact.} \quad (2.3)$$

Here  $E$  is a Banach space and  $\not\hookrightarrow$  excludes a bounded linear embedding with a positive lower bound. In particular, if  $e_n$  denotes the standard unit vector of  $c_0$ , then  $e_n \rightarrow 0$  and compactness of  $T$  implies  $\|Te_n\| \rightarrow 0$ .

### 3. Weighted graph spaces: the reduction criterion

The purpose of this section is to specify what the later constructions must supply. Two gliding-hump arguments provide the linear obstruction. A change of variables then matches the metrics. We keep the weights variable, since their total sum controls the additional distortion. The graph construction and change of variables follow [9, Sections 2–3].

**Lemma 3.1** (A gliding-hump argument in  $\ell_1(E)$ ). *If a Banach space  $E$  contains no linear copy of  $c_0$ , neither does  $\ell_1(E)$ .*

*Proof.* Suppose that  $S : c_0 \rightarrow \ell_1(E)$  is bounded and linear and that

$$\|Sa\| \geq b \|a\|_\infty \quad (a \in c_0)$$

for some  $b > 0$ . For each  $i$ , the coordinate map  $\pi_i : \ell_1(E) \rightarrow E$  is bounded. Applying (2.3) to  $\pi_i S : c_0 \rightarrow E$  shows that  $\pi_i S$  is compact. Hence

$$\|(Se_n)_i\| = \|\pi_i Se_n\| \rightarrow 0 \quad (n \rightarrow \infty), \quad (3.1)$$

because  $e_n \rightarrow 0$  in  $c_0$ . At the same time,  $\|Se_n\| \geq b$  for every  $n$ .

Let  $P_m$  denote projection onto the first  $m$  coordinates of  $\ell_1(E)$ , with  $P_0 = 0$ . We choose integers

$$0 = n_0 < n_1 < n_2 < \cdots, \quad 0 = m_0 < m_1 < m_2 < \cdots$$

such that, for every  $j \geq 1$ ,

$$\|P_{m_{j-1}} Se_{n_j}\| < \frac{b}{8}, \quad \|(I - P_{m_j}) Se_{n_j}\| < \frac{b}{8}. \quad (3.2)$$

Here is the induction. Suppose  $n_1, \dots, n_{j-1}$  and  $m_1, \dots, m_{j-1}$  have been chosen. By (3.1),

$$\|P_{m_{j-1}} Se_n\| = \sum_{i=1}^{m_{j-1}} \|(Se_n)_i\| \rightarrow 0.$$

Choose  $n_j > n_{j-1}$  for which this sum is less than  $b/8$ . For this fixed  $n_j$ , membership in  $\ell_1(E)$  gives

$$\|(I - P_m) Se_{n_j}\| = \sum_{i>m} \|(Se_{n_j})_i\| \rightarrow 0 \quad (m \rightarrow \infty).$$

Choose  $m_j > m_{j-1}$  for which this tail is less than  $b/8$ . No earlier choice has changed, and both new inequalities in (3.2) hold. This also starts the induction at  $j = 1$ , when the first sum is empty.

Define

$$z_j = (P_{m_j} - P_{m_{j-1}})Se_{n_j}.$$

The two discarded parts give

$$\begin{aligned} Se_{n_j} - z_j &= P_{m_{j-1}}Se_{n_j} + (I - P_{m_j})Se_{n_j}, \\ \|Se_{n_j} - z_j\| &\leq \|P_{m_{j-1}}Se_{n_j}\| + \|(I - P_{m_j})Se_{n_j}\| < \frac{b}{8} + \frac{b}{8} = \frac{b}{4}, \\ \|z_j\| &\geq \|Se_{n_j}\| - \|Se_{n_j} - z_j\| > b - \frac{b}{4} = \frac{3b}{4}. \end{aligned}$$

For  $u_N = \sum_{j=1}^N e_{n_j}$ , distinctness of the indices gives  $\|u_N\|_{c_0} = 1$ . The nonzero coordinates of  $z_j$  lie in  $(m_{j-1}, m_j]$ , so

$$\left\| \sum_{j=1}^N z_j \right\|_{\ell_1(E)} = \sum_{j=1}^N \sum_{i=m_{j-1}+1}^{m_j} \|(z_j)_i\| = \sum_{j=1}^N \|z_j\|_{\ell_1(E)}.$$

Consequently,

$$\begin{aligned} \|S\| &\geq \|Su_N\| = \left\| \sum_{j=1}^N Se_{n_j} \right\| \\ &\geq \left\| \sum_{j=1}^N z_j \right\| - \sum_{j=1}^N \|Se_{n_j} - z_j\| \\ &> N \frac{3b}{4} - N \frac{b}{4} = \frac{Nb}{2}. \end{aligned}$$

This is impossible for arbitrarily large  $N$ . □

The next argument makes the opposite comparison. Disjoint vectors in  $c_0(Y)$  add by taking a maximum, whereas a sum of orthonormal vectors in  $\ell_2$  has norm  $\sqrt{N}$ .

**Lemma 3.2.** *Let  $Y$  be a Banach space and let  $T : \ell_2 \rightarrow c_0(Y)$  be bounded and linear. Suppose there is an orthonormal sequence  $(e_k)$  in  $\ell_2$  such that  $\|(Te_k)_i\| \rightarrow 0$  as  $k \rightarrow \infty$  for every  $i \geq 1$ . Then  $T$  is not bounded below.*

*Proof.* Put  $y_k = Te_k$ , and let  $P_m : c_0(Y) \rightarrow c_0(Y)$  be projection onto the first  $m$  coordinates, with  $P_0 = 0$ . We choose increasing integers  $k_j, m_j$ , starting from  $k_0 = m_0 = 0$ , such that

$$\|P_{m_{j-1}}y_{k_j}\| < 2^{-j}, \quad \|(I - P_{m_j})y_{k_j}\| < 2^{-j} \quad (j \geq 1).$$

Suppose the choices through  $j - 1$  satisfy these inequalities. Since there are only finitely many coordinates  $i \leq m_{j-1}$ ,

$$\|P_{m_{j-1}}Te_k\| = \max(\{0\} \cup \{\|(Te_k)_i\| : 1 \leq i \leq m_{j-1}\}) \rightarrow 0 \quad (k \rightarrow \infty).$$

The formula includes  $m_{j-1} = 0$ , when the projection is zero. We can therefore choose  $k_j > k_{j-1}$  such that

$$\|P_{m_{j-1}}y_{k_j}\| < 2^{-j}. \tag{3.3}$$

Now  $y_{k_j}$  is a fixed member of  $c_0(Y)$ , so its tail norm tends to zero. Choose  $m_j > m_{j-1}$  with

$$\|(I - P_{m_j})y_{k_j}\| = \sup_{i > m_j} \|(y_{k_j})_i\| < 2^{-j}. \quad (3.4)$$

The choices satisfy both new inequalities, without changing any earlier one. This completes the induction. Define

$$z_j = (P_{m_j} - P_{m_{j-1}})y_{k_j}.$$

Its nonzero coordinates lie in  $m_{j-1} < i \leq m_j$ . The two discarded parts are disjoint, and the norm is a supremum, not a sum. Therefore

$$\|y_{k_j} - z_j\| = \max \left\{ \|P_{m_{j-1}}y_{k_j}\|, \|(I - P_{m_j})y_{k_j}\| \right\} < 2^{-j}.$$

Also  $\|z_j\| \leq \|y_{k_j}\| \leq \|T\|$ .

For  $u_N = \sum_{j=1}^N e_{k_j}$ , orthonormality gives  $\|u_N\|_2 = \sqrt{N}$ . The disjoint intervals of the  $z_j$  give

$$\left\| \sum_{j=1}^N z_j \right\|_{c_0(Y)} = \max_{1 \leq j \leq N} \max_{m_{j-1} < i \leq m_j} \|(z_j)_i\| = \max_{j \leq N} \|z_j\| \leq \|T\|.$$

Consequently,

$$\begin{aligned} \|Tu_N\| &= \left\| \sum_{j=1}^N y_{k_j} \right\| \\ &\leq \left\| \sum_{j=1}^N z_j \right\| + \sum_{j=1}^N \|y_{k_j} - z_j\| < \|T\| + \sum_{j=1}^{\infty} 2^{-j} = \|T\| + 1. \end{aligned}$$

A lower bound  $\|Tx\| \geq a\|x\|$  with  $a > 0$  would imply  $a\sqrt{N} < \|T\| + 1$  for every  $N$ , which is impossible.  $\square$

### 3.1 Weighted sequences and the graph norm

Let  $A, Y$  be Banach spaces, let  $Q : A \rightarrow Y$  be bounded and linear, and fix positive weights  $\lambda_i$  with  $\theta = \sum_{i \geq 1} \lambda_i < \infty$ . First define

$$\ell_1(\lambda; A) = \{s = (s_i)_{i \geq 1} \in A^{\mathbb{N}} : \sum_i \lambda_i \|s_i\| < \infty\}, \quad \|s\|_{\lambda} = \sum_i \lambda_i \|s_i\|.$$

The diagonal map and its inverse are

$$T : \ell_1(\lambda; A) \longrightarrow \ell_1(A), \quad T(s) = (\lambda_i s_i)_i, \quad T^{-1}(x) = (\lambda_i^{-1} x_i)_i.$$

Their norms satisfy

$$\|T(s)\|_1 = \sum_i \lambda_i \|s_i\| = \|s\|_{\lambda}, \quad \|T^{-1}(x)\|_{\lambda} = \sum_i \|x_i\| = \|x\|_1.$$

Thus this weighted space is complete. Its original coordinates need not be bounded: for  $\lambda_i = 2^{-i}$ , the sequence  $s_i = 2^i i^{-2} a$  belongs to it for each  $a \in A$ .

Since  $A$  and  $Y$  are specified as the domain and codomain of  $Q : A \rightarrow Y$ , we omit them from the notation  $Z_Q^\lambda$ . The graph space is

$$Z_Q^\lambda = \{s = (s_i) \in \ell_1(\lambda; A) : (Qs_i)_i \in c_0(Y)\},$$

$$\|s\|_{Z_Q^\lambda} = \max \left\{ \sum_i \lambda_i \|s_i\|, \sup_i \|Qs_i\| \right\}. \quad (3.5)$$

The second membership condition is convergence to zero. Merely asking that the displayed supremum be finite would define a different space.

**Lemma 3.3.** *The space  $Z_Q^\lambda$  is complete. If  $A$  and  $Y$  are separable, then  $Z_Q^\lambda$  is also separable. The coordinate maps*

$$p_i : Z_Q^\lambda \rightarrow A, \quad p_i(s) = s_i, \quad J_Q : Z_Q^\lambda \rightarrow c_0(Y), \quad J_Q(s) = (Qs_i)_i \quad (3.6)$$

are bounded linear operators, with  $\|p_i\| \leq \lambda_i^{-1}$  and  $\|J_Q\| \leq 1$ .

*Proof.* The map  $s \mapsto (s, (Qs_i)_i)$  is an isometric embedding into  $\ell_1(\lambda; A) \oplus_\infty c_0(Y)$ . To show its range is closed, suppose its values at  $s^{(n)}$  converge to  $(s, y)$  in that direct sum. For every fixed  $i$ ,

$$\|s_i^{(n)} - s_i\| \leq \frac{1}{\lambda_i} \|s^{(n)} - s\|_\lambda \longrightarrow 0, \quad Qs_i^{(n)} \longrightarrow Qs_i.$$

Convergence in the other component also gives

$$\|Qs_i^{(n)} - y_i\| \leq \sup_j \|Qs_j^{(n)} - y_j\| \longrightarrow 0.$$

Uniqueness of the limit yields  $y_i = Qs_i$  for every  $i$ . Since  $s \in \ell_1(\lambda; A)$  and  $y \in c_0(Y)$ , it follows that  $s \in Z_Q^\lambda$  and  $(s, y) = (s, (Qs_i)_i)$ . The range is closed, which proves completeness.

If  $A, Y$  are separable, so are  $\ell_1(A)$  and  $c_0(Y)$  and their finite direct sum. Its metric subspace  $Z_Q^\lambda$  is therefore separable. Finally, the two estimates follow directly from

$$\lambda_i \|p_i(s)\| \leq \sum_j \lambda_j \|s_j\| \leq \|s\|_{Z_Q^\lambda}, \quad \|J_Q(s)\|_{c_0(Y)} = \sup_i \|Qs_i\| \leq \|s\|_{Z_Q^\lambda}. \quad \square$$

### 3.2 The linear obstruction

A bounded linear operator is called *completely continuous* here if it sends every weakly null sequence to a norm-null sequence. This is not being used as a synonym for compactness.

**Proposition 3.4.** *Suppose  $A$  contains no linear copy of  $c_0$ , and  $Q : A \rightarrow Y$  is bounded, linear, and completely continuous. Then  $Z_Q^\lambda$  contains no linear copy of  $c_0(\ell_2)$ .*

*Proof.* Suppose  $S : c_0(\ell_2) \rightarrow Z_Q^\lambda$  is an embedding satisfying  $\|Sx\| \geq b_0 \|x\|$  with  $b_0 > 0$ . Recall the two terms of the norm:

$$\|Sx\|_{Z_Q^\lambda} = \max \left\{ \sum_i \lambda_i \|(Sx)_i\|, \|J_Q(Sx)\|_{c_0(Y)} \right\}, \quad J_Q((z_i)) = (Qz_i)_i.$$

We will choose a unit vector  $t_n \in \ell_2$  for each  $n$  so that, on  $R(c_0)$  where  $R(a) = (a_n t_n)_n$ , the second term is at most  $(b_0/2) \|a\|_\infty$ . The first term must then be at least  $b_0 \|a\|_\infty$ .

**Choose a direction in each  $\ell_2$  coordinate.**

Let  $I_n : \ell_2 \rightarrow c_0(\ell_2)$  insert a vector into coordinate  $n$ , with zero elsewhere, and define

$$T_n = J_Q \circ S \circ I_n : \ell_2 \longrightarrow c_0(Y).$$

Fix  $n$  and an orthonormal sequence  $(e_k)$  in  $\ell_2$ . It is weakly null. For every  $i$ , the bounded linear map  $p_i S I_n : \ell_2 \rightarrow A$  gives  $p_i S I_n e_k \rightarrow 0$ . Complete continuity of  $Q$  gives

$$\|(T_n e_k)_i\| = \|Q p_i S I_n e_k\| \longrightarrow 0$$

as  $k \rightarrow \infty$  for every  $i$ . Lemma 3.2, applied to this  $T_n$  and the sequence  $(e_k)$ , shows that  $T_n$  is not bounded below. Thus

$$\inf_{\|t\|=1} \|T_n t\| = 0.$$

For each  $n$ , choose  $t_n \in \ell_2$  satisfying

$$\|t_n\| = 1, \quad \|T_n t_n\| \leq \frac{b_0}{2^{n+1}}.$$

**The chosen directions give a scalar copy of  $c_0$ .**

For  $a = (a_n) \in c_0$ , the sequence  $R(a) = (a_n t_n)_n$  belongs to  $c_0(\ell_2)$  and satisfies  $\|R(a)\| = \sup_n |a_n|$ . For a finitely supported sequence  $a = (a_n)$ , the definition of  $R$  gives  $R(a) = \sum_n a_n I_n(t_n)$ , a finite sum. Linearity and  $T_n = J_Q \circ S \circ I_n$  then give

$$J_Q(S(R(a))) = \sum_n a_n J_Q(S(I_n(t_n))) = \sum_n a_n T_n(t_n).$$

Since each  $\|T_n(t_n)\| \leq b_0 2^{-n-1}$  and  $\sum_{n \geq 1} 2^{-n-1} = 1/2$ , it follows that

$$\|J_Q(S(R(a)))\| \leq \sum_n |a_n| \|T_n(t_n)\| \leq b_0 \|a\|_\infty \sum_{n \geq 1} 2^{-n-1} = \frac{b_0}{2} \|a\|_\infty.$$

Since the finitely supported sequences are dense in  $c_0$  and the operator  $J_Q \circ S \circ R$  is bounded, the estimate extends to all of  $c_0$  with the same bound.

Since  $R$  is an isometry,  $\|S(R(a))\| \geq b_0 \|a\|_\infty$ . For  $a \neq 0$  the second norm term is strictly smaller than this lower bound. Therefore

$$\sum_i \lambda_i \|(S(R(a)))_i\| \geq b_0 \|a\|_\infty.$$

The inequality also holds at  $a = 0$ . Define

$$L : c_0 \longrightarrow \ell_1(A), \quad L(a) = (\lambda_i(S(R(a)))_i)_{i \geq 1}.$$

The defining weighted sum of  $S(R(a)) \in Z_Q^\lambda$  is finite, so  $L(a)$  does belong to  $\ell_1(A)$ . The lower bound just obtained and the upper graph-norm bound give

$$b_0 \|a\|_\infty \leq \|L(a)\|_1 = \sum_i \lambda_i \|(S(R(a)))_i\| \leq \|S(R(a))\|_{Z_Q^\lambda} \leq \|S\| \|a\|_\infty.$$

This is a linear embedding of  $c_0$  into  $\ell_1(A)$ , contrary to Lemma 3.1. □

The infinite-dimensional Hilbert coordinates are essential to this argument: they supply the orthonormal weakly null sequences used to choose  $t_n$ . The assertion does not exclude ordinary  $c_0$  from  $Z_Q^\lambda$ .

### 3.3 The nonlinear change of variables

We can now state what the Hilbert-space construction must provide. Throughout this proposition,  $u$  belongs to  $\mathcal{H}$ ,  $v$  belongs to  $\ell_2$ , and  $\mu, \nu$  belong to  $\mathcal{F}(\mathcal{H} \oplus_2 \ell_2)$ . In particular, the two direct-sum norms that occur below are

$$\|(u, v)\| = \left( \|u\|^2 + \|v\|^2 \right)^{1/2}, \quad \|(\mu, \nu)\| = \|\mu\| + \|\nu\|.$$

The first is a Hilbert norm; the second is the sum norm used in the weighted sequence space.

**Proposition 3.5** (Quantitative graph criterion). *Let  $\mathcal{H}$  be a separable real Hilbert space, and suppose that  $h : \mathcal{H} \oplus_2 \ell_2 \rightarrow \mathcal{H}$  is a bijection fixing zero. Suppose that  $0 < a \leq b$  and*

$$a \|(u - u', v - v')\| \leq \|h(u, v) - h(u', v')\| \leq b \|(u - u', v - v')\|$$

*for all  $(u, v), (u', v') \in \mathcal{H} \oplus_2 \ell_2$ . Define  $D : \mathcal{H} \oplus_2 \ell_2 \rightarrow \mathcal{H}$  by  $D(u, v) = h(u, v) - u$ , and let  $Q = \widehat{D} : \mathcal{F}(\mathcal{H} \oplus_2 \ell_2) \rightarrow \mathcal{H}$  be its linearization. Assume that  $Q$  is completely continuous. The absence of  $c_0$  from  $\mathcal{F}(\mathcal{H} \oplus_2 \ell_2)$  is already provided by Theorem 2.6. Let  $\lambda_i > 0$  and  $0 < \theta = \sum_{i \geq 1} \lambda_i \leq 1$ . Define the bounded linear operator*

$$\tilde{Q} : \mathcal{F}(\mathcal{H} \oplus_2 \ell_2) \oplus_1 \ell_2 \rightarrow \mathcal{H} \oplus_2 \ell_2 \quad \text{by} \quad \tilde{Q}(\mu, v) = (Q\mu, v),$$

*and let  $X = Z_{\tilde{Q}}^\lambda$  and  $Y = Z_Q^\lambda$ . Then  $X, Y$  are separable Banach spaces and there is a bijection  $\Psi : X \rightarrow Y$  fixing zero such that*

$$\text{Lip}(\Psi) \leq \max \{1 + \theta, b\}, \quad \text{Lip}(\Psi^{-1}) \leq \max \left\{ 1 + \frac{2\theta}{a}, \frac{1}{a} \right\}. \quad (3.7)$$

*Moreover,  $X$  contains  $c_0(\ell_2)$  linearly isometrically, whereas  $Y$  contains no linear copy of  $c_0(\ell_2)$ . Hence  $X$  and  $Y$  are not linearly isomorphic.*

*Proof.* The map  $D$  is Lipschitz, since

$$\|D(u, v) - D(u', v')\| \leq \|h(u, v) - h(u', v')\| + \|u - u'\| \leq (b + 1) \|(u - u', v - v')\|.$$

It fixes zero. Proposition 2.1 therefore defines  $Q$ , with  $\|Q\| = \text{Lip}(D) \leq b + 1$ . The operator  $\tilde{Q}$  is bounded because

$$\|\tilde{Q}(\mu, v)\| = \left( \|Q\mu\|^2 + \|v\|^2 \right)^{1/2} \leq \|Q\| \|\mu\| + \|v\| \leq \max \{\|Q\|, 1\} (\|\mu\| + \|v\|).$$

The space  $\mathcal{H} \oplus_2 \ell_2$  and its free space are separable. Thus Lemma 3.3, first with  $A = \mathcal{F}(\mathcal{H} \oplus_2 \ell_2) \oplus_1 \ell_2$  and operator  $\tilde{Q}$ , and then with  $A = \mathcal{F}(\mathcal{H} \oplus_2 \ell_2)$  and operator  $Q$ , shows that  $X$  and  $Y$  are complete and separable.

We first construct a bijection between the spaces in which the individual entries of  $X$  and  $Y$  take their values. Let  $\delta : \mathcal{H} \oplus_2 \ell_2 \rightarrow \mathcal{F}(\mathcal{H} \oplus_2 \ell_2)$  be the canonical evaluation map, written as  $\delta(u, v)$  on an ordered pair. Recall both identities that will be needed:

$$\begin{aligned} \|\delta(u, v) - \delta(u', v')\| &= \|(u - u', v - v')\|, \\ Q(\delta(u, v)) &= h(u, v) - u. \end{aligned} \quad (3.8)$$

The first is the isometry of the evaluation map. The second is the linearization identity  $Q \circ \delta = D$  for the particular function  $D(u, v) = h(u, v) - u$ .

Define  $B : \mathcal{F}(\mathcal{H} \oplus_2 \ell_2) \oplus_1 \ell_2 \rightarrow \mathcal{F}(\mathcal{H} \oplus_2 \ell_2)$  by

$$B(\mu, v) = \mu + \delta(Q\mu, v). \quad (3.9)$$

Here  $Q\mu \in \mathcal{H}$ , so  $\delta(Q\mu, v)$  belongs to the same free space as  $\mu$ . We chose this formula so that the second identity in (3.8), evaluated at  $(Q\mu, v)$ , cancels the term  $Q\mu$ :

$$\begin{aligned} Q(B(\mu, v)) &= Q(\mu + \delta(Q\mu, v)) \\ &= Q\mu + Q(\delta(Q\mu, v)) \\ &= Q\mu + (h(Q\mu, v) - Q\mu) \\ &= h(Q\mu, v). \end{aligned} \tag{3.10}$$

Equation (3.10) determines  $(Q\mu, v)$  from  $B(\mu, v)$  by applying  $h^{-1} \circ Q$ . This suggests the following inverse, whose two compositions we now verify.

For  $\nu \in \mathcal{F}(\mathcal{H} \oplus_2 \ell_2)$ , let  $(u, v) = h^{-1}(Q\nu)$ , so that  $h(u, v) = Q\nu$ , and define

$$\begin{aligned} C : \mathcal{F}(\mathcal{H} \oplus_2 \ell_2) &\longrightarrow \mathcal{F}(\mathcal{H} \oplus_2 \ell_2) \oplus_1 \ell_2, \\ C(\nu) &= (\nu - \delta(u, v), v). \end{aligned} \tag{3.11}$$

To compute  $B(C(\nu))$  from (3.9), we need the image under  $Q$  of its first argument  $\nu - \delta(u, v)$ . Linearity of  $Q$  and (3.8) give

$$\begin{aligned} Q(\nu - \delta(u, v)) &= Q\nu - Q(\delta(u, v)) \\ &= Q\nu - (h(u, v) - u) \\ &= Q\nu - Q\nu + u = u. \end{aligned}$$

We now substitute this value into the definition of  $B$ :

$$\begin{aligned} B(C(\nu)) &= B(\nu - \delta(u, v), v) \\ &= \nu - \delta(u, v) + \delta(Q(\nu - \delta(u, v)), v) \\ &= \nu - \delta(u, v) + \delta(u, v) = \nu. \end{aligned}$$

For the other composition, start with an arbitrary pair  $(\mu, v)$ . Equation (3.10) determines exactly which pair the definition of  $C$  recovers:

$$h^{-1}(Q(B(\mu, v))) = h^{-1}(h(Q\mu, v)) = (Q\mu, v).$$

In the definition of  $C$  in (3.11), substitute  $\nu = B(\mu, v)$ . The recovered components are  $(Q\mu, v)$ , so

$$\begin{aligned} C(B(\mu, v)) &= (B(\mu, v) - \delta(Q\mu, v), v) \\ &= (\mu + \delta(Q\mu, v) - \delta(Q\mu, v), v) = (\mu, v). \end{aligned}$$

We have proved both compositions, so  $B$  is bijective and  $C = B^{-1}$ . The computation of  $Q(\nu - \delta(u, v))$  also gives the identity needed for the second term of the graph norm:

$$\begin{aligned} \tilde{Q}(C(\nu)) &= \tilde{Q}(\nu - \delta(u, v), v) \\ &= (Q(\nu - \delta(u, v)), v) \\ &= (u, v) = h^{-1}(Q\nu). \end{aligned} \tag{3.12}$$

In all these calculations only  $h$  was inverted. No inverse or surjectivity of  $Q$  has been assumed. We next pass from single entries to sequences. For clarity, the membership conditions and the two norms are, explicitly,

$$\begin{aligned} s = ((\mu_i, v_i))_{i \geq 1} \in X &\iff \sum_i \lambda_i (\|\mu_i\| + \|v_i\|) < \infty \text{ and } (Q\mu_i, v_i) \longrightarrow (0, 0), \\ \nu = (\nu_i)_{i \geq 1} \in Y &\iff \sum_i \lambda_i \|\nu_i\| < \infty \text{ and } Q\nu_i \longrightarrow 0, \end{aligned}$$

$$\begin{aligned}\|s\|_X &= \max \left\{ \sum_i \lambda_i (\|\mu_i\| + \|v_i\|), \sup_i \|(Q\mu_i, v_i)\| \right\}, \\ \|\nu\|_Y &= \max \left\{ \sum_i \lambda_i \|\nu_i\|, \sup_i \|Q\nu_i\| \right\}.\end{aligned}$$

Define  $\Psi$  on  $X$  by

$$\Psi((\mu_i, v_i)_{i \geq 1}) = (B(\mu_i, v_i))_{i \geq 1} = (\mu_i + \delta(Q\mu_i, v_i))_{i \geq 1}.$$

We must check that the resulting sequence lies in  $Y$ . Since  $(Q\mu_i, v_i) \rightarrow (0, 0)$ , its norms have a finite supremum. Using  $\|\delta(Q\mu_i, v_i)\| = \|(Q\mu_i, v_i)\|$  and  $\sum_i \lambda_i = \theta$ , we obtain

$$\begin{aligned}\sum_i \lambda_i \|B(\mu_i, v_i)\| &\leq \sum_i \lambda_i \|\mu_i\| + \sum_i \lambda_i \|(Q\mu_i, v_i)\| \\ &\leq \sum_i \lambda_i \|\mu_i\| + \theta \sup_i \|(Q\mu_i, v_i)\| < \infty.\end{aligned}$$

For the other condition, (3.10) and continuity of  $h$  at  $(0, 0)$  give

$$Q(B(\mu_i, v_i)) = h(Q\mu_i, v_i) \longrightarrow h(0, 0) = 0.$$

Therefore  $\Psi(s) \in Y$ .

Conversely, let  $\nu = (\nu_i)_{i \geq 1} \in Y$ , and define its proposed inverse sequence by

$$(u_i, v_i) = h^{-1}(Q\nu_i), \quad (C(\nu_i))_{i \geq 1} = (\nu_i - \delta(u_i, v_i), v_i)_{i \geq 1}.$$

The lower bound for  $h$ , evaluated between  $(u_i, v_i)$  and  $(0, 0)$ , gives

$$\|(u_i, v_i)\| \leq a^{-1} \|h(u_i, v_i)\| = a^{-1} \|Q\nu_i\| \longrightarrow 0.$$

In particular  $\sup_i \|(u_i, v_i)\| < \infty$ . The norm of a pair in  $\mathcal{F}(\mathcal{H} \oplus_2 \ell_2) \oplus_1 \ell_2$  is the sum of its component norms, whereas  $\|v_i\| \leq \|(u_i, v_i)\|$  in the Hilbert sum. Consequently

$$\begin{aligned}\sum_i \lambda_i \|C(\nu_i)\| &= \sum_i \lambda_i (\|\nu_i - \delta(u_i, v_i)\| + \|v_i\|) \\ &\leq \sum_i \lambda_i \|\nu_i\| + \sum_i \lambda_i (\|\delta(u_i, v_i)\| + \|v_i\|) \\ &\leq \sum_i \lambda_i \|\nu_i\| + 2\theta \sup_i \|(u_i, v_i)\| < \infty.\end{aligned}$$

The second condition follows from the already proved component identity:

$$\tilde{Q}(C(\nu_i)) = (Q(\nu_i - \delta(u_i, v_i)), v_i) = (u_i, v_i) \longrightarrow (0, 0).$$

Thus  $(C(\nu_i))_i \in X$ . At every index,  $B(C(\nu_i)) = \nu_i$  and  $C(B(\mu_i, v_i)) = (\mu_i, v_i)$ . The two sequence maps are therefore mutual inverses. Since  $\delta(0, 0) = 0$  and  $h^{-1}(0) = (0, 0)$ , both fix zero.

It remains to estimate their Lipschitz constants. Let  $s = ((\mu_i, v_i))_i$  and  $t = ((\mu'_i, v'_i))_i$  belong to  $X$ . The definition of the  $X$ -norm gives

$$\sum_i \lambda_i (\|\mu_i - \mu'_i\| + \|v_i - v'_i\|) \leq \|s - t\|_X, \quad \|(Q(\mu_i - \mu'_i), v_i - v'_i)\| \leq \|s - t\|_X \quad (i \geq 1).$$

The definition of  $B$  and the isometry of  $\delta$  give

$$\begin{aligned}\|B(\mu_i, v_i) - B(\mu'_i, v'_i)\| &= \|\mu_i - \mu'_i + \delta(Q\mu_i, v_i) - \delta(Q\mu'_i, v'_i)\| \\ &\leq \|\mu_i - \mu'_i\| + \|\delta(Q\mu_i, v_i) - \delta(Q\mu'_i, v'_i)\| \\ &= \|\mu_i - \mu'_i\| + \|(Q(\mu_i - \mu'_i), v_i - v'_i)\|.\end{aligned}$$

Multiplying by  $\lambda_i$  and summing gives the first term of the  $Y$ -norm:

$$\begin{aligned} \sum_i \lambda_i \|B(\mu_i, v_i) - B(\mu'_i, v'_i)\| &\leq \sum_i \lambda_i \|\mu_i - \mu'_i\| + \sum_i \lambda_i \|(Q(\mu_i - \mu'_i), v_i - v'_i)\| \\ &\leq \|s - t\|_X + \theta \|s - t\|_X = (1 + \theta) \|s - t\|_X. \end{aligned} \quad (3.13)$$

For the second term, (3.10) identifies the two images under  $Q$  before we use the upper bound for  $h$ :

$$\begin{aligned} \sup_i \|Q(B(\mu_i, v_i)) - Q(B(\mu'_i, v'_i))\| &= \sup_i \|h(Q\mu_i, v_i) - h(Q\mu'_i, v'_i)\| \\ &\leq b \sup_i \|(Q(\mu_i - \mu'_i), v_i - v'_i)\| \\ &\leq b \|s - t\|_X. \end{aligned} \quad (3.14)$$

The  $Y$ -norm is the maximum of these two terms. Hence

$$\|\Psi(s) - \Psi(t)\|_Y \leq \max\{1 + \theta, b\} \|s - t\|_X.$$

Notice that the additional term in (3.13) is controlled by  $\theta$ , not by  $\|Q\|$ .

For the inverse estimate, take  $\nu = (\nu_i)_i, \nu' = (\nu'_i)_i \in Y$ , and define  $(u_i, v_i) = h^{-1}(Q\nu_i)$  and  $(u'_i, v'_i) = h^{-1}(Q\nu'_i)$ . The lower bound for  $h$  now reads

$$\begin{aligned} a \|(u_i - u'_i, v_i - v'_i)\| &\leq \|h(u_i, v_i) - h(u'_i, v'_i)\| \\ &= \|Q\nu_i - Q\nu'_i\| = \|Q(\nu_i - \nu'_i)\| \leq \|\nu - \nu'\|_Y. \end{aligned}$$

In particular,

$$\|(u_i - u'_i, v_i - v'_i)\| \leq \frac{1}{a} \|\nu - \nu'\|_Y \quad (i \geq 1).$$

Using the sum norm for the values of  $C$  and then the isometry of  $\delta$ , we get

$$\begin{aligned} \|C(\nu_i) - C(\nu'_i)\| &= \|\nu_i - \nu'_i - \delta(u_i, v_i) + \delta(u'_i, v'_i)\| + \|v_i - v'_i\| \\ &\leq \|\nu_i - \nu'_i\| + \|\delta(u_i, v_i) - \delta(u'_i, v'_i)\| + \|v_i - v'_i\| \\ &\leq \|\nu_i - \nu'_i\| + 2 \|(u_i - u'_i, v_i - v'_i)\|. \end{aligned}$$

The two parts of the  $X$ -norm of the inverse difference therefore satisfy

$$\begin{aligned} \sum_i \lambda_i \|C(\nu_i) - C(\nu'_i)\| &\leq \sum_i \lambda_i \|\nu_i - \nu'_i\| + 2 \sum_i \lambda_i \|(u_i - u'_i, v_i - v'_i)\| \\ &\leq \|\nu - \nu'\|_Y + \frac{2\theta}{a} \|\nu - \nu'\|_Y, \end{aligned} \quad (3.15)$$

$$\begin{aligned} \sup_i \|\tilde{Q}(C(\nu_i)) - \tilde{Q}(C(\nu'_i))\| &= \sup_i \|(u_i - u'_i, v_i - v'_i)\| \\ &\leq a^{-1} \|\nu - \nu'\|_Y. \end{aligned} \quad (3.16)$$

For the equality in the last calculation we used  $\tilde{Q}(C(\nu_i)) = (u_i, v_i)$  and its primed version, both proved in (3.12). Taking the maximum of these two bounds proves the inverse estimate in (3.7).

Finally, define

$$\iota : c_0(\ell_2) \longrightarrow X, \quad \iota((v_i)_i) = ((0, v_i))_i.$$

If  $(v_i)_i \in c_0(\ell_2)$ , then

$$\tilde{Q}(0, v_i) = (0, v_i) \longrightarrow (0, 0), \quad \sum_i \lambda_i \|v_i\| \leq \theta \sup_i \|v_i\| < \infty.$$

These are the two membership conditions for  $X$ . Since  $\theta \leq 1$ ,

$$\|\iota((v_i)_i)\|_X = \max \left\{ \sum_i \lambda_i \|v_i\|, \sup_i \|v_i\| \right\} = \sup_i \|v_i\|.$$

This proves that  $\iota$  is a linear isometry. On the other side, Proposition 3.4, with  $A = \mathcal{F}(\mathcal{H} \oplus_2 \ell_2)$  and the operator  $Q$  above, excludes  $c_0(\ell_2)$  from  $Y = Z_Q^\lambda$ . The complete continuity of  $Q$  is assumed here. The other hypothesis follows by applying Theorem 2.6 to the real Hilbert space  $\mathcal{H} \oplus_2 \ell_2$ . If  $X$  and  $Y$  were linearly isomorphic, composing that isomorphism with  $\iota$  would give a linear copy of  $c_0(\ell_2)$  in  $Y$ . This is impossible.  $\square$

**Corollary 3.6** (No additional distortion from the graph construction). *In Proposition 3.5, suppose  $0 < a < 1 < b$ . If*

$$0 < \theta \leq \min \left\{ 1, b - 1, \frac{1 - a}{2} \right\},$$

*then  $\text{Lip}(\Psi) \leq b$  and  $\text{Lip}(\Psi^{-1}) \leq a^{-1}$ .*

*Proof.* The bound  $\theta \leq b - 1$  gives  $1 + \theta \leq b$ . The bound  $2\theta \leq 1 - a$  gives

$$1 + \frac{2\theta}{a} \leq 1 + \frac{1 - a}{a} = \frac{1}{a}.$$

Substitution in the two maxima in (3.7) gives the asserted bounds.  $\square$

The linear input for the free space is already available from Theorem 2.6. It remains to construct a nearly isometric bijection  $h : \mathcal{H} \oplus_2 \ell_2 \rightarrow \mathcal{H}$  for which the linearization of  $(u, v) \mapsto h(u, v) - u$  is completely continuous. Section 4 gives a sufficient condition directly on the increments of this map; Section 5 constructs it.

## 4. Local orthogonality and complete continuity

We prove a sufficient condition for the linearization  $Q = \hat{D}$  to be completely continuous. The condition is imposed on  $D(x) - D(y)$  in neighborhoods of finitely many distinct points. The free-space results used in the proof are stated in Section 2.

### 4.1 The condition on increments

The following theorem is the increment form of [9, Theorem 4.1]. We first apply compact reduction, so no separability assumption on the whole metric space is required.

**Theorem 4.1** (Localized linearization). *Let  $(M, d, 0)$  be a complete pointed metric space, let  $H$  be a real Hilbert space, and let  $D : M \rightarrow H$  be Lipschitz with  $D(0) = 0$ . For every finite family of distinct points  $z_1, \dots, z_m \in M$ , suppose there exist open neighborhoods  $O_j$  of  $z_j$ , a finite-dimensional subspace  $F \subset H$ , and closed subspaces  $\mathcal{K}_j \subset F^\perp$  which are pairwise orthogonal, such that*

$$D(x) - D(y) \in F \oplus \mathcal{K}_j \quad (x, y \in O_j, \ 1 \leq j \leq m). \quad (4.1)$$

*Then the linearization  $Q = \hat{D} : \mathcal{F}(M) \rightarrow H$  is completely continuous.*

In Section 5, the target will be  $\mathcal{H} = (\bigoplus_{n \geq 1} H_n)_2$ , where each  $H_n$  is finite-dimensional. Let  $P_l$  be the orthogonal projection onto the first  $l$  blocks and let  $Q_l = I - P_l$ . The construction there gives

$$Q_l D(O_j) \subset \mathcal{K}_j \subset Q_l \mathcal{H}, \quad \mathcal{K}_j \perp \mathcal{K}_{j'} \quad (j \neq j').$$

For  $x, y \in O_j$ , the required implication is

$$\begin{aligned} D(x) - D(y) &= P_l(D(x) - D(y)) + Q_l D(x) - Q_l D(y), \\ P_l(D(x) - D(y)) &\in P_l \mathcal{H}, \\ Q_l D(x) - Q_l D(y) &\in \mathcal{K}_j. \end{aligned}$$

Thus (4.1) holds with  $F = P_l \mathcal{H}$ . The theorem itself requires only this condition on differences.

## 4.2 Tests confined to shrinking balls

We shall use the following consequence of the cutoff and summation lemmas. Its proof extracts functions with disjoint nonzero sets from functions supported in shrinking balls; compare [9, Lemma 4.2].

**Lemma 4.2** (Shrinking-ball tests). *Let  $\mu_n \rightarrow 0$  in  $\mathcal{F}(M)$ , let  $r_n > 0$  with  $r_n \rightarrow 0$ , and fix  $z \in M$ . Suppose  $f_n(z) = 0$ ,  $\text{Lip}(f_n) \leq L$ , and  $f_n(x) = 0$  whenever  $d(x, z) \geq r_n$ . Then  $\langle f_n, \mu_n \rangle \rightarrow 0$ .*

*Proof.* If the conclusion fails, there are a subsequence and  $\delta > 0$  on which  $|\langle f_n, \mu_n \rangle| \geq \delta$ . Use that subsequence for  $\mu_n, f_n, r_n$  and retain the same notation. We have  $L > 0$ , since  $L = 0$  and  $f_n(z) = 0$  would give  $f_n = 0$ .

We construct  $n_1 < n_2 < \dots$ , positive radii  $s_j$ , and functions  $u_j$  such that

$$\begin{aligned} 0 < s_j < r_{n_j}, \quad r_{n_{j+1}} &< \frac{s_j}{2}, \\ \{x \in M : u_j(x) \neq 0\} &\subset \{x \in M : s_j/2 < d(x, z) < r_{n_j}\}, \\ \text{Lip}(u_j) &\leq 3L, \quad |\langle u_j, \mu_{n_j} \rangle| > \frac{\delta}{2}. \end{aligned}$$

Set  $n_1 = 1$ . Suppose  $n_1, \dots, n_j$  and  $s_i, u_i$  for  $i < j$  have been chosen with these properties. In particular, if  $j > 1$ ,  $r_{n_j} < s_{j-1}/2$ . We next construct  $s_j, u_j$  and then choose  $n_{j+1}$ .

### 1. Make the inner part zero at the finite evaluation points.

By density of the evaluation span, choose

$$\nu_j = \sum_{k=1}^{N_j} a_{jk} \delta_M(x_{jk}), \quad \|\mu_{n_j} - \nu_j\| < \frac{\delta}{6L}.$$

There are only finitely many points in  $\{0, x_{j1}, \dots, x_{jN_j}\} \setminus \{z\}$ , and each has positive distance from  $z$ . We may therefore choose  $s_j$  with

$$0 < s_j < r_{n_j}, \quad s_j < d(x, z) \quad (x \in \{0, x_{j1}, \dots, x_{jN_j}\} \setminus \{z\}).$$

If that set is empty, the first condition alone suffices. Recall the cutoff from Lemma 2.3:

$$\chi_{z, s_j}(x) = \max \left\{ 0, \min \left\{ 1, 2 - \frac{2d(x, z)}{s_j} \right\} \right\}, \quad \begin{cases} \chi_{z, s_j}(x) = 1, & d(x, z) \leq s_j/2, \\ \chi_{z, s_j}(x) = 0, & d(x, z) \geq s_j. \end{cases}$$

Define  $p_j(x) = \chi_{z,s_j}(x)f_{n_j}(x)$ . For each  $x \in \{0, x_{j1}, \dots, x_{jN_j}\}$ ,

$$\begin{aligned} x = z &\implies p_j(x) = \chi_{z,s_j}(z)f_{n_j}(z) = 0, \\ x \neq z &\implies d(x, z) > s_j \implies \chi_{z,s_j}(x) = 0 \implies p_j(x) = 0. \end{aligned}$$

Consequently, using the pairing convention of Subsection 2.1,

$$\langle p_j, \nu_j \rangle = \sum_{k=1}^{N_j} a_{jk}(p_j(x_{jk}) - p_j(0)) = 0.$$

Since  $f_{n_j}(z) = 0$  and  $\text{Lip}(f_{n_j}) \leq L$ , applying Lemma 2.3 to  $f_{n_j}$  with center  $z$  and radius  $s_j$  gives

$$\text{Lip}(p_j) = \text{Lip}(\chi_{z,s_j}f_{n_j}) \leq 3L.$$

Hence

$$\begin{aligned} |\langle p_j, \mu_{n_j} \rangle| &= |\langle p_j, \mu_{n_j} - \nu_j \rangle| \\ &\leq \text{Lip}(p_j) \|\mu_{n_j} - \nu_j\| < 3L \frac{\delta}{6L} = \frac{\delta}{2}. \end{aligned}$$

Set

$$u_j = f_{n_j} - p_j = (1 - \chi_{z,s_j})f_{n_j}.$$

The second product estimate in the same lemma gives  $\text{Lip}(u_j) \leq 3L$ , and

$$\begin{aligned} |\langle u_j, \mu_{n_j} \rangle| &= |\langle f_{n_j}, \mu_{n_j} \rangle - \langle p_j, \mu_{n_j} \rangle| \\ &\geq |\langle f_{n_j}, \mu_{n_j} \rangle| - |\langle p_j, \mu_{n_j} \rangle| > \delta - \frac{\delta}{2} = \frac{\delta}{2}. \end{aligned}$$

We record these estimates as

$$|\langle u_j, \mu_{n_j} \rangle| > \delta/2, \quad \text{Lip}(u_j) \leq 3L. \quad (4.2)$$

## 2. Verify disjointness and continue the induction.

The two zero regions follow from the definitions:

$$\begin{aligned} d(x, z) \leq s_j/2 &\implies 1 - \chi_{z,s_j}(x) = 0 \implies u_j(x) = 0, \\ d(x, z) \geq r_{n_j} &\implies f_{n_j}(x) = 0 \implies u_j(x) = 0. \end{aligned}$$

Since  $r_n \rightarrow 0$ , choose  $n_{j+1} > n_j$  with  $r_{n_{j+1}} < s_j/2$ . This verifies all the conditions at the new index without changing any earlier choice. The radii along the chosen subsequence satisfy

$$r_{n_{j+1}} < s_j/2 < r_{n_j}.$$

For  $i > j$ , successive applications give  $r_{n_i} \leq r_{n_{j+1}} < s_j/2$ . Thus

$$\begin{aligned} u_i(x) \neq 0 &\implies d(x, z) < r_{n_i} < s_j/2, \\ u_j(x) \neq 0 &\implies d(x, z) > s_j/2. \end{aligned}$$

The nonzero sets of the  $u_j$  are therefore pairwise disjoint.

### 3. Apply the summation lemma.

Lemma 2.4, applied to these nonzero sets and the bound  $3L$ , gives

$$\sum_{j \geq 1} |u_j(x) - u_j(y)| \leq 2(3L)d(x, y) = 6Ld(x, y).$$

Apply Lemma 2.2 to the functions  $u_j$  with  $C = 6L$ . The pairing convention already subtracts  $u_j(0)$ , so its conclusion is

$$A : \mathcal{F}(M) \longrightarrow \ell_1, \quad A\mu = (\langle u_j, \mu \rangle)_{j \geq 1}, \quad \|A\| \leq 6L, \quad \|A\mu_n\|_1 \longrightarrow 0.$$

At  $n = n_j$ , this contradicts (4.2), since

$$\frac{\delta}{2} < |\langle u_j, \mu_{n_j} \rangle| = |(A\mu_{n_j})_j| \leq \|A\mu_{n_j}\|_1 \longrightarrow 0. \quad \square$$

**Corollary 4.3.** *Let  $\mu_n \rightarrow 0$  in  $\mathcal{F}(M)$  and fix  $z \in M$ . Suppose that there are constants  $r_0 > 0$  and  $L \geq 0$  and, for every  $n \geq 1$  and  $0 < r \leq r_0$ , a function  $f_{n,r} : M \rightarrow \mathbb{R}$  such that*

$$\begin{aligned} f_{n,r}(z) &= 0, & \text{Lip}(f_{n,r}) &\leq L, \\ f_{n,r}(x) &= 0 & (x \in M, d(x, z) \geq r). \end{aligned}$$

Then

$$\lim_{r \downarrow 0} \limsup_{n \rightarrow \infty} |\langle f_{n,r}, \mu_n \rangle| = 0.$$

*Proof.* Otherwise there are  $a > 0$  and  $0 < r_j \leq r_0$ , with  $r_j \downarrow 0$ , such that

$$\limsup_{n \rightarrow \infty} |\langle f_{n,r_j}, \mu_n \rangle| \geq 2a \quad (j \geq 1).$$

For each fixed  $j$ , the set  $\{n : |\langle f_{n,r_j}, \mu_n \rangle| \geq a\}$  is infinite: if it were finite, the upper limit would be at most  $a$ . Starting with  $n_0 = 0$ , choose successively

$$n_j > n_{j-1}, \quad |\langle f_{n_j,r_j}, \mu_{n_j} \rangle| \geq a.$$

Apply Lemma 4.2 to the vectors  $\mu_{n_j}$ , functions  $f_{n_j,r_j}$ , radii  $r_j$ , and the same center  $z$ . Its hypotheses hold because  $\mu_{n_j} \rightarrow 0$  and

$$f_{n_j,r_j}(z) = 0, \quad \text{Lip}(f_{n_j,r_j}) \leq L, \quad d(x, z) \geq r_j \implies f_{n_j,r_j}(x) = 0.$$

Its conclusion contradicts the choice of  $n_j$ :

$$a \leq |\langle f_{n_j,r_j}, \mu_{n_j} \rangle| \longrightarrow 0. \quad \square$$

### 4.3 Local estimates on one compact set

If  $f_n(x) \rightarrow 0$  for every  $x$  and  $\text{Lip}(f_n) \leq L$ , the convergence is uniform on a compact set  $K$ . Indeed, for a finite  $\eta$ -net  $\{x_1, \dots, x_m\} \subset K$  and  $x \in K$ , choose  $j$  with  $d(x, x_j) < \eta$ . Then

$$|f_n(x)| \leq |f_n(x_j)| + |f_n(x) - f_n(x_j)| \leq \max_{1 \leq j \leq m} |f_n(x_j)| + L\eta.$$

Taking the supremum gives

$$\sup_K |f_n| \leq \max_{1 \leq j \leq m} |f_n(x_j)| + L\eta. \quad (4.3)$$

The maximum involves finitely many fixed points, so it tends to zero. Consequently  $0 \leq \limsup_n \sup_K |f_n| \leq L\eta$  for every  $\eta > 0$ , and  $\sup_K |f_n| \rightarrow 0$ .

**Lemma 4.4** (Compact local-to-global estimate). *Let  $K$  be a nonempty compact metric space and let  $V_1, \dots, V_m$  be a relatively open cover. There exists  $\lambda > 0$  such that any two points of  $K$  at distance less than  $\lambda$  belong to one common  $V_j$ . For any such  $\lambda$ , if  $f : K \rightarrow \mathbb{R}$  satisfies  $\sup_K |f| \leq \eta$  and  $\text{Lip}(f|_{V_j}) \leq c_j$ , then*

$$\text{Lip}(f) \leq \max \left\{ \max_{j \leq m} c_j, \frac{2\eta}{\lambda} \right\}. \quad (4.4)$$

*Proof.* For each  $x \in K$ , choose  $r_x > 0$  and  $j(x)$  such that  $B_K(x, 2r_x) \subset V_{j(x)}$ . Compactness gives points  $x_1, \dots, x_s$  with

$$K = \bigcup_{t=1}^s B_K(x_t, r_{x_t}), \quad \lambda = \min_{1 \leq t \leq s} r_{x_t} > 0.$$

If  $a, b \in K$  and  $d(a, b) < \lambda$ , choose  $t$  with  $a \in B_K(x_t, r_{x_t})$ . Then

$$d(x_t, b) \leq d(x_t, a) + d(a, b) < r_{x_t} + \lambda \leq 2r_{x_t},$$

so  $a, b \in B_K(x_t, 2r_{x_t}) \subset V_{j(x_t)}$ . Let  $a, b \in K$  be distinct. If  $d(a, b) < \lambda$ , the preceding argument gives an index  $j$  with  $a, b \in V_j$ . Hence

$$\frac{|f(a) - f(b)|}{d(a, b)} \leq \text{Lip}(f|_{V_j}) \leq c_j \leq \max_{1 \leq j \leq m} c_j.$$

If  $d(a, b) \geq \lambda$ , then

$$\frac{|f(a) - f(b)|}{d(a, b)} \leq \frac{|f(a)| + |f(b)|}{d(a, b)} \leq \frac{2\eta}{d(a, b)} \leq \frac{2\eta}{\lambda}.$$

Thus in either case

$$\frac{|f(a) - f(b)|}{d(a, b)} \leq \max \left\{ \max_{1 \leq j \leq m} c_j, \frac{2\eta}{\lambda} \right\}.$$

Taking the supremum over distinct  $a, b \in K$  proves (4.4). If  $K$  is a singleton, the same conclusion holds with  $\text{Lip}(f) = 0$ .  $\square$

#### 4.4 Proof: a finite exceptional set and a small remainder

*Proof of Theorem 4.1.* Put  $L = \text{Lip}(D) = \|Q\|$ . If  $L = 0$ , then  $D(0) = 0$  implies  $D = 0$  and  $Q = 0$ . Suppose  $L > 0$  and  $Q$  is not completely continuous. Passing to a subsequence, we obtain

$$\mu_i \rightharpoonup 0 \text{ in } \mathcal{F}(M), \quad \|Q\mu_i\| \geq \epsilon_0 > 0 \quad (i \geq 1).$$

We keep this sequence and this lower bound fixed.

## 1. Scalar functions and compact reduction.

Define

$$\begin{aligned} b_i &= \frac{Q\mu_i}{\|Q\mu_i\|} \in H, & d_i(x) &= \langle D(x), b_i \rangle \quad (x \in M), \\ B &= \sup_i \|\mu_i\| < \infty. \end{aligned} \tag{4.5}$$

Weak convergence implies the last bound. For any fixed  $b \in H$ ,  $\mu \mapsto \langle Q\mu, b \rangle$  is a bounded linear functional, so

$$|\langle b_i, b \rangle| = \frac{|\langle Q\mu_i, b \rangle|}{\|Q\mu_i\|} \leq \frac{|\langle Q\mu_i, b \rangle|}{\epsilon_0} \longrightarrow 0.$$

Thus  $\|b_i\| = 1$  and  $b_i \rightharpoonup 0$ . In particular,  $d_i(x) \rightarrow 0$  for each  $x \in M$ . Moreover,

$$d_i(0) = 0, \quad |d_i(x) - d_i(y)| = |\langle D(x) - D(y), b_i \rangle| \leq Ld(x, y).$$

The linearization identity  $Q\delta_M(x) = D(x)$  gives

$$\langle d_i, \delta_M(x) \rangle = d_i(x) = \langle D(x), b_i \rangle = \langle Q\delta_M(x), b_i \rangle.$$

Both sides define continuous linear functionals on  $\mathcal{F}(M)$ , and the evaluations have dense linear span. Hence

$$\langle d_i, \mu_i \rangle = \langle Q\mu_i, b_i \rangle = \|Q\mu_i\| \geq \epsilon_0. \tag{4.6}$$

Choose

$$\tau = \frac{\epsilon_0}{56L}.$$

Apply Theorem 2.5 with the complete domain  $M$ , the weakly null sequence  $(\mu_i)$ , and this tolerance  $\tau$ . It gives a compact  $K \subset M$  containing 0. The distance estimate in that theorem permits us to choose  $\nu_i \in \mathcal{F}(K)$  with

$$\|\mu_i - \nu_i\| < \tau, \quad \|\nu_i\| \leq \|\mu_i\| + \|\nu_i - \mu_i\| \leq B + \tau. \tag{4.7}$$

We have proved that  $d_i(x) \rightarrow 0$  for every  $x \in K$  and  $\text{Lip}(d_i|_K) \leq L$ . Since  $K$  is compact, the uniform-convergence conclusion following (4.3), applied to  $(d_i|_K)_i$ , gives

$$\epsilon_i := \sup_{x \in K} |d_i(x)| \longrightarrow 0. \tag{4.8}$$

## 2. Local Lipschitz constants and the exceptional set.

A compact metric space has a countable base; choose  $\mathcal{B} = \{O_1, O_2, \dots\}$  for the relative topology of  $K$ . For instance, relative balls with rational radii and centers in the union of finite  $1/n$ -nets give such a base. A Lipschitz constant on an empty set or a singleton is understood to be zero.

We first choose a subsequence on which every  $\text{Lip}(d_i|_{O_s})$  converges. Set  $I_0 = \mathbb{N}$ . Inductively, assume  $I_{r-1}$  is infinite and that the limits for  $1 \leq s < r$  exist through  $I_{r-1}$ . Since

$$0 \leq \text{Lip}(d_i|_{O_r}) \leq L \quad (i \in I_{r-1}),$$

choose an infinite  $I_r \subset I_{r-1}$  on which this real sequence converges. Each earlier limit persists because  $I_r \subset I_{r-1}$ . Choose  $i_r \in I_r$  with  $i_r > i_{r-1}$ , starting with  $i_0 = 0$ . For each fixed  $s$ ,

$$r \geq s \implies i_r \in I_r \subset I_s,$$

so  $\text{Lip}(d_{i_r}|_{O_s})$  converges. Restrict  $\mu_i, \nu_i, b_i, d_i, \epsilon_i$  to this common subsequence and use the same notation. Set

$$a_O = \lim_{i \rightarrow \infty} \text{Lip}(d_i|_O) \quad (O \in \mathcal{B}), \quad \eta = \frac{\epsilon_0}{8(B + \tau)}, \quad (4.9)$$

$$Z = \{z \in K : a_O \geq \eta \text{ for every } O \in \mathcal{B} \text{ containing } z\}.$$

Take any distinct  $z_1, \dots, z_m \in Z$ . Apply the hypothesis of Theorem 4.1 to these points. It gives open  $O_j^{\text{amb}} \subset M$  containing  $z_j$ , a finite-dimensional  $F \subset H$ , and pairwise orthogonal  $\mathcal{K}_j \subset F^\perp$ , with (4.1) on each  $O_j^{\text{amb}}$ . Choose  $V_j \in \mathcal{B}$  with

$$z_j \in V_j \subset O_j^{\text{amb}} \cap K.$$

Recall that (4.1) and  $V_j \subset O_j^{\text{amb}} \cap K$  give

$$D(x) - D(y) \in F \oplus \mathcal{K}_j \quad (x, y \in V_j).$$

Let  $P$  and  $R_j$  be the orthogonal projections onto  $F$  and  $\mathcal{K}_j$ , respectively. Since these subspaces are orthogonal,

$$D(x) - D(y) = P(D(x) - D(y)) + R_j(D(x) - D(y)) \quad (x, y \in V_j).$$

For any orthogonal projection  $P$ , the vectors  $Pu$  and  $(I - P)v$  are orthogonal. Thus

$$\langle Pu, v \rangle = \langle Pu, Pv \rangle + \langle Pu, (I - P)v \rangle = \langle Pu, Pv \rangle \quad (u, v \in H).$$

Applying this identity to  $P$  and to  $R_j$  gives

$$\begin{aligned} d_i(x) - d_i(y) &= \langle D(x) - D(y), b_i \rangle \\ &= \langle P(D(x) - D(y)), b_i \rangle + \langle R_j(D(x) - D(y)), b_i \rangle \\ &= \langle P(D(x) - D(y)), Pb_i \rangle + \langle R_j(D(x) - D(y)), R_j b_i \rangle. \end{aligned}$$

Consequently, using the projection norm bounds and then  $\text{Lip}(D) = L$ ,

$$\begin{aligned} |d_i(x) - d_i(y)| &\leq \|P(D(x) - D(y))\| \|Pb_i\| + \|R_j(D(x) - D(y))\| \|R_j b_i\| \\ &\leq \|D(x) - D(y)\| (\|Pb_i\| + \|R_j b_i\|) \\ &\leq Ld(x, y) (\|Pb_i\| + \|R_j b_i\|). \end{aligned}$$

Divide by  $d(x, y) > 0$  and take the supremum over distinct points of  $V_j$ :

$$\text{Lip}(d_i|_{V_j}) \leq L(\|Pb_i\| + \|R_j b_i\|).$$

Since  $L > 0$ , division by  $L$  and subtraction of  $\|Pb_i\|$  give

$$\|R_j b_i\| \geq \frac{\text{Lip}(d_i|_{V_j})}{L} - \|Pb_i\|. \quad (4.10)$$

For an orthonormal basis  $e_1, \dots, e_d$  of  $F$ , weak convergence of  $b_i$  gives

$$\|Pb_i\|^2 = \sum_{r=1}^d |\langle b_i, e_r \rangle|^2 \longrightarrow 0.$$

Since  $z_j \in V_j \cap Z$ , we have  $a_{V_j} \geq \eta$ . Therefore

$$\liminf_i \|R_j b_i\| \geq \frac{a_{V_j}}{L} \geq \frac{\eta}{L}.$$

The mutually orthogonal ranges of  $R_1, \dots, R_m$  imply

$$\sum_{j=1}^m \|R_j b_i\|^2 \leq \|b_i\|^2 = 1.$$

Combining these estimates for the finite sum gives

$$m \frac{\eta^2}{L^2} \leq \sum_{j=1}^m \liminf_i \|R_j b_i\|^2 \leq \liminf_i \sum_{j=1}^m \|R_j b_i\|^2 \leq 1.$$

If  $Z$  had more than  $L^2/\eta^2$  points, we could choose  $m$  distinct points in  $Z$  with  $m > L^2/\eta^2$ , contradicting this inequality. Hence

$$|Z| \leq L^2/\eta^2 < \infty. \quad (4.11)$$

### 3. Cut off neighborhoods of the exceptional points.

If  $|Z| \geq 2$ , set

$$r_0 = \frac{1}{3} \min\{d(z, z') : z, z' \in Z, z \neq z'\} > 0;$$

if  $|Z| \leq 1$ , choose any  $r_0 > 0$ . For  $0 < r < r_0$ , the balls  $B(z, r)$ ,  $z \in Z$ , are pairwise disjoint. Indeed, a point  $x \in B(z, r) \cap B(z', r)$  with  $z \neq z'$  would give

$$d(z, z') \leq d(z, x) + d(x, z') < 2r < 3r_0 \leq d(z, z'),$$

which is impossible.

For  $z \in Z$ , use the cutoff

$$\chi_{z,r}(x) = \max \left\{ 0, \min \left\{ 1, 2 - \frac{2d(x, z)}{r} \right\} \right\}, \quad \begin{cases} \chi_{z,r}(x) = 1, & d(x, z) \leq r/2, \\ \chi_{z,r}(x) = 0, & d(x, z) \geq r, \end{cases}$$

and define

$$\begin{aligned} f_{i,z,r}(x) &= \chi_{z,r}(x)(d_i(x) - d_i(z)), \\ g_{i,r}(x) &= \sum_{z \in Z} f_{i,z,r}(x), \quad R_{i,r}(x) = d_i(x) - g_{i,r}(x). \end{aligned}$$

The function  $d_i - d_i(z)$  vanishes at  $z$  and has Lipschitz constant at most  $L$ . Lemma 2.3, with this function and radius  $r$ , gives  $\text{Lip}(f_{i,z,r}) \leq 3L$ . Also

$$f_{i,z,r}(z) = 0, \quad \{x : f_{i,z,r}(x) \neq 0\} \subset B(z, r).$$

For fixed  $i, r$ , these nonzero sets are disjoint. Applying Lemma 2.4 with bound  $3L$ , we obtain

$$\begin{aligned} |g_{i,r}(x) - g_{i,r}(y)| &\leq \sum_{z \in Z} |f_{i,z,r}(x) - f_{i,z,r}(y)| \\ &\leq 6Ld(x, y). \end{aligned}$$

Consequently,

$$\text{Lip}(g_{i,r}) \leq 6L, \quad \text{Lip}(R_{i,r}) \leq \text{Lip}(d_i) + \text{Lip}(g_{i,r}) \leq 7L. \quad (4.12)$$

Apply Corollary 4.3 for each fixed  $z \in Z$ , with vectors  $\mu_i$ , functions  $f_{i,z,r}$ , and common bound  $3L$ . It gives

$$\lim_{r \downarrow 0} \limsup_i |\langle f_{i,z,r}, \mu_i \rangle| = 0.$$

If  $Z \neq \emptyset$ , choose  $0 < r < r_0$  small enough that

$$\limsup_i |\langle f_{i,z,r}, \mu_i \rangle| \leq \frac{\epsilon_0}{4|Z|} \quad (z \in Z).$$

One radius satisfies all these conditions because  $Z$  is finite. For this radius,

$$\begin{aligned} \limsup_i |\langle g_{i,r}, \mu_i \rangle| &\leq \sum_{z \in Z} \limsup_i |\langle f_{i,z,r}, \mu_i \rangle| \\ &\leq |Z| \frac{\epsilon_0}{4|Z|} = \frac{\epsilon_0}{4}. \end{aligned}$$

If  $Z = \emptyset$ , the sum defining  $g_{i,r}$  is zero, so any  $0 < r < r_0$  gives the same conclusion. In either case fix this radius; then

$$\limsup_i |\langle g_{i,r}, \mu_i \rangle| \leq \epsilon_0/4. \quad (4.13)$$

#### 4. Estimate the remainder on neighborhoods in $K$ .

Put  $m = |Z|$  and  $\Phi(x) = \sum_{z \in Z} \chi_{z,r}(x)$ . Disjointness of the balls and  $0 \leq \chi_{z,r} \leq 1$  imply  $0 \leq \Phi \leq 1$ . At every  $x \in M$ ,

$$\begin{aligned} R_{i,r}(x) &= d_i(x) - \sum_{z \in Z} \chi_{z,r}(x)(d_i(x) - d_i(z)) \\ &= (1 - \Phi(x))d_i(x) + \sum_{z \in Z} \chi_{z,r}(x)d_i(z). \end{aligned} \quad (4.14)$$

Since  $Z \subset K$  and  $\epsilon_i = \sup_K |d_i|$ , for  $x \in K$  this gives

$$\begin{aligned} |R_{i,r}(x)| &\leq (1 - \Phi(x))\epsilon_i + \sum_{z \in Z} \chi_{z,r}(x)\epsilon_i \\ &= (1 - \Phi(x) + \Phi(x))\epsilon_i = \epsilon_i. \end{aligned} \quad (4.15)$$

If  $x \in B(z, r/2)$  for some  $z \in Z$ , then  $\chi_{z,r}(x) = 1$  and every other cutoff is zero. Thus

$$R_{i,r}(x) = d_i(z) \quad (x \in B(z, r/2)), \quad \text{Lip}(R_{i,r}|_{K \cap B(z, r/2)}) = 0.$$

For  $p \in K \setminus Z$ , the negation of the defining condition in (4.9) gives  $O_p \in \mathcal{B}$  with

$$p \in O_p, \quad a_{O_p} = \lim_i \text{Lip}(d_i|_{O_p}) < \eta.$$

For distinct  $x, y \in O_p$ , subtract (4.14) at these points and add and subtract  $(1 - \Phi(x))d_i(y)$ :

$$\begin{aligned} R_{i,r}(x) - R_{i,r}(y) &= (1 - \Phi(x))d_i(x) - (1 - \Phi(y))d_i(y) \\ &\quad + \sum_{z \in Z} (\chi_{z,r}(x) - \chi_{z,r}(y))d_i(z) \\ &= (1 - \Phi(x))d_i(x) - (1 - \Phi(x))d_i(y) \\ &\quad + (1 - \Phi(x))d_i(y) - (1 - \Phi(y))d_i(y) \\ &\quad + \sum_{z \in Z} (\chi_{z,r}(x) - \chi_{z,r}(y))d_i(z) \\ &= (1 - \Phi(x))(d_i(x) - d_i(y)) \\ &\quad + (\Phi(y) - \Phi(x))d_i(y) \\ &\quad + \sum_{z \in Z} (\chi_{z,r}(x) - \chi_{z,r}(y))d_i(z) \\ &= (1 - \Phi(x))(d_i(x) - d_i(y)) \\ &\quad + \sum_{z \in Z} (\chi_{z,r}(x) - \chi_{z,r}(y))(d_i(z) - d_i(y)). \end{aligned}$$

The last equality uses  $\Phi(y) - \Phi(x) = -\sum_{z \in Z} (\chi_{z,r}(x) - \chi_{z,r}(y))$ . By Lemma 2.3 and  $z, y \in K$ ,

$$|\chi_{z,r}(x) - \chi_{z,r}(y)| \leq \frac{2}{r}d(x, y), \quad |d_i(z) - d_i(y)| \leq |d_i(z)| + |d_i(y)| \leq 2\epsilon_i.$$

It follows that

$$\begin{aligned} |R_{i,r}(x) - R_{i,r}(y)| &\leq \text{Lip}(d_i|_{O_p})d(x, y) + \sum_{z \in Z} \frac{2}{r}d(x, y) 2\epsilon_i \\ &= \left( \text{Lip}(d_i|_{O_p}) + \frac{4m}{r}\epsilon_i \right) d(x, y). \end{aligned}$$

Taking difference quotients gives

$$\text{Lip}(R_{i,r}|_{O_p}) \leq \text{Lip}(d_i|_{O_p}) + \frac{4m}{r}\epsilon_i. \quad (4.16)$$

Here  $m, r$  are fixed,  $\epsilon_i \rightarrow 0$ , and the right side tends to  $a_{O_p} < \eta$ .

## 5. Apply the compact local-to-global estimate and conclude.

The relatively open sets

$$\{K \cap B(z, r/2) : z \in Z\} \cup \{O_p : p \in K \setminus Z\}$$

cover  $K$ . By compactness, finitely many of the sets  $O_p$ , together with the finitely many balls, still cover  $K$ . Write them as  $O_{p_1}, \dots, O_{p_s}$ ;  $s = 0$  is allowed when the balls already cover  $K$ . Choose a Lebesgue number  $\lambda > 0$  for this finite cover as in Lemma 4.4.

For each  $1 \leq j \leq s$ , (4.16) bounds  $\text{Lip}(R_{i,r}|_{O_{p_j}})$  by

$$\text{Lip}(d_i|_{O_{p_j}}) + \frac{4m}{r}\epsilon_i \rightarrow a_{O_{p_j}} < \eta.$$

There are only finitely many  $j$ . On each  $K \cap B(z, r/2)$  the Lipschitz constant is zero, and  $2\epsilon_i/\lambda \rightarrow 0$ . Hence there is  $i_0$  such that, for  $i \geq i_0$ , every member of this cover has restricted Lipschitz constant at most  $\eta$ , and  $2\epsilon_i/\lambda \leq \eta$ . Apply Lemma 4.4 to  $f = R_{i,r}|_K$ , this fixed cover, local bounds  $c_j = \eta$ , and the uniform bound  $\sup_K |R_{i,r}| \leq \epsilon_i$  from (4.15). We obtain

$$\text{Lip}(R_{i,r}|_K) \leq \max \left\{ \eta, \frac{2\epsilon_i}{\lambda} \right\} = \eta \quad (i \geq i_0).$$

In particular,

$$\limsup_i \text{Lip}(R_{i,r}|_K) \leq \eta. \quad (4.17)$$

Recall from (4.7) that  $\nu_i \in \mathcal{F}(K)$ ,  $\|\nu_i\| \leq B + \tau$ , and  $\|\mu_i - \nu_i\| < \tau$ . The pairing subtracts  $R_{i,r}(0)$ , and  $0 \in K$ . For the pairing with  $\nu_i$  use the restricted Lipschitz norm in (2.2); for the pairing with  $\mu_i - \nu_i \in \mathcal{F}(M)$  use the global bound (4.12). These two bounds give

$$\begin{aligned} |\langle R_{i,r}, \mu_i \rangle| &\leq |\langle R_{i,r}, \nu_i \rangle| + |\langle R_{i,r}, \mu_i - \nu_i \rangle| \\ &\leq \text{Lip}(R_{i,r}|_K) \|\nu_i\| + \text{Lip}(R_{i,r}) \|\mu_i - \nu_i\| \\ &\leq (B + \tau) \text{Lip}(R_{i,r}|_K) + 7L\tau. \end{aligned}$$

Since  $\eta = \epsilon_0/(8(B + \tau))$  and  $\tau = \epsilon_0/(56L)$ , taking upper limits gives

$$\begin{aligned} \limsup_i |\langle R_{i,r}, \mu_i \rangle| &\leq (B + \tau)\eta + 7L\tau \\ &= (B + \tau) \frac{\epsilon_0}{8(B + \tau)} + 7L \frac{\epsilon_0}{56L} = \frac{\epsilon_0}{4}. \end{aligned}$$

Finally  $d_i = R_{i,r} + g_{i,r}$ , so (4.13) yields

$$\begin{aligned} \epsilon_0 &\leq \limsup_i |\langle d_i, \mu_i \rangle| && \text{by (4.6)} \\ &\leq \limsup_i |\langle R_{i,r}, \mu_i \rangle| + \limsup_i |\langle g_{i,r}, \mu_i \rangle| \\ &\leq \frac{\epsilon_0}{4} + \frac{\epsilon_0}{4} = \frac{\epsilon_0}{2}, \end{aligned}$$

which contradicts  $\epsilon_0 > 0$ .  $\square$

## 5. The Hilbert-space construction

We now construct the map required by Proposition 3.5. The only part retained from the fixed-direction construction in [9, Section 5] is the linear isometry in Lemma 5.2. The unit direction in the  $n$ th block will subsequently depend on the output vector  $y \in \mathcal{H}$ . We first define an inverse map

$$\Phi : \mathcal{H} \longrightarrow \mathcal{H} \oplus_2 \ell_2, \quad \Phi(y) = L_{w(y)}^{-1} y,$$

prove its two-sided distance estimate directly, and then solve  $\Phi(y) = (u, v)$  one block at a time. The desired map is  $h = \Phi^{-1}$ .

Throughout this section,

$$0 < c < \frac{1}{4}.$$

The number  $c$  controls every deviation from a fixed-direction isometry.

**Theorem 5.1.** *There are nonzero finite-dimensional real Hilbert spaces  $H_n$  such that, with*

$$\mathcal{H} = \left( \bigoplus_{n \geq 1} H_n \right)_2,$$

*there is a bijection*

$$h : \mathcal{H} \oplus_2 \ell_2 \longrightarrow \mathcal{H}, \quad h(0, 0) = 0,$$

*for which*

$$\begin{aligned} \|h(u, v)\| &= \|(u, v)\|, \\ (1 - 4c) \|(u - u', v - v')\| &\leq \|h(u, v) - h(u', v')\| \leq (1 + 4c) \|(u - u', v - v')\| \end{aligned} \quad (5.1)$$

*for all  $(u, v), (u', v') \in \mathcal{H} \oplus_2 \ell_2$ .*

*Define*

$$D : \mathcal{H} \oplus_2 \ell_2 \longrightarrow \mathcal{H}, \quad D(u, v) = h(u, v) - u.$$

*Let  $P_l : \mathcal{H} \rightarrow \mathcal{H}$  be the orthogonal projection onto the first  $l$  blocks and put  $Q_l = I - P_l$ . For every finite family of distinct points*

$$(u^{(1)}, v^{(1)}), \dots, (u^{(m)}, v^{(m)}) \in \mathcal{H} \oplus_2 \ell_2,$$

*there are open neighborhoods  $O_j$  of these points, an integer  $l \geq 1$ , and pairwise orthogonal closed subspaces*

$$\mathcal{K}_j \subset Q_l \mathcal{H}$$

*such that*

$$Q_l D(O_j) \subset \mathcal{K}_j \quad (1 \leq j \leq m). \quad (5.2)$$

### 5.1 A rearrangement with one fixed direction in each block

For the moment, let  $H_n$  be arbitrary nonzero finite-dimensional Hilbert spaces, and choose a unit vector  $\xi_n \in H_n$  for each  $n$ . For  $u = (u_n) \in \mathcal{H}$ , put

$$\alpha_n = \langle \xi_n, u_n \rangle, \quad u_n^\perp = u_n - \alpha_n \xi_n.$$

Then

$$\langle \xi_n, u_n^\perp \rangle = \langle \xi_n, u_n \rangle - \alpha_n \|\xi_n\|^2 = 0.$$

The rearrangement keeps  $u_n^\perp$  in block  $n$ , sends the scalar  $\alpha_m$  to block  $2m$ , and sends the additional scalar  $v_m$  to block  $2m-1$ . Thus the first four output blocks are

| block | output vector                  |
|-------|--------------------------------|
| 1     | $u_1^\perp + v_1 \xi_1$        |
| 2     | $u_2^\perp + \alpha_1 \xi_2$   |
| 3     | $u_3^\perp + v_2 \xi_3$        |
| 4     | $u_4^\perp + \alpha_2 \xi_4$ . |

For every  $m \geq 1$ , define

$$C_{2m} = \alpha_m, \quad C_{2m-1} = v_m,$$

and set

$$(L_\xi(u, v))_n = u_n^\perp + C_n \xi_n. \quad (5.3)$$

**Lemma 5.2** (Frozen rearrangement). *Formula (5.3) defines a surjective linear isometry*

$$L_\xi : \mathcal{H} \oplus_2 \ell_2 \longrightarrow \mathcal{H}.$$

For  $y = (y_n) \in \mathcal{H}$ , put

$$\beta_n = \langle \xi_n, y_n \rangle.$$

Then

$$L_\xi^{-1}y = (u, v),$$

where

$$u_m = y_m - \beta_m \xi_m + \beta_{2m} \xi_m, \quad v_m = \beta_{2m-1}. \quad (5.4)$$

*Proof.* Since  $u_n^\perp \perp \xi_n$  and  $\|\xi_n\| = 1$ ,

$$\|(L_\xi(u, v))_n\|^2 = \|u_n^\perp\|^2 + |C_n|^2, \quad \|u_n^\perp\|^2 = \|u_n\|^2 - |\alpha_n|^2.$$

Moreover,

$$\begin{aligned} \sum_{n \geq 1} |C_n|^2 &= \sum_{m \geq 1} |C_{2m}|^2 + \sum_{m \geq 1} |C_{2m-1}|^2 \\ &= \sum_{m \geq 1} |\alpha_m|^2 + \sum_{m \geq 1} |v_m|^2. \end{aligned}$$

Therefore

$$\begin{aligned} \|L_\xi(u, v)\|^2 &= \sum_{n \geq 1} (\|u_n\|^2 - |\alpha_n|^2) + \sum_{m \geq 1} |\alpha_m|^2 + \sum_{m \geq 1} |v_m|^2 \\ &= \|u\|^2 + \|v\|^2 = \|(u, v)\|^2. \end{aligned}$$

The formulas are linear, so  $L_\xi$  is a linear isometry and hence is injective.

Fix  $y \in \mathcal{H}$  and define  $(u, v)$  by (5.4). Since  $|\beta_n| \leq \|y_n\|$ ,  $(\beta_n) \in \ell_2$ . The vectors  $y_m - \beta_m \xi_m$  and  $\beta_{2m} \xi_m$  are orthogonal, and hence

$$\begin{aligned} \sum_{m \geq 1} (\|u_m\|^2 + |v_m|^2) &= \sum_{m \geq 1} (\|y_m\|^2 - |\beta_m|^2 + |\beta_{2m}|^2 + |\beta_{2m-1}|^2) \\ &= \sum_{m \geq 1} \|y_m\|^2. \end{aligned}$$

Thus  $(u, v) \in \mathcal{H} \oplus_2 \ell_2$ . For this pair,

$$\begin{aligned} \alpha_m &= \langle \xi_m, u_m \rangle \\ &= \langle \xi_m, y_m - \beta_m \xi_m + \beta_{2m} \xi_m \rangle \\ &= \beta_m - \beta_m + \beta_{2m} = \beta_{2m}. \end{aligned}$$

Consequently,

$$u_m^\perp = u_m - \alpha_m \xi_m = y_m - \beta_m \xi_m,$$

and the scalar used in block  $n$  is  $C_n = \beta_n$ . Substitution into (5.3) gives

$$(L_\xi(u, v))_n = (y_n - \beta_n \xi_n) + \beta_n \xi_n = y_n.$$

Hence  $L_\xi$  is onto, and (5.4) is its inverse.  $\square$

## 5.2 Changing the fixed directions

The construction below replaces the fixed unit vector  $\xi_n$  in each block by a unit vector  $w_n(y)$  depending on the output point  $y$ . We shall compare two such choices by applying the explicit inverse formula (5.4) coordinate by coordinate.

**Lemma 5.3.** *Let  $\xi_n, \zeta_n \in H_n$  be unit vectors for every  $n \geq 1$ . Then, for every  $y = (y_n) \in \mathcal{H}$ ,*

$$\|L_\xi^{-1}y - L_\zeta^{-1}y\| \leq 4 \left( \sum_{n \geq 1} (\|y_n\|^2 + \|y_{2n}\|^2) \|\xi_n - \zeta_n\|^2 \right)^{1/2}. \quad (5.5)$$

*Proof.* For  $\omega \in \{\xi, \zeta\}$ , write

$$L_\omega^{-1}y = (U^\omega(y), V^\omega(y)).$$

The inverse formula (5.4) reads

$$\begin{aligned} U_n^\omega(y) &= y_n + (\langle \omega_{2n}, y_{2n} \rangle - \langle \omega_n, y_n \rangle) \omega_n, \\ V_n^\omega(y) &= \langle \omega_{2n-1}, y_{2n-1} \rangle. \end{aligned}$$

Set  $\delta_n = \langle \xi_n - \zeta_n, y_n \rangle$ , and define

$$E^2 = \sum_{n \geq 1} (\|y_n\|^2 + \|y_{2n}\|^2) \|\xi_n - \zeta_n\|^2.$$

Since  $\|\xi_n - \zeta_n\| \leq 2$ , the series defining  $E$  is finite. Moreover,

$$\sum_{n \geq 1} |\delta_n|^2 \leq \sum_{n \geq 1} \|y_n\|^2 \|\xi_n - \zeta_n\|^2 \leq E^2.$$

Subtract the two formulas for the  $\mathcal{H}$ -component before introducing any abbreviations:

$$\begin{aligned} U_n^\xi(y) - U_n^\zeta(y) &= (\langle \xi_{2n}, y_{2n} \rangle - \langle \xi_n, y_n \rangle) \xi_n \\ &\quad - (\langle \zeta_{2n}, y_{2n} \rangle - \langle \zeta_n, y_n \rangle) \zeta_n \\ &= (\langle \xi_{2n}, y_{2n} \rangle - \langle \xi_n, y_n \rangle) (\xi_n - \zeta_n) \\ &\quad + \left[ (\langle \xi_{2n}, y_{2n} \rangle - \langle \xi_n, y_n \rangle) - (\langle \zeta_{2n}, y_{2n} \rangle - \langle \zeta_n, y_n \rangle) \right] \zeta_n \\ &= (\langle \xi_{2n}, y_{2n} \rangle - \langle \xi_n, y_n \rangle) (\xi_n - \zeta_n) + (\delta_{2n} - \delta_n) \zeta_n. \end{aligned}$$

Subtracting the two formulas for the  $\ell_2$ -component gives

$$V_n^\xi(y) - V_n^\zeta(y) = \langle \xi_{2n-1} - \zeta_{2n-1}, y_{2n-1} \rangle = \delta_{2n-1}.$$

Now put

$$\begin{aligned} A_n &= (\langle \xi_{2n}, y_{2n} \rangle - \langle \xi_n, y_n \rangle) (\xi_n - \zeta_n), \\ B_n &= (\delta_{2n} - \delta_n) \zeta_n, \end{aligned}$$

and define

$$X = ((A_n)_n, 0), \quad Y = ((B_n)_n, (\delta_{2n-1})_n) \in \mathcal{H} \oplus_2 \ell_2.$$

The two component calculations give the vector identity

$$L_\xi^{-1}y - L_\zeta^{-1}y = X + Y.$$

Since  $0 \leq (\|y_{2n}\| - \|y_n\|)^2$ , we have  $(\|y_{2n}\| + \|y_n\|)^2 \leq 2(\|y_{2n}\|^2 + \|y_n\|^2)$ . Hence

$$\begin{aligned} \|X\|^2 &= \sum_{n \geq 1} \|A_n\|^2 \\ &\leq \sum_{n \geq 1} (\|y_{2n}\| + \|y_n\|)^2 \|\xi_n - \zeta_n\|^2 \\ &\leq 2 \sum_{n \geq 1} (\|y_{2n}\|^2 + \|y_n\|^2) \|\xi_n - \zeta_n\|^2 = 2E^2, \end{aligned}$$

so  $\|X\| \leq \sqrt{2} E$ .

For  $Y$ , separate the even-odd reindexing from the subtraction by writing

$$\begin{aligned} Y_{\text{eo}} &= ((\delta_{2n}\zeta_n)_n, (\delta_{2n-1})_n), \\ Y_{\text{all}} &= ((\delta_n\zeta_n)_n, 0). \end{aligned}$$

Then  $Y = Y_{\text{eo}} - Y_{\text{all}}$ , and

$$\begin{aligned} \|Y_{\text{eo}}\|^2 &= \sum_{n \geq 1} |\delta_{2n}|^2 + \sum_{n \geq 1} |\delta_{2n-1}|^2 = \sum_{j \geq 1} |\delta_j|^2, \\ \|Y_{\text{all}}\|^2 &= \sum_{n \geq 1} |\delta_n|^2. \end{aligned}$$

Therefore

$$\|Y\| \leq \|Y_{\text{eo}}\| + \|Y_{\text{all}}\| = 2 \left( \sum_{n \geq 1} |\delta_n|^2 \right)^{1/2} \leq 2E.$$

Finally,

$$\begin{aligned} \|L_\xi^{-1}y - L_\zeta^{-1}y\| &= \|X + Y\| \\ &\leq \|X\| + \|Y\| \\ &\leq \sqrt{2} E + 2E \\ &\leq 4E, \end{aligned}$$

which is exactly (5.5). □

### 5.3 Unit directions from finitely many coordinates

For  $k \geq 1$ , put

$$\mathcal{A}_k = \{-k, -k+1, \dots, k\}^k, \quad S_k = \mathbb{R}^{\mathcal{A}_k}, \quad S_0 = \mathbb{R}.$$

Every  $S_k$  has its Euclidean norm and its standard coordinate orthonormal basis. Define

$$H_n = \bigoplus_{k=0}^{n-1} S_k^{(n)}, \quad \dim H_n = 1 + \sum_{k=1}^{n-1} (2k+1)^k, \quad \mathcal{H} = \left( \bigoplus_{n \geq 1} H_n \right)_2. \quad (5.6)$$

Here  $S_k^{(n)}$  is an orthogonal copy of  $S_k$  in  $H_n$ , and  $\iota_{n,k} : S_k \rightarrow H_n$  is its isometric inclusion. Enumerate the coordinate orthonormal basis of  $\mathcal{H}$  block by block as  $(e_j)_{j \geq 1}$ . Since each block is nonzero, the first  $k$  basis vectors occur among the first  $k$  blocks:

$$e_1, \dots, e_k \in P_k \mathcal{H} \quad (k \geq 1). \quad (5.7)$$

Throughout the rest of this section,  $P_l$  denotes the orthogonal projection onto the first  $l$  blocks,  $P_0 = 0$ , and  $Q_l = I - P_l$ .

For  $y \in \mathcal{H}$ , define

$$p_j(y) = \frac{\langle y, e_j \rangle}{1 + |\langle y, e_j \rangle|} \quad (j \geq 1). \quad (5.8)$$

We need only two properties: every  $p_j$  is 1-Lipschitz, and the family  $(p_j)$  separates points.

Let  $q(s) = s/(1 + |s|)$ . If  $s$  and  $t$  have the same sign, then

$$q(t) - q(s) = \frac{t - s}{(1 + |s|)(1 + |t|)}.$$

If  $s \leq 0 \leq t$ , then

$$q(t) - q(s) = \frac{t}{1+t} + \frac{|s|}{1+|s|} \leq t + |s| = t - s.$$

Interchanging  $s$  and  $t$  covers the remaining order. Thus  $|q(t) - q(s)| \leq |t - s|$ , and the same formulas show that  $q$  is strictly increasing. Consequently,

$$|p_j(y)| < 1, \quad |p_j(y) - p_j(z)| \leq |\langle y - z, e_j \rangle| \leq \|y - z\|. \quad (5.9)$$

If  $y \neq z$ , some basis coordinate differs, and strict monotonicity of  $q$  then gives  $p_j(y) \neq p_j(z)$ . Hence  $(p_j)$  separates points of  $\mathcal{H}$ .

For  $a = (a_1, \dots, a_k) \in \mathcal{A}_k$ , define

$$G_{k,a}(y) = \max \left\{ 0, 1 - \max_{1 \leq j \leq k} |kp_j(y) - a_j| \right\}, \quad (5.10)$$

$$G_k(y) = (G_{k,a}(y))_{a \in \mathcal{A}_k} \in S_k.$$

For each  $j \leq k$ , choose an integer  $a_j \in \{-k, \dots, k\}$  with  $|kp_j(y) - a_j| \leq 1/2$ . Such an integer exists because  $kp_j(y) \in (-k, k)$ . For the resulting  $a = (a_1, \dots, a_k)$ ,

$$G_{k,a}(y) = 1 - \max_{1 \leq j \leq k} |kp_j(y) - a_j| \geq \frac{1}{2}.$$

Therefore

$$\|G_k(y)\| \geq \frac{1}{2} \quad (y \in \mathcal{H}). \quad (5.11)$$

The positive-part map, the absolute value, and a finite maximum are all 1-Lipschitz in the scalar variables involved. Thus

$$\begin{aligned}
 |G_{k,a}(y) - G_{k,a}(z)| &\leq \left| \max_{j \leq k} |kp_j(y) - a_j| - \max_{j \leq k} |kp_j(z) - a_j| \right| \\
 &\leq \max_{j \leq k} ||kp_j(y) - a_j| - |kp_j(z) - a_j|| \\
 &\leq k \max_{j \leq k} |p_j(y) - p_j(z)| \\
 &\leq k \|y - z\|.
 \end{aligned}$$

Since  $S_k = \mathbb{R}^{\mathcal{A}_k}$  has its Euclidean norm, summing the squares over  $a \in \mathcal{A}_k$  gives

$$\begin{aligned}
 \|G_k(y) - G_k(z)\|^2 &= \sum_{a \in \mathcal{A}_k} |G_{k,a}(y) - G_{k,a}(z)|^2 \\
 &\leq |\mathcal{A}_k| k^2 \|y - z\|^2.
 \end{aligned} \tag{5.12}$$

Hence  $\|G_k(y) - G_k(z)\| \leq k\sqrt{|\mathcal{A}_k|} \|y - z\|$ .

For nonzero vectors  $A, B$  in a normed space,

$$\begin{aligned}
 \left\| \frac{A}{\|A\|} - \frac{B}{\|B\|} \right\| &= \left\| \frac{A - B}{\|A\|} + B \left( \frac{1}{\|A\|} - \frac{1}{\|B\|} \right) \right\| \\
 &\leq \frac{2\|A - B\|}{\|A\|}.
 \end{aligned}$$

Interchanging  $A$  and  $B$  and taking the smaller of the two upper bounds gives

$$\left\| \frac{A}{\|A\|} - \frac{B}{\|B\|} \right\| \leq \frac{2\|A - B\|}{\max\{\|A\|, \|B\|\}}. \tag{5.13}$$

Define unit vectors in the orthogonal summands of  $H_n$  by

$$W_{n,0} = \iota_{n,0}(1), \quad W_{n,k}(y) = \iota_{n,k} \left( \frac{G_k(y)}{\|G_k(y)\|} \right) \quad (1 \leq k < n). \tag{5.14}$$

Using (5.11), (5.12), and (5.13), we obtain

$$\begin{aligned}
 \|W_{n,k}(y) - W_{n,k}(z)\| &\leq \frac{2\|G_k(y) - G_k(z)\|}{\max\{\|G_k(y)\|, \|G_k(z)\|\}} \\
 &\leq 4k\sqrt{|\mathcal{A}_k|} \|y - z\|.
 \end{aligned}$$

Thus, with

$$L_0 = 0, \quad L_k = 4k(2k+1)^{k/2} \quad (k \geq 1), \tag{5.15}$$

we have

$$\|W_{n,k}(y)\| = 1, \quad \|W_{n,k}(y) - W_{n,k}(z)\| \leq L_k \|y - z\| \quad (0 \leq k < n).$$

Only the finiteness of each  $L_k$  will be used.

Fix  $n \geq 2$ ,  $1 \leq k < n$ , and  $y, z \in \mathcal{H}$ . Suppose that, for some  $a \in \mathcal{A}_k$ , the coordinate indexed by  $a$  is nonzero in both  $W_{n,k}(y)$  and  $W_{n,k}(z)$ . Since the normalization in (5.14) multiplies every coordinate by a positive scalar, the coordinate indexed by  $a$  is nonzero in both  $G_k(y)$  and  $G_k(z)$ . Equation (5.10) then gives

$$|kp_j(y) - a_j| < 1, \quad |kp_j(z) - a_j| < 1 \quad (1 \leq j \leq k),$$

and hence

$$|p_j(y) - p_j(z)| < \frac{2}{k} \quad (1 \leq j \leq k). \quad (5.16)$$

Finally,  $W_{n,k}$  uses only  $p_1, \dots, p_k$ . For  $1 \leq j \leq k < n$ , equation (5.7) gives  $P_{n-1}e_j = e_j$ , and therefore

$$\begin{aligned} p_j(P_{n-1}y) &= \frac{\langle P_{n-1}y, e_j \rangle}{1 + |\langle P_{n-1}y, e_j \rangle|} \\ &= \frac{\langle y, e_j \rangle}{1 + |\langle y, e_j \rangle|} = p_j(y). \end{aligned}$$

It follows from (5.10) and (5.14) that

$$W_{n,k}(P_{n-1}y) = W_{n,k}(y) \quad (0 \leq k < n). \quad (5.17)$$

This dependence on the first  $n - 1$  blocks is used in the coordinate recursion in Subsection 5.8.

## 5.4 Tail norms and a weighted estimate

For  $y = (y_j) \in \mathcal{H}$ , define

$$r_n(y) = \|Q_{n-1}y\| = \left( \sum_{j \geq n} \|y_j\|^2 \right)^{1/2}. \quad (5.18)$$

The reverse triangle inequality and the orthogonal block decomposition give

$$\begin{aligned} |r_n(y) - r_n(z)| &\leq \|y - z\|, \\ r_n(y)^2 - r_{n+1}(y)^2 &= \|y_n\|^2, \\ r_n(y) &\downarrow 0. \end{aligned} \quad (5.19)$$

For positive  $r$ , define

$$\gamma_c(r) = \begin{cases} \frac{c 2^m}{8(m+1)}, & 2^{-(m+1)} < r \leq 2^{-m}, \quad m = 0, 1, \dots, \\ 0, & r > 1. \end{cases} \quad (5.20)$$

The coefficient sequence is nondecreasing because

$$\frac{2^{m+1}}{m+2} - \frac{2^m}{m+1} = \frac{2^m m}{(m+1)(m+2)} \geq 0.$$

It tends to infinity, since for  $m \geq 2$ ,

$$\frac{2^{m+1}/(m+2)}{2^m/(m+1)} = \frac{2(m+1)}{m+2} \geq \frac{3}{2}.$$

Hence  $\gamma_c$  is nonincreasing as a function of  $r$ , and  $\gamma_c(r) \rightarrow \infty$  as  $r \downarrow 0$ .

Fix  $y \in \mathcal{H}$ . For  $m \geq 0$ , let

$$I_m = \{n \geq 1 : 2^{-(m+1)} < r_n(y) \leq 2^{-m}\}.$$

Because  $r_n(y)$  is nonincreasing and tends to zero, each nonempty  $I_m$  is a finite interval of consecutive integers. If  $I_m = \{s, \dots, t\}$ , then (5.19) gives

$$\begin{aligned} \sum_{n \in I_m} \|y_n\|^2 &= \sum_{n=s}^t (r_n(y)^2 - r_{n+1}(y)^2) \\ &= r_s(y)^2 - r_{t+1}(y)^2 \\ &\leq 4^{-m}. \end{aligned}$$

The same upper bound is trivial when  $I_m$  is empty.

We shall also use

$$\sum_{j=1}^{\infty} \frac{1}{j^2} \leq 2. \quad (5.21)$$

Indeed, for  $N \geq 2$ ,

$$\sum_{j=1}^N \frac{1}{j^2} \leq 1 + \sum_{j=2}^N \frac{1}{j(j-1)} = 2 - \frac{1}{N} \leq 2.$$

Using  $\sum_{n \in I_m} \|y_n\|^2 \leq 4^{-m}$ , proved in the preceding calculation, and then (5.21), we obtain

$$\begin{aligned} \sum_{n: r_n(y) > 0} \|y_n\|^2 \gamma_c(r_n(y))^2 &= \sum_{m \geq 0} \frac{c^2 4^m}{64(m+1)^2} \sum_{n \in I_m} \|y_n\|^2 \\ &\leq \frac{c^2}{64} \sum_{m \geq 0} \frac{1}{(m+1)^2} \\ &\leq \frac{c^2}{32}. \end{aligned} \quad (5.22)$$

No value  $\gamma_c(0)$  is used. If  $r_n(y) = 0$ , then  $y_j = 0$  for every  $j \geq n$ .

For every  $n$  with  $r_{2n}(y) > 0$ , we have  $r_n(y) \geq r_{2n}(y)$  and therefore  $\gamma_c(r_n(y)) \leq \gamma_c(r_{2n}(y))$ . If  $r_{2n}(y) = 0$ , then  $y_{2n} = 0$ . Hence

$$\begin{aligned} \sum_{n: r_n(y) > 0} \|y_{2n}\|^2 \gamma_c(r_n(y))^2 &\leq \sum_{n: r_{2n}(y) > 0} \|y_{2n}\|^2 \gamma_c(r_{2n}(y))^2 \\ &\leq \sum_{j: r_j(y) > 0} \|y_j\|^2 \gamma_c(r_j(y))^2 \\ &\leq \frac{c^2}{32}, \end{aligned}$$

The last inequality is (5.22). Adding this estimate to (5.22) gives

$$\sum_{n: r_n(y) > 0} (\|y_n\|^2 + \|y_{2n}\|^2) \gamma_c(r_n(y))^2 \leq \frac{c^2}{16}. \quad (5.23)$$

### 5.5 A scalar parameter and separated transition intervals

Put  $\sigma_0 = 0$  and  $\sigma_m = \sum_{j=1}^m j^{-1}$  for  $m \geq 1$ . Define  $\tau_c(r) = 0$  for  $r \geq 1$ . If  $2^{-(m+1)} \leq r \leq 2^{-m}$ , define

$$\tau_c(r) = \frac{c}{16}\sigma_m + \frac{c2^m}{8(m+1)}(2^{-m} - r). \quad (5.24)$$

At the two dyadic endpoints,

$$\begin{aligned} \tau_c(2^{-m}) &= \frac{c}{16}\sigma_m, \\ \tau_c(2^{-(m+1)}) &= \frac{c}{16}\left(\sigma_m + \frac{1}{m+1}\right) = \frac{c}{16}\sigma_{m+1}. \end{aligned}$$

Thus the affine formulas agree wherever two dyadic intervals meet, and  $\tau_c$  is continuous and strictly decreasing on  $(0, 1]$ . The harmonic sums diverge because

$$\sigma_{2^M} \geq 1 + \sum_{j=1}^M \frac{2^{j-1}}{2^j} = 1 + \frac{M}{2}.$$

Since  $\tau_c(r) \geq (c/16)\sigma_m$  on  $2^{-(m+1)} \leq r \leq 2^{-m}$ , while  $m \rightarrow \infty$  as  $r \downarrow 0$ , we have  $\tau_c(r) \rightarrow \infty$  as  $r \downarrow 0$ .

To compare the directions corresponding to two different radii, we shall use

$$|\tau_c(a) - \tau_c(b)| \leq \gamma_c(\min\{a, b\})|a - b| \quad (a, b > 0). \quad (5.25)$$

We prove this directly from the dyadic affine formulas. By symmetry, assume  $0 < a < b$ . If  $a \geq 1$ , then  $\tau_c(a) = \tau_c(b) = 0$ , so the estimate is immediate. If  $a < 1 \leq b$ , then  $\tau_c(b) = \tau_c(1) = 0$  and  $1 - a \leq b - a$ , so it is enough to prove the estimate with  $b = 1$ . We may therefore assume  $0 < a < b \leq 1$ .

Choose the unique integers  $m, n \geq 0$  such that

$$2^{-(m+1)} < a \leq 2^{-m}, \quad 2^{-(n+1)} < b \leq 2^{-n}.$$

Then  $n \leq m$ . If  $n = m$ , the two points lie in one dyadic interval, and (5.24) gives

$$\tau_c(a) - \tau_c(b) = \frac{c2^m}{8(m+1)}(b - a) = \gamma_c(a)(b - a).$$

Suppose now that  $n < m$ . Splitting only at the actual dyadic points between  $a$  and  $b$ , we have

$$\begin{aligned} \tau_c(a) - \tau_c(b) &= [\tau_c(a) - \tau_c(2^{-m})] \\ &\quad + \sum_{j=n+1}^{m-1} [\tau_c(2^{-(j+1)}) - \tau_c(2^{-j})] \\ &\quad + [\tau_c(2^{-(n+1)}) - \tau_c(b)] \\ &= \frac{c2^m}{8(m+1)}(2^{-m} - a) \\ &\quad + \sum_{j=n+1}^{m-1} \frac{c2^j}{8(j+1)}(2^{-j} - 2^{-(j+1)}) \\ &\quad + \frac{c2^n}{8(n+1)}(b - 2^{-(n+1)}). \end{aligned}$$

The coefficients  $c2^j/[8(j+1)]$  are nondecreasing in  $j$ , as shown immediately after (5.20). Hence, for every  $n \leq j \leq m$ ,

$$\frac{c2^j}{8(j+1)} \leq \frac{c2^m}{8(m+1)} = \gamma_c(a).$$

Therefore

$$\begin{aligned} \tau_c(a) - \tau_c(b) &\leq \gamma_c(a) \left[ (2^{-m} - a) + \sum_{j=n+1}^{m-1} (2^{-j} - 2^{-(j+1)}) + (b - 2^{-(n+1)}) \right] \\ &= \gamma_c(a)(b - a). \end{aligned}$$

This proves (5.25).

Choose integers  $N_1 < N_2 < \dots$  and put

$$T_k = \tau_c(2^{-N_k}) = \frac{c}{16} \sigma_{N_k}. \quad (5.26)$$

We require

$$\gamma_c(2^{-N_k}) \geq 2 \max\{L_{k-1}, L_k\}, \quad T_{k+1} > T_k + 9 \quad (k \geq 1). \quad (5.27)$$

Choose  $N_1 \geq 1$  so that  $\gamma_c(2^{-N_1}) \geq 2 \max\{L_0, L_1\}$ . If  $N_1, \dots, N_k$  have already been chosen, choose  $N_{k+1} > N_k$  so large that

$$\frac{c2^{N_{k+1}}}{8(N_{k+1} + 1)} \geq 2 \max\{L_k, L_{k+1}\}, \quad \frac{c}{16} \sigma_{N_{k+1}} > T_k + 9.$$

As  $N \rightarrow \infty$ , both  $c2^N/[8(N+1)]$  and  $(c/16)\sigma_N$  tend to infinity. Hence the new choice establishes (5.27) at index  $k+1$  and leaves all previous conditions unchanged.

## 5.6 Interpolating the previously constructed directions

Recall from (5.14) and (5.15) that, for every  $0 \leq k < n$ ,

$$W_{n,k}(y) \in S_k^{(n)}, \quad \|W_{n,k}(y)\| = 1, \quad \|W_{n,k}(y) - W_{n,k}(z)\| \leq L_k \|y - z\|,$$

and that the summands  $S_0^{(n)}, \dots, S_{n-1}^{(n)}$  are mutually orthogonal.

For  $1 \leq k < n$ , fixed  $y$ , and  $0 \leq s, s' \leq 1$ , set

$$A_s = (1-s)W_{n,k-1}(y) + sW_{n,k}(y), \quad A_{s'} = (1-s')W_{n,k-1}(y) + s'W_{n,k}(y).$$

Since

$$\|A_s\|^2 = (1-s)^2 + s^2 \geq \frac{1}{2}, \quad \|A_s - A_{s'}\| = \sqrt{2}|s - s'|,$$

the normalization estimate (5.13) gives

$$\left\| \frac{A_s}{\|A_s\|} - \frac{A_{s'}}{\|A_{s'}\|} \right\| \leq \frac{2\sqrt{2}|s - s'|}{1/\sqrt{2}} = 4|s - s'|. \quad (5.28)$$

For  $n = 1$ , set  $\Omega_1(y, r) = W_{1,0}$ . For  $n \geq 2$  and  $1 \leq k < n$ , put  $\lambda_k(r) = (\tau_c(r) - T_k)/8$  whenever  $T_k \leq \tau_c(r) \leq T_k + 8$ . Define  $\Omega_n : \mathcal{H} \times [0, \infty) \rightarrow H_n$  by

$$\Omega_n(y, r) = \begin{cases} W_{n,0}, & r > 0, \tau_c(r) \leq T_1, \\ \frac{(1 - \lambda_k(r))W_{n,k-1}(y) + \lambda_k(r)W_{n,k}(y)}{\sqrt{(1 - \lambda_k(r))^2 + \lambda_k(r)^2}}, & r > 0, T_k \leq \tau_c(r) \leq T_k + 8, \\ & 1 \leq k < n, \\ W_{n,k}(y), & r > 0, T_k + 8 \leq \tau_c(r) \leq T_{k+1}, \\ & 1 \leq k < n-1, \\ W_{n,n-1}(y), & r = 0 \text{ or } (r > 0, \tau_c(r) \geq T_{n-1} + 8). \end{cases} \quad (5.29)$$

At  $\tau_c(r) = T_k$  and  $\tau_c(r) = T_k + 8$ , the transition formula equals  $W_{n,k-1}(y)$  and  $W_{n,k}(y)$ , respectively, so the adjacent formulas agree. Every value has norm one. Moreover, (5.17) gives  $W_{n,k}(P_{n-1}y) = W_{n,k}(y)$  whenever  $k < n$ ; hence

$$\Omega_n(P_{n-1}y, r) = \Omega_n(y, r), \quad \|\Omega_n(y, r)\| = 1. \quad (5.30)$$

**(I) Control in the first variable.** Fix  $n \geq 2$ ,  $r > 0$ , and  $y, z \in \mathcal{H}$ . We prove

$$\|\Omega_n(y, r) - \Omega_n(z, r)\| \leq \frac{\gamma_c(r)}{2} \|y - z\|. \quad (5.31)$$

The four cases are those in the positive-radius part of (5.29).

(I-a)  $\tau_c(r) \leq T_1$ . By (5.29),  $\Omega_n(y, r) = W_{n,0} = \Omega_n(z, r)$ , so (5.31) holds.

(I-b)  $T_k \leq \tau_c(r) \leq T_k + 8$  for some  $1 \leq k < n$ . Orthogonality of  $S_{k-1}^{(n)}$  and  $S_k^{(n)}$  gives

$$\begin{aligned} & ((1 - \lambda_k(r))^2 + \lambda_k(r)^2) \|\Omega_n(y, r) - \Omega_n(z, r)\|^2 \\ &= (1 - \lambda_k(r))^2 \|W_{n,k-1}(y) - W_{n,k-1}(z)\|^2 \\ &\quad + \lambda_k(r)^2 \|W_{n,k}(y) - W_{n,k}(z)\|^2 \\ &\leq ((1 - \lambda_k(r))^2 + \lambda_k(r)^2) \max\{L_{k-1}, L_k\}^2 \|y - z\|^2. \end{aligned}$$

The coefficient on the left is positive. Dividing by it and using (5.27) gives

$$\begin{aligned} \|\Omega_n(y, r) - \Omega_n(z, r)\| &\leq \max\{L_{k-1}, L_k\} \|y - z\| \\ &\leq \frac{\gamma_c(2^{-N_k})}{2} \|y - z\| \\ &\leq \frac{\gamma_c(r)}{2} \|y - z\|. \end{aligned}$$

The last inequality follows because  $\tau_c(r) \geq T_k = \tau_c(2^{-N_k})$  implies  $r \leq 2^{-N_k}$ , and  $\gamma_c$  is nonincreasing.

(I-c)  $T_k + 8 \leq \tau_c(r) \leq T_{k+1}$  for some  $1 \leq k < n - 1$ . Using (5.29), (5.15), and (5.27),

$$\begin{aligned} \|\Omega_n(y, r) - \Omega_n(z, r)\| &= \|W_{n,k}(y) - W_{n,k}(z)\| \\ &\leq L_k \|y - z\| \\ &\leq \frac{\gamma_c(2^{-N_k})}{2} \|y - z\| \\ &\leq \frac{\gamma_c(r)}{2} \|y - z\|, \end{aligned}$$

where the last inequality follows from  $\tau_c(r) \geq T_k = \tau_c(2^{-N_k})$ .

(I-d)  $\tau_c(r) \geq T_{n-1} + 8$ . Using (5.29), (5.15), and (5.27) at  $k = n - 1$ ,

$$\begin{aligned} \|\Omega_n(y, r) - \Omega_n(z, r)\| &= \|W_{n,n-1}(y) - W_{n,n-1}(z)\| \\ &\leq L_{n-1} \|y - z\| \\ &\leq \frac{\gamma_c(2^{-N_{n-1}})}{2} \|y - z\| \\ &\leq \frac{\gamma_c(r)}{2} \|y - z\|. \end{aligned}$$

Thus (5.31) holds in all four cases. For  $n = 1$ , it is immediate because  $\Omega_1$  is constant.

**(II) Control in the second variable.** Fix  $n \geq 2$ ,  $y \in \mathcal{H}$ , and  $r, r' > 0$ . We prove

$$\|\Omega_n(y, r) - \Omega_n(y, r')\| \leq \frac{\gamma_c(\min\{r, r'\})}{2} |r - r'|. \quad (5.32)$$

By symmetry, assume  $\tau_c(r) \leq \tau_c(r')$ .

(II-a)  $\tau_c(r), \tau_c(r') \leq T_1$ . Then (5.29) gives  $\Omega_n(y, r) = W_{n,0} = \Omega_n(y, r')$ .

(II-b)  $T_k \leq \tau_c(r), \tau_c(r') \leq T_k + 8$  for some  $1 \leq k < n$ . Applying (5.28) with  $s = \lambda_k(r)$  and  $s' = \lambda_k(r')$  gives

$$\begin{aligned} \|\Omega_n(y, r) - \Omega_n(y, r')\| &\leq 4|\lambda_k(r) - \lambda_k(r')| \\ &= \frac{1}{2}|\tau_c(r) - \tau_c(r')| \\ &\leq \frac{\gamma_c(\min\{r, r'\})}{2} |r - r'|, \end{aligned} \quad (5.33)$$

where the last line is (5.25).

(II-c)  $T_k + 8 \leq \tau_c(r), \tau_c(r') \leq T_{k+1}$  for some  $1 \leq k < n - 1$ . Then (5.29) gives  $\Omega_n(y, r) = W_{n,k}(y) = \Omega_n(y, r')$ .

(II-d)  $\tau_c(r), \tau_c(r') \geq T_{n-1} + 8$ . Then (5.29) gives  $\Omega_n(y, r) = W_{n,n-1}(y) = \Omega_n(y, r')$ .

(II-e) None of (II-a)–(II-d) applies. Put  $a = \tau_c(r)$  and  $b = \tau_c(r')$ . Then  $a < b$ , since  $a = b$  would fall under one of (II-a)–(II-d). We use the first and last indices satisfying  $\max\{a, T_j\} < \min\{b, T_j + 8\}$ :

$$\begin{aligned} p &= \min\{1 \leq j < n : \max\{a, T_j\} < \min\{b, T_j + 8\}\}, \\ q &= \max\{1 \leq j < n : \max\{a, T_j\} < \min\{b, T_j + 8\}\}. \end{aligned}$$

The index set is nonempty. Otherwise  $a, b$  would lie together in  $[0, T_1]$ , in some  $[T_j + 8, T_{j+1}]$ , or in  $[T_{n-1} + 8, \infty)$ , so one of (II-a), (II-c), or (II-d) would apply. The strict inequality in the definition of  $p, q$  excludes intervals that meet  $[a, b]$  only at one point. Figure 1 illustrates  $p = k$  and  $q = k + 2$ .

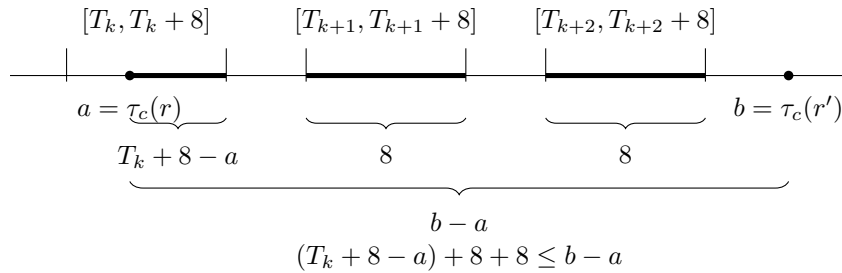


Figure 1: The horizontal axis records the value of  $\tau_c$ . Here  $p = k$ ,  $q = k + 2$ ,  $T_k < a < T_k + 8$ , and  $b > T_{k+2} + 8$ . The three full intervals are drawn with equal widths. The thick segments contribute the differences  $T_k + 8 - a$ , 8, and 8 to the sum in (II-e-2).

The definitions of  $p, q$  give  $a < T_p + 8$  and  $T_q < b$ . If  $p > 1$ , they also give  $T_{p-1} + 8 \leq a$ ; if  $q < n - 1$ , they give  $b \leq T_{q+1}$ . Consequently, in (5.29),  $\Omega_n(y, r)$  is given by the normalized segment between  $W_{n,p-1}(y)$  and  $W_{n,p}(y)$  with coefficient  $(\max\{a, T_p\} - T_p)/8$ . Indeed, this coefficient is 0 when  $a \leq T_p$ , in which case  $\Omega_n(y, r) = W_{n,p-1}(y)$ ; when  $T_p < a < T_p + 8$ , it is  $(a - T_p)/8 = \lambda_p(r)$ . At  $r'$ , the normalized segment between  $W_{n,q-1}(y)$  and  $W_{n,q}(y)$  has coefficient  $(\min\{b, T_q + 8\} - T_q)/8$ : for  $T_q < b < T_q + 8$  this is  $\lambda_q(r')$ , and for  $b \geq T_q + 8$  it is 1, giving  $\Omega_n(y, r') = W_{n,q}(y)$ .

(II-e-1)  $p = q$ . Both values use the segment between  $W_{n,p-1}(y)$  and  $W_{n,p}(y)$ . Apply (5.28) with  $k = p$  and

$$s = \frac{\max\{a, T_p\} - T_p}{8}, \quad s' = \frac{\min\{b, T_p + 8\} - T_p}{8}.$$

The definition of  $p$  gives  $0 \leq s < s' \leq 1$ , and hence

$$\begin{aligned} \|\Omega_n(y, r) - \Omega_n(y, r')\| &\leq 4(s' - s) \\ &= \frac{1}{2}(\min\{b, T_p + 8\} - \max\{a, T_p\}) \\ &\leq \frac{1}{2}(b - a). \end{aligned}$$

(II-e-2)  $p < q$ . For every  $p < j < q$ , the separation condition (5.27) gives

$$a < T_p + 8 < T_j < T_j + 8 < T_q < b.$$

Thus all the transitions with  $p < j < q$  occur in full between  $a$  and  $b$ . For the first transition, apply (5.28) at  $k = p$  with  $s = (\max\{a, T_p\} - T_p)/8$  and  $s' = 1$ :

$$\begin{aligned} \|\Omega_n(y, r) - W_{n,p}(y)\| &\leq 4\left(1 - \frac{\max\{a, T_p\} - T_p}{8}\right) \\ &= \frac{1}{2}(T_p + 8 - \max\{a, T_p\}). \end{aligned}$$

For each  $p < j < q$ , use  $k = j$ ,  $s = 0$ , and  $s' = 1$  in (5.28):

$$\|W_{n,j-1}(y) - W_{n,j}(y)\| \leq 4 = \frac{1}{2}((T_j + 8) - T_j).$$

For the last transition, apply (5.28) with  $k = q$ ,  $s = 0$ , and  $s' = (\min\{b, T_q + 8\} - T_q)/8$ :

$$\begin{aligned} \|W_{n,q-1}(y) - \Omega_n(y, r')\| &\leq 4 \frac{\min\{b, T_q + 8\} - T_q}{8} \\ &= \frac{1}{2}(\min\{b, T_q + 8\} - T_q). \end{aligned}$$

Now the actual vector differences telescope:

$$\begin{aligned} \Omega_n(y, r) - \Omega_n(y, r') &= (\Omega_n(y, r) - W_{n,p}(y)) \\ &\quad + \sum_{j=p+1}^{q-1} (W_{n,j-1}(y) - W_{n,j}(y)) \\ &\quad + (W_{n,q-1}(y) - \Omega_n(y, r')). \end{aligned}$$

The sum is zero when  $q = p + 1$ . Substituting the three estimates gives

$$\begin{aligned} \|\Omega_n(y, r) - \Omega_n(y, r')\| &\leq \|\Omega_n(y, r) - W_{n,p}(y)\| + \sum_{j=p+1}^{q-1} \|W_{n,j-1}(y) - W_{n,j}(y)\| \\ &\quad + \|W_{n,q-1}(y) - \Omega_n(y, r')\| \\ &\leq \frac{1}{2}(T_p + 8 - \max\{a, T_p\}) + \frac{1}{2} \sum_{j=p+1}^{q-1} ((T_j + 8) - T_j) \\ &\quad + \frac{1}{2}(\min\{b, T_q + 8\} - T_q). \end{aligned}$$

To compare this sum with  $b - a$ , write the intervening differences explicitly:

$$\begin{aligned} b - a &= (\max\{a, T_p\} - a) + (T_p + 8 - \max\{a, T_p\}) \\ &\quad + \sum_{j=p}^{q-1} (T_{j+1} - (T_j + 8)) \\ &\quad + \sum_{j=p+1}^{q-1} ((T_j + 8) - T_j) \\ &\quad + (\min\{b, T_q + 8\} - T_q) + (b - \min\{b, T_q + 8\}). \end{aligned}$$

Every term is nonnegative; in particular,  $T_{j+1} - (T_j + 8) > 1$  by (5.27). The identity therefore gives

$$\begin{aligned} T_p + 8 - \max\{a, T_p\} + \sum_{j=p+1}^{q-1} ((T_j + 8) - T_j) + \min\{b, T_q + 8\} - T_q \\ \leq b - a. \end{aligned}$$

It follows in both (II-e-1) and (II-e-2), using (5.25), that

$$\begin{aligned} \|\Omega_n(y, r) - \Omega_n(y, r')\| &\leq \frac{1}{2}(b - a) \\ &= \frac{1}{2}|\tau_c(r) - \tau_c(r')| \\ &\leq \frac{\gamma_c(\min\{r, r'\})}{2}|r - r'|. \end{aligned}$$

This proves (5.32); for  $n = 1$  it is immediate.

Recall from (5.18) that  $r_n(y) = (\sum_{j \geq n} \|y_j\|^2)^{1/2}$ , and define

$$w_n(y) = \Omega_n(y, r_n(y)). \quad (5.34)$$

If  $r_n(y), r_n(z) > 0$ , then (5.31), (5.32), and  $|r_n(y) - r_n(z)| \leq \|y - z\|$  from (5.19) give

$$\begin{aligned} \|w_n(y) - w_n(z)\| &\leq \|\Omega_n(y, r_n(y)) - \Omega_n(z, r_n(y))\| \\ &\quad + \|\Omega_n(z, r_n(y)) - \Omega_n(z, r_n(z))\| \\ &\leq \frac{\gamma_c(r_n(y))}{2} \|y - z\| + \frac{\gamma_c(\min\{r_n(y), r_n(z)\})}{2} |r_n(y) - r_n(z)| \\ &\leq \gamma_c(\min\{r_n(y), r_n(z)\}) \|y - z\|. \end{aligned} \quad (5.35)$$

If one tail radius is zero, the corresponding coefficient  $\|y_n\|^2 + \|y_{2n}\|^2$  or  $\|z_n\|^2 + \|z_{2n}\|^2$  in the weighted comparisons below is zero, so no value  $\gamma_c(0)$  is needed.

## 5.7 The inverse map and its two-point estimate

For  $y = (y_n) \in \mathcal{H}$ , put  $\beta_n(y) = \langle w_n(y), y_n \rangle$  and define

$$\begin{aligned} \Phi(y) &= (U(y), V(y)) \in \mathcal{H} \oplus_2 \ell_2, \\ U_n(y) &= y_n + (\beta_{2n}(y) - \beta_n(y))w_n(y), \\ V_n(y) &= \beta_{2n-1}(y). \end{aligned} \quad (5.36)$$

For a fixed  $y$ , the family  $(w_n(y))_n$  consists of one unit vector in each block. Applying Lemma 5.2 with  $\xi_n = w_n(y)$  gives

$$\Phi(y) = L_{w(y)}^{-1}y, \quad \Phi(y) \in \mathcal{H} \oplus_2 \ell_2, \quad \|\Phi(y)\| = \|y\|. \quad (5.37)$$

The family  $w(y)$  need not be square-summable; Lemma 5.2 requires only one unit direction in each block.

Fix  $y, z \in \mathcal{H}$ . For every block  $n$ , choose one comparison direction from the point having the larger tail norm:

$$\xi_n = \begin{cases} w_n(z), & r_n(y) \leq r_n(z), \\ w_n(y), & r_n(y) > r_n(z). \end{cases} \quad (5.38)$$

The family  $\xi = (\xi_n)$  defines the single linear isometry  $L_\xi^{-1}$  used to compare both  $\Phi(y)$  and  $\Phi(z)$ .

For the  $y$ -comparison, the indices with  $r_n(y) > r_n(z)$  contribute zero because  $\xi_n = w_n(y)$ . If  $r_n(y) = 0$ , then  $y_n = y_{2n} = 0$ . On the remaining indices, apply (5.35), and then apply the weighted estimate (5.23) to  $y$ :

$$\begin{aligned} & \sum_{n \geq 1} (\|y_n\|^2 + \|y_{2n}\|^2) \|w_n(y) - \xi_n\|^2 \\ &= \sum_{\{n: 0 < r_n(y) \leq r_n(z)\}} (\|y_n\|^2 + \|y_{2n}\|^2) \|w_n(y) - w_n(z)\|^2 \\ &\leq \|y - z\|^2 \sum_{n: r_n(y) > 0} (\|y_n\|^2 + \|y_{2n}\|^2) \gamma_c(r_n(y))^2 \\ &\leq \frac{c^2}{16} \|y - z\|^2. \end{aligned}$$

Lemma 5.3, applied to the direction families  $w(y)$  and  $\xi$  at the vector  $y$ , yields

$$\begin{aligned} \|\Phi(y) - L_\xi^{-1}y\| &= \|L_{w(y)}^{-1}y - L_\xi^{-1}y\| \\ &\leq 4 \left( \frac{c^2}{16} \|y - z\|^2 \right)^{1/2} \\ &= c \|y - z\|. \end{aligned} \quad (5.39)$$

For the  $z$ -comparison, the indices with  $r_n(z) \geq r_n(y)$  contribute zero because  $\xi_n = w_n(z)$ , and the indices with  $r_n(z) = 0$  have  $z_n = z_{2n} = 0$ . Thus the nonzero terms satisfy  $0 < r_n(z) < r_n(y)$ . For every index satisfying this inequality, (5.35) gives

$$\|w_n(z) - w_n(y)\| \leq \gamma_c(r_n(z)) \|y - z\|.$$

Consequently, (5.23), applied to  $z$ , gives

$$\begin{aligned} & \sum_{n \geq 1} (\|z_n\|^2 + \|z_{2n}\|^2) \|w_n(z) - \xi_n\|^2 \\ &= \sum_{\{n: 0 < r_n(z) < r_n(y)\}} (\|z_n\|^2 + \|z_{2n}\|^2) \|w_n(z) - w_n(y)\|^2 \\ &\leq \|y - z\|^2 \sum_{n: r_n(z) > 0} (\|z_n\|^2 + \|z_{2n}\|^2) \gamma_c(r_n(z))^2 \\ &\leq \frac{c^2}{16} \|y - z\|^2. \end{aligned}$$

Applying Lemma 5.3 to  $w(z)$  and  $\xi$  at  $z$  gives

$$\|\Phi(z) - L_\xi^{-1}z\| \leq c \|y - z\|. \quad (5.40)$$

Since  $L_\xi^{-1}$  is linear,

$$\begin{aligned}
& \left\| \Phi(y) - \Phi(z) - L_\xi^{-1}(y - z) \right\| \\
&= \left\| (\Phi(y) - L_\xi^{-1}y) - (\Phi(z) - L_\xi^{-1}z) \right\| \\
&\leq \left\| \Phi(y) - L_\xi^{-1}y \right\| + \left\| \Phi(z) - L_\xi^{-1}z \right\| \\
&\leq c \|y - z\| + c \|y - z\| \\
&= 2c \|y - z\|,
\end{aligned} \tag{5.41}$$

where the final inequality is (5.39) together with (5.40). Since  $L_\xi^{-1}$  is an isometry, (5.41) gives

$$\begin{aligned}
\|\Phi(y) - \Phi(z)\| &\leq \|L_\xi^{-1}(y - z)\| + \|\Phi(y) - \Phi(z) - L_\xi^{-1}(y - z)\| \\
&\leq (1 + 2c) \|y - z\|, \\
\|\Phi(y) - \Phi(z)\| &\geq \|L_\xi^{-1}(y - z)\| - \|\Phi(y) - \Phi(z) - L_\xi^{-1}(y - z)\| \\
&\geq (1 - 2c) \|y - z\|.
\end{aligned}$$

Thus

$$(1 - 2c) \|y - z\| \leq \|\Phi(y) - \Phi(z)\| \leq (1 + 2c) \|y - z\|. \tag{5.42}$$

In particular,  $\Phi$  is continuous and injective.

## 5.8 Surjectivity by a coordinate recursion

Fix  $(u, v) \in \mathcal{H} \oplus_2 \ell_2$ . We construct  $y = (y_n) \in \mathcal{H}$  and prove  $\Phi(y) = (u, v)$ .

Suppose that, before stage  $n$ , the blocks  $y_1, \dots, y_{n-1}$ , the unit vectors  $\xi_1, \dots, \xi_{n-1}$ , and the scalars  $\alpha_j = \langle \xi_j, u_j \rangle$ ,  $1 \leq j < n$ , have been defined. The induction invariant is the partial-norm identity

$$\sum_{j=1}^N \|y_j\|^2 = \sum_{j=1}^N \|u_j\|^2 - \sum_{j=\lfloor N/2 \rfloor + 1}^N |\alpha_j|^2 + \sum_{j=1}^{\lfloor N/2 \rfloor} |v_j|^2 \quad (0 \leq N < n). \tag{5.43}$$

The identity is true at stage  $n = 1$ , because every sum with  $N = 0$  is zero. Since  $|\alpha_j| \leq \|u_j\|$ , the invariant implies

$$\sum_{j < n} \|y_j\|^2 \leq \|u\|^2 + \|v\|^2.$$

Therefore the following radius is well-defined:

$$\begin{aligned}
y^{[n-1]} &= (y_1, \dots, y_{n-1}, 0, 0, \dots), \\
s_n &= \left( \|u\|^2 + \|v\|^2 - \sum_{j < n} \|y_j\|^2 \right)^{1/2}.
\end{aligned} \tag{5.44}$$

Both quantities in (5.44) depend only on data already constructed.

Define

$$\xi_n = \Omega_n(y^{[n-1]}, s_n), \quad \alpha_n = \langle \xi_n, u_n \rangle. \tag{5.45}$$

The scalar inserted into block  $n$  is

$$C_n = \begin{cases} \alpha_m, & n = 2m, \\ v_m, & n = 2m - 1. \end{cases} \tag{5.46}$$

If  $n = 2m$ , then  $m < n$ , so  $\alpha_m$  was constructed at an earlier stage. If  $n = 2m - 1$ , then  $v_m$  is part of the prescribed input. Thus (5.46) is not circular. Set

$$y_n = u_n + (C_n - \alpha_n)\xi_n. \quad (5.47)$$

We now verify the invariant at the new index. Since  $\|\xi_n\| = 1$ , we have  $|\alpha_n| = |\langle \xi_n, u_n \rangle| \leq \|u_n\|$ ; moreover,  $\langle u_n - \alpha_n \xi_n, \xi_n \rangle = 0$ . Hence (5.47) gives

$$\begin{aligned} \|y_n\|^2 &= \|u_n - \alpha_n \xi_n\|^2 + |C_n|^2 \\ &= \|u_n\|^2 - |\alpha_n|^2 + |C_n|^2. \end{aligned} \quad (5.48)$$

Summing (5.48) from 1 to any constructed index  $N$  gives

$$\begin{aligned} \sum_{j=1}^N \|y_j\|^2 &= \sum_{j=1}^N \|u_j\|^2 - \sum_{j=1}^N |\alpha_j|^2 + \sum_{j=1}^N |C_j|^2 \\ &= \sum_{j=1}^N \|u_j\|^2 - \sum_{j=1}^N |\alpha_j|^2 + \sum_{m=1}^{\lfloor N/2 \rfloor} |\alpha_m|^2 + \sum_{m=1}^{\lceil N/2 \rceil} |v_m|^2 \\ &= \sum_{j=1}^N \|u_j\|^2 - \sum_{m=\lfloor N/2 \rfloor + 1}^N |\alpha_m|^2 + \sum_{m=1}^{\lceil N/2 \rceil} |v_m|^2. \end{aligned}$$

The equality just obtained is precisely (5.43) with  $N = n$ . Every instance with  $N < n$  remains true because no previously constructed block is changed. The induction therefore defines all blocks  $y_n$ .

The invariant also gives

$$\sum_{j=1}^N \|y_j\|^2 \leq \|u\|^2 + \|v\|^2 \quad (N \geq 1),$$

so  $y \in \mathcal{H}$ . Furthermore,

$$0 \leq \sum_{m=\lfloor N/2 \rfloor + 1}^N |\alpha_m|^2 \leq \sum_{m>\lfloor N/2 \rfloor} \|u_m\|^2 \longrightarrow 0.$$

Letting  $N \rightarrow \infty$  in (5.43) yields

$$\|y\|^2 = \|u\|^2 + \|v\|^2. \quad (5.49)$$

Consequently,

$$\begin{aligned} s_n^2 &= \|u\|^2 + \|v\|^2 - \sum_{j<n} \|y_j\|^2 \\ &= \|y\|^2 - \sum_{j<n} \|y_j\|^2 \\ &= \sum_{j \geq n} \|y_j\|^2 \\ &= r_n(y)^2. \end{aligned}$$

Both  $s_n$  and  $r_n(y)$  are nonnegative, so  $s_n = r_n(y)$ . Also  $y^{[n-1]} = P_{n-1}y$ . The head-dependence identity (5.30) therefore gives

$$\begin{aligned} \xi_n &= \Omega_n(y^{[n-1]}, s_n) \\ &= \Omega_n(P_{n-1}y, r_n(y)) \\ &= \Omega_n(y, r_n(y)) \\ &= w_n(y). \end{aligned} \quad (5.50)$$

Using (5.47), (5.50), and  $\alpha_n = \langle \xi_n, u_n \rangle$ , we obtain

$$\begin{aligned}\beta_n(y) &= \langle w_n(y), y_n \rangle \\ &= \langle \xi_n, u_n + (C_n - \alpha_n)\xi_n \rangle \\ &= \alpha_n + C_n - \alpha_n \\ &= C_n.\end{aligned}$$

Thus  $\beta_{2m}(y) = \alpha_m$  and  $\beta_{2m-1}(y) = v_m$ . We now evaluate both components in (5.36). For every  $m \geq 1$ ,

$$\begin{aligned}U_m(y) &= y_m + (\beta_{2m}(y) - \beta_m(y))w_m(y) \\ &= u_m + (C_m - \alpha_m)\xi_m + (\alpha_m - C_m)\xi_m \\ &= u_m, \\ V_m(y) &= \beta_{2m-1}(y) = v_m.\end{aligned}$$

Therefore  $\Phi(y) = (u, v)$ . Since  $(u, v)$  was arbitrary,  $\Phi$  is surjective. Together with injectivity from (5.42), this proves that  $\Phi$  is a bijection.

Define

$$h = \Phi^{-1} : \mathcal{H} \oplus_2 \ell_2 \longrightarrow \mathcal{H}. \quad (5.51)$$

The norm identity (5.37) gives  $\|h(u, v)\| = \|(u, v)\|$  and  $h(0, 0) = 0$ . Apply (5.42) to  $y = h(u, v)$  and  $z = h(u', v')$ . Since  $\Phi(y) = (u, v)$  and  $\Phi(z) = (u', v')$ ,

$$(1 - 2c) \|h(u, v) - h(u', v')\| \leq \|(u - u', v - v')\| \leq (1 + 2c) \|h(u, v) - h(u', v')\|.$$

Division by the positive factors gives

$$\frac{1}{1 + 2c} \|(u - u', v - v')\| \leq \|h(u, v) - h(u', v')\| \leq \frac{1}{1 - 2c} \|(u - u', v - v')\|. \quad (5.52)$$

Finally,

$$\begin{aligned}\frac{1}{1 + 2c} \geq 1 - 4c &\iff 1 - (1 - 4c)(1 + 2c) = 2c + 8c^2 \geq 0, \\ \frac{1}{1 - 2c} \leq 1 + 4c &\iff (1 + 4c)(1 - 2c) - 1 = 2c(1 - 4c) \geq 0.\end{aligned}$$

Substituting these two estimates into the lower and upper bounds in (5.52) gives (5.1).

## 5.9 Orthogonal coordinate supports on distinct neighborhoods

It remains to prove (5.2). We first establish a separation statement in  $\mathcal{H}$  for the directions  $w_n(y)$ .

Fix distinct points  $y^{(1)}, \dots, y^{(m)} \in \mathcal{H}$ , where  $m \geq 2$ . For every pair  $1 \leq a < b \leq m$ , choose an index  $j_{a,b}$  such that

$$d_{a,b} = |p_{j_{a,b}}(y^{(a)}) - p_{j_{a,b}}(y^{(b)})| > 0.$$

Choose an integer  $K$  with

$$K \geq \max_{a < b} j_{a,b}, \quad \frac{2}{K} < \frac{d_{a,b}}{2} \quad (1 \leq a < b \leq m). \quad (5.53)$$

Such a  $K$  exists because there are only finitely many pairs, and every  $d_{a,b}$  is positive. We fix this one  $K$  for all pairs. For notational symmetry, set  $j_{a,b} = j_{b,a}$  and  $d_{a,b} = d_{b,a}$  when  $a > b$ .

For  $1 \leq a \leq m$ , define

$$U_a^0 = \bigcap_{b \neq a} \left\{ y \in \mathcal{H} : |p_{j_{a,b}}(y) - p_{j_{a,b}}(y^{(a)})| < \frac{d_{a,b}}{4} \right\}.$$

Each  $U_a^0$  is an open neighborhood of  $y^{(a)}$ . If  $y \in U_a^0$ ,  $z \in U_b^0$ , and  $a \neq b$ , then

$$\begin{aligned} |p_{j_{a,b}}(y) - p_{j_{a,b}}(z)| &\geq d_{a,b} - |p_{j_{a,b}}(y) - p_{j_{a,b}}(y^{(a)})| \\ &\quad - |p_{j_{a,b}}(z) - p_{j_{a,b}}(y^{(b)})| \\ &> d_{a,b} - \frac{d_{a,b}}{4} - \frac{d_{a,b}}{4} \\ &= \frac{d_{a,b}}{2} \\ &> \frac{2}{K}. \end{aligned} \tag{5.54}$$

We now locate the coordinates of the directions used by the inverse map. Recall the orthogonal decomposition and the unit vectors from (5.6) and (5.14):

$$H_n = \bigoplus_{k=0}^{n-1} S_k^{(n)}, \quad W_{n,k}(x) \in S_k^{(n)}, \quad \|W_{n,k}(x)\| = 1.$$

Here  $W_{n,0}$  is constant. The actual direction is obtained by inserting the tail radius into the interpolation in (5.29):

$$r_n(x) = \left( \sum_{j \geq n} \|x_j\|^2 \right)^{1/2}, \quad w_n(x) = \Omega_n(x, r_n(x)).$$

Thus  $w_n(x)$  is either one  $W_{n,j}(x)$  or a normalized linear combination of  $W_{n,j-1}(x)$  and  $W_{n,j}(x)$  from two adjacent orthogonal summands. At radius zero it is  $W_{n,n-1}(x)$ . The next case check shows that, on sufficiently high blocks, every summand that can occur has index at least  $K$ .

Choose  $\rho > 0$  such that  $\tau_c(\rho) > T_K + 8$ ; this is possible because  $\tau_c(r) \rightarrow \infty$  as  $r \downarrow 0$ . Since  $r_n(y^{(a)}) \rightarrow 0$  for each of the finitely many centers, choose  $l \geq K$  such that

$$r_{l+1}(y^{(a)}) < \frac{\rho}{2} \quad (1 \leq a \leq m),$$

and set  $U_a = U_a^0 \cap B(y^{(a)}, \rho/2)$ . For  $n > l$  and  $y \in U_a$ ,

$$\begin{aligned} r_n(y) &\leq r_{l+1}(y) \\ &\leq r_{l+1}(y^{(a)}) + \|y - y^{(a)}\| \\ &< \rho. \end{aligned}$$

We now prove the required coordinate restriction by the cases in (5.29).

(A) *The case  $r_n(y) = 0$ .* Since  $n > l \geq K$ , we have  $n - 1 \geq K$ , and therefore

$$w_n(y) = \Omega_n(y, 0) = W_{n,n-1}(y) \in S_{n-1}^{(n)} \subseteq \bigoplus_{j=K}^{n-1} S_j^{(n)}.$$

(B) *The case  $r_n(y) > 0$ .* The inequality  $r_n(y) < \rho$  and the monotonicity of  $\tau_c$  give

$$\tau_c(r_n(y)) > \tau_c(\rho) > T_K + 8.$$

The four positive-radius cases in (5.29) are handled as follows.

(B-a) *The initial interval*  $\tau_c(r_n(y)) \leq T_1$ . This case is impossible because  $T_1 < T_K + 8$ .

(B-b) *A transition interval*  $T_k \leq \tau_c(r_n(y)) \leq T_k + 8$ . If  $k \leq K$ , then

$$\tau_c(r_n(y)) \leq T_k + 8 \leq T_K + 8,$$

contrary to the strict inequality above. Hence  $k > K$ , and

$$w_n(y) \in S_{k-1}^{(n)} \oplus S_k^{(n)} \subseteq \bigoplus_{j=K}^{n-1} S_j^{(n)}.$$

(B-c) *An intermediate interval*  $T_k + 8 \leq \tau_c(r_n(y)) \leq T_{k+1}$ . If  $k < K$ , then  $k + 1 \leq K$  and

$$\tau_c(r_n(y)) \leq T_{k+1} \leq T_K < T_K + 8,$$

again a contradiction. Thus  $k \geq K$ , and

$$w_n(y) = W_{n,k}(y) \in S_k^{(n)} \subseteq \bigoplus_{j=K}^{n-1} S_j^{(n)}.$$

(B-d) *The final interval*  $\tau_c(r_n(y)) \geq T_{n-1} + 8$ . Here

$$w_n(y) = W_{n,n-1}(y) \in S_{n-1}^{(n)} \subseteq \bigoplus_{j=K}^{n-1} S_j^{(n)},$$

because  $n - 1 \geq l \geq K$ .

All the cases in (5.29) have now been exhausted. Hence

$$w_n(y) \in \bigoplus_{j=K}^{n-1} S_j^{(n)} \quad (n > l, y \in U_1 \cup \dots \cup U_m). \quad (5.55)$$

We now prove that no basis coordinate can be used by two distinct neighborhoods. Write  $\mathcal{E}_{n,k}$  for the standard coordinate orthonormal basis of  $S_k^{(n)}$ , and put

$$\mathcal{E}_n = \bigsqcup_{k=0}^{n-1} \mathcal{E}_{n,k}.$$

Thus  $\mathcal{E}_n$  is the fixed coordinate orthonormal basis of  $H_n$ . Our claim is that, for every  $a \neq b$ , every  $n > l$ , every  $y \in U_a$  and  $z \in U_b$ , and every  $e \in \mathcal{E}_n$ ,

$$\langle w_n(y), e \rangle \langle w_n(z), e \rangle = 0.$$

Fix  $a \neq b$ ,  $n > l$ ,  $y \in U_a$ , and  $z \in U_b$ . Suppose, to the contrary, that some  $e \in \mathcal{E}_n$  satisfies

$$\langle w_n(y), e \rangle \neq 0, \quad \langle w_n(z), e \rangle \neq 0.$$

There is a unique  $k \in \{0, \dots, n-1\}$  such that  $e \in \mathcal{E}_{n,k}$ . If  $k < K$ , then (5.55) and orthogonality would give  $\langle w_n(y), e \rangle = 0$ . Consequently,

$$K \leq k < n, \quad j_{a,b} \leq K \leq k.$$

Here  $k$  is determined by the assumed common coordinate  $e$ ; it is not chosen to be large. Its lower bound follows from (5.55).

Let  $\pi_{n,k} : H_n \rightarrow S_k^{(n)}$  be the orthogonal projection. For either  $x = y$  or  $x = z$ , recall that

$$r = r_n(x) = \left( \sum_{q \geq n} \|x_q\|^2 \right)^{1/2}, \quad w_n(x) = \Omega_n(x, r).$$

We compute its  $S_k^{(n)}$  component from (5.29). If  $w_n(x) = W_{n,j}(x)$  for one  $0 \leq j < n$ , its projection is  $W_{n,k}(x)$  when  $k = j$ , and zero otherwise. This includes  $r = 0$ . In a transition interval  $T_j \leq \tau_c(r) \leq T_j + 8$ , put

$$\lambda = \lambda_j(r) = \frac{\tau_c(r) - T_j}{8} \in [0, 1], \quad d = \sqrt{(1 - \lambda)^2 + \lambda^2} > 0.$$

Then

$$w_n(x) = \frac{1 - \lambda}{d} W_{n,j-1}(x) + \frac{\lambda}{d} W_{n,j}(x).$$

The two vectors belong to the orthogonal summands  $S_{j-1}^{(n)}$  and  $S_j^{(n)}$ . Hence

$$\pi_{n,k} w_n(x) = \begin{cases} \frac{1 - \lambda}{d} W_{n,k}(x), & k = j - 1, \\ \frac{\lambda}{d} W_{n,k}(x), & k = j, \\ 0, & k \notin \{j - 1, j\}. \end{cases}$$

The transition index  $j$  and the coefficients may differ between  $y$  and  $z$ ; the coordinate index  $k$ , determined by  $e$ , is the same. In every case,  $\pi_{n,k} w_n(x) = \eta_{n,k}(x) W_{n,k}(x)$  for some scalar  $\eta_{n,k}(x) \geq 0$ , possibly zero. Since  $e \in S_k^{(n)}$ ,

$$\langle w_n(x), e \rangle = \langle \pi_{n,k} w_n(x), e \rangle = \eta_{n,k}(x) \langle W_{n,k}(x), e \rangle.$$

The assumed nonzero coordinates therefore imply

$$\langle W_{n,k}(y), e \rangle \neq 0, \quad \langle W_{n,k}(z), e \rangle \neq 0.$$

Write  $\alpha = (\alpha_1, \dots, \alpha_k) \in \{-k, \dots, k\}^k$  for the index of  $e$  in  $\mathcal{E}_{n,k}$ . By (5.10) and (5.14),  $W_{n,k}(x)$  is the positive normalization of the vector with coordinates

$$G_{k,\alpha}(x) = \max \left\{ 0, 1 - \max_{1 \leq j \leq k} |kp_j(x) - \alpha_j| \right\}.$$

Here  $p_j(x) = \langle x, e_j \rangle / (1 + |\langle x, e_j \rangle|)$ . The common nonzero coordinate therefore forces

$$|kp_j(y) - \alpha_j| < 1, \quad |kp_j(z) - \alpha_j| < 1 \quad (1 \leq j \leq k).$$

Because  $j_{a,b} \leq K \leq k$ , we can use  $j = j_{a,b}$  and obtain

$$\begin{aligned} |p_{j_{a,b}}(y) - p_{j_{a,b}}(z)| &\leq \frac{|kp_{j_{a,b}}(y) - \alpha_{j_{a,b}}| + |kp_{j_{a,b}}(z) - \alpha_{j_{a,b}}|}{k} \\ &< \frac{2}{k} \leq \frac{2}{K}. \end{aligned}$$

This contradicts (5.54). Thus no common nonzero coordinate exists, for any  $a \neq b$ ,  $n > l$ ,  $y \in U_a$ , and  $z \in U_b$ . In particular, the coordinate supports are disjoint uniformly over all points in the two neighborhoods, not just for their centers.

We now collect all coordinates used by each neighborhood, over all blocks  $n > l$  and all summands with  $K \leq k < n$ . For  $1 \leq a \leq m$ , define

$$\begin{aligned}\mathcal{B}_a &= \bigcup_{n>l} \bigcup_{k=K}^{n-1} \{e \in \mathcal{E}_{n,k} : \langle w_n(y), e \rangle \neq 0 \text{ for some } y \in U_a\}, \\ \mathcal{K}_a &= \overline{\text{span}} \mathcal{B}_a.\end{aligned}\tag{5.56}$$

The indices  $n$  and  $k$  are bound by the two unions; no individual  $k$  is retained in  $\mathcal{K}_a$ . By (5.55), these unions include every nonzero coordinate of every  $w_n(y)$  with  $n > l$  and  $y \in U_a$ .

If  $e \in \mathcal{B}_a \cap \mathcal{B}_b$ , its block index  $n$  is unique, and membership in the two sets supplies  $y \in U_a$  and  $z \in U_b$  for which the same coordinate is nonzero. This contradicts what we just proved. Hence  $\mathcal{B}_a \cap \mathcal{B}_b = \emptyset$  when  $a \neq b$ . All the sets  $\mathcal{B}_a$  are subsets of the one fixed coordinate orthonormal basis  $\bigcup_{n \geq 1} \mathcal{E}_n$ , and use only blocks  $n > l$ . Their closed linear spans therefore satisfy

$$\mathcal{K}_a \perp \mathcal{K}_b \quad (a \neq b), \quad \mathcal{K}_a \subset Q_l \mathcal{H} \quad (1 \leq a \leq m).$$

Now return to the distinct inputs in the statement of Theorem 5.1. Put

$$y^{(a)} = h(u^{(a)}, v^{(a)}) \quad (1 \leq a \leq m).$$

The points  $y^{(a)}$  are distinct because  $h$  is injective. Apply the construction in (5.53)–(5.56) to  $y^{(1)}, \dots, y^{(m)}$ , and define

$$O_a = h^{-1}(U_a).$$

The map  $h$  is continuous by (5.52), so  $O_a$  is an open neighborhood of  $(u^{(a)}, v^{(a)})$ .

Fix  $(u, v) \in O_a$  and put  $y = h(u, v)$ . Then  $\Phi(y) = (u, v)$ . The  $\mathcal{H}$  component in (5.36) gives

$$u_n = y_n + (\beta_{2n}(y) - \beta_n(y))w_n(y).$$

Therefore

$$\begin{aligned}D(u, v)_n &= h(u, v)_n - u_n \\ &= y_n - u_n \\ &= (\beta_n(y) - \beta_{2n}(y))w_n(y).\end{aligned}\tag{5.57}$$

For  $n > l$ , equation (5.57) expresses  $D(u, v)_n$  as a scalar multiple of  $w_n(y)$ . Every nonzero coordinate of  $w_n(y)$  belongs to the generating set of  $\mathcal{K}_a$ ; hence  $D(u, v)_n$  uses only basis coordinates that generate  $\mathcal{K}_a$ . Thus

$$(P_N - P_l)D(u, v) \in \mathcal{K}_a \quad (N > l).$$

Moreover,

$$\begin{aligned}\|Q_l D(u, v) - (P_N - P_l)D(u, v)\|^2 &= \|Q_N D(u, v)\|^2 \\ &= \sum_{n>N} \|D(u, v)_n\|^2 \\ &\longrightarrow 0.\end{aligned}$$

Since  $\mathcal{K}_a$  is closed,

$$Q_l D(u, v) \in \mathcal{K}_a.$$

This proves

$$Q_l D(O_a) \subset \mathcal{K}_a \quad (1 \leq a \leq m).$$

If the original finite family contains only one point, take  $l = 1$ ,  $O_1 = \mathcal{H} \oplus_2 \ell_2$ , and  $\mathcal{K}_1 = Q_1 \mathcal{H}$ . Thus (5.2) holds in every case, and Theorem 5.1 is proved.  $\square$

**Corollary 5.4.** *For the bijection  $h$  constructed in Theorem 5.1, define*

$$D(u, v) = h(u, v) - u, \quad Q = \widehat{D} : \mathcal{F}(\mathcal{H} \oplus_2 \ell_2) \longrightarrow \mathcal{H}.$$

*Then  $D$  is Lipschitz and  $Q$  is completely continuous.*

*Proof.* The upper estimate in (5.1) gives

$$\begin{aligned} \|D(u, v) - D(u', v')\| &\leq \|h(u, v) - h(u', v')\| + \|u - u'\| \\ &\leq (2 + 4c) \|(u - u', v - v')\|. \end{aligned}$$

Thus  $D$  is Lipschitz and fixes zero, so its linearization  $Q = \widehat{D}$  is defined. For the neighborhoods and subspaces in (5.2), if  $x, x' \in O_j$ , then

$$\begin{aligned} D(x) - D(x') &= P_l(D(x) - D(x')) + (Q_l D(x) - Q_l D(x')) \\ &\in P_l \mathcal{H} \oplus \mathcal{K}_j. \end{aligned}$$

The space  $P_l \mathcal{H}$  is finite-dimensional, and the subspaces  $\mathcal{K}_j \subset (P_l \mathcal{H})^\perp$  are pairwise orthogonal. Theorem 4.1, applied with  $F = P_l \mathcal{H}$ , therefore shows that  $Q$  is completely continuous.  $\square$

## 6. The final spaces and the choice of parameters

*Proof of Theorem 1.1.* Fix  $\varepsilon > 0$  and put

$$c = \frac{\varepsilon}{4(1 + \varepsilon)}, \quad a = 1 - 4c = \frac{1}{1 + \varepsilon}, \quad b = 1 + 4c. \quad (6.1)$$

Then  $0 < c < 1/4$ ,  $0 < a < 1 < b$ , and  $b \leq 1 + \varepsilon$ .

Let  $\mathcal{H}$ ,  $h$ ,  $D$ , and  $Q = \widehat{D}$  be supplied by Theorem 5.1 and Corollary 5.4. Thus

$$a \|(u - u', v - v')\| \leq \|h(u, v) - h(u', v')\| \leq b \|(u - u', v - v')\|,$$

and  $Q : \mathcal{F}(\mathcal{H} \oplus_2 \ell_2) \rightarrow \mathcal{H}$  is completely continuous. Since  $\mathcal{H} \oplus_2 \ell_2$  is a real Hilbert space, Theorem 2.6 also gives

$$c_0 \not\rightarrow \mathcal{F}(\mathcal{H} \oplus_2 \ell_2).$$

Set

$$\theta = 2c, \quad \lambda_i = \theta 2^{-i} \quad (i \geq 1), \quad \tilde{Q}(\mu, v) = (Q\mu, v),$$

and define

$$X_\varepsilon = Z_{\tilde{Q}}^\lambda, \quad Y_\varepsilon = Z_Q^\lambda.$$

We spell out the graph-space definitions from (3.5) for these particular operators. Every  $\mu_i, \nu_i$  below belongs to  $\mathcal{F}(\mathcal{H} \oplus_2 \ell_2)$ , and every  $v_i$  belongs to  $\ell_2$ . Thus

$$\begin{aligned} X_\varepsilon &= \left\{ ((\mu_i, v_i))_{i \geq 1} : \sum_i \lambda_i (\|\mu_i\| + \|v_i\|) < \infty, \quad (Q\mu_i, v_i) \longrightarrow (0, 0) \right\}, \\ Y_\varepsilon &= \left\{ (\nu_i)_{i \geq 1} : \sum_i \lambda_i \|\nu_i\| < \infty, \quad Q\nu_i \longrightarrow 0 \right\}. \end{aligned}$$

The convergence conditions are in  $\mathcal{H} \oplus_2 \ell_2$  and  $\mathcal{H}$ , respectively. The norms on these two spaces are

$$\|(\mu_i, v_i)_i\|_{X_\varepsilon} = \max \left\{ \sum_i \lambda_i (\|\mu_i\| + \|v_i\|), \sup_i (\|Q\mu_i\|^2 + \|v_i\|^2)^{1/2} \right\},$$

$$\|(\nu_i)_i\|_{Y_\varepsilon} = \max \left\{ \sum_i \lambda_i \|\nu_i\|, \sup_i \|Q\nu_i\| \right\}.$$

In particular, these definitions require convergence to zero of the images, not merely boundedness of the displayed suprema. The weights are positive and  $\sum_{i \geq 1} \lambda_i = \theta = 2c < 1$ . Proposition 3.5 therefore gives separable Banach spaces  $X_\varepsilon, Y_\varepsilon$  and a bijection  $\Psi_\varepsilon : X_\varepsilon \rightarrow Y_\varepsilon$ . Here  $\delta : \mathcal{H} \oplus_2 \ell_2 \rightarrow \mathcal{F}(\mathcal{H} \oplus_2 \ell_2)$  is the evaluation map, and the bijection is explicitly

$$\Psi_\varepsilon((\mu_i, v_i)_{i \geq 1}) = (\mu_i + \delta(Q\mu_i, v_i))_{i \geq 1}. \quad (6.2)$$

For  $\nu = (\nu_i) \in Y_\varepsilon$ , write  $(u_i, v_i) = h^{-1}(Q\nu_i)$ . The inverse from the same proposition is

$$\Psi_\varepsilon^{-1}(\nu) = (\nu_i - \delta(u_i, v_i), v_i)_{i \geq 1}. \quad (6.3)$$

The linear obstruction can also be read directly in this notation. For  $v = (v_i)_i \in c_0(\ell_2)$ , set

$$\iota(v) = ((0, v_i))_i.$$

Since  $(0, v_i) \rightarrow (0, 0)$  and  $\sum_i \lambda_i \|v_i\| \leq \theta \sup_i \|v_i\| < \infty$ , this vector belongs to  $X_\varepsilon$ . Moreover,  $\theta < 1$  gives

$$\|\iota(v)\|_{X_\varepsilon} = \max \left\{ \sum_i \lambda_i \|v_i\|, \sup_i \|v_i\| \right\} = \sup_i \|v_i\| = \|v\|_{c_0(\ell_2)}.$$

Thus  $\iota$  is a linear isometry. In the other space, Proposition 3.4 applies to  $A = \mathcal{F}(\mathcal{H} \oplus_2 \ell_2)$  and the completely continuous operator  $Q : A \rightarrow \mathcal{H}$ : the hypothesis  $c_0 \not\hookrightarrow A$  was checked above. It follows that  $Y_\varepsilon = Z_Q^\lambda$  contains no linear copy of  $c_0(\ell_2)$ . If a bounded linear isomorphism  $T : X_\varepsilon \rightarrow Y_\varepsilon$  existed, then

$$\|T\iota(v)\|_{Y_\varepsilon} \geq \frac{1}{\|T^{-1}\|} \|\iota(v)\|_{X_\varepsilon} = \frac{1}{\|T^{-1}\|} \|v\|_{c_0(\ell_2)} \quad (v \in c_0(\ell_2)),$$

so  $T\iota$  would be precisely the forbidden linear embedding. This proves the nonisomorphism assertion of Theorem 1.1.

It remains only to evaluate the constants. Here

$$\theta = 2c \leq 1, \quad \theta \leq b - 1 = 4c, \quad \theta = \frac{1-a}{2}.$$

Corollary 3.6 yields

$$\text{Lip}(\Psi_\varepsilon) \leq b \leq 1 + \varepsilon, \quad \text{Lip}(\Psi_\varepsilon^{-1}) \leq a^{-1} = 1 + \varepsilon.$$

The inverse estimate gives, for  $s, t \in X_\varepsilon$ ,

$$\frac{1}{1 + \varepsilon} \|s - t\|_{X_\varepsilon} \leq \|\Psi_\varepsilon(s) - \Psi_\varepsilon(t)\|_{Y_\varepsilon},$$

which, together with the upper estimate, proves (1.1). Finally,  $\Psi_\varepsilon(0) = 0$  because  $\delta(0, 0) = 0$  and  $h^{-1}(0) = (0, 0)$ .  $\square$

## Attribution

The counterexample, the graph-space reduction, the nonlinear change of variables, and the use of moving directions are due to OpenAI [9]. This note uses the fixed manuscript identified in the bibliography. The realization in Section 5 instead makes the directions depend on the output and proves bijectivity by a coordinate recursion. Its two-point estimates replace the differentiability and inversion arguments of the source. This reconstruction makes no claim of independent solution or priority. The free-space proofs are in the companion [1], with the original sources [5, 2, 6, 3, 4].

## Use of ChatGPT

The manuscript was drafted and revised using ChatGPT.

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