

A reader's reconstruction of the chord argument for Tingley's problem

Version 1

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Publication responsibility: Ryotaro Tanaka

This note explains and reorganizes the argument in the cited OpenAI preprint; it is not an announcement of an independent solution.

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Abstract

This article gives an exposition of the proof in OpenAI's preprint *A positive solution to Tingley's problem*. We reorganize its presentation to make the geometric ideas and logical dependencies explicit. The first step identifies the precise target: vanishing of a radial distance defect makes the natural extension a surjective isometry fixing zero, so the Mazur–Ulam theorem gives linearity. We then describe the canonical sphere isometry between ultrapowers, including preservation and attainment of the defect. The geometric argument produces aligned chords by common supporting functionals and Darbo's theorem. A comparison of the two chord coordinates yields the final contradiction. The fixed-point background is collected in a separate section for the reader.

1. Introduction and the zero-defect target

This article gives a detailed exposition of the argument in OpenAI's preprint *A positive solution to Tingley's problem*, dated 23 September 2026 [7]. Its purpose is to make the proof easier to read and verify, with the geometric ideas stated before their technical implementation and the classical tools cited where they are used. The underlying argument is attributed to that preprint; the present article reorganizes its presentation and simplifies several steps.

Two features of the presentation are the direct construction of a convex invariant set by successive convex enlargement, and the use of explicit support estimates followed by a direct comparison of the two chord coordinates.

Reading guide. We assume familiarity with basic normed-space theory, convexity, and the Hahn–Banach theorem. Sections 1 to 3 give the reduction and geometric preparation; Section 4 collects the fixed-point background. The core chord argument occupies Sections 5 and 6. Readers already familiar with Darbo's theorem may use Theorem 4.3 and proceed to Section 5, returning to the preceding section as needed.

For a nonzero real normed space X , let

$$S_X = \{x \in X : \|x\| = 1\}.$$

Tingley's problem asks whether every surjective isometry $f : S_X \rightarrow S_Y$ extends to a real-linear isometry [9]. The conclusion, stated here for normed spaces using the completion reduction below, is the following.

Theorem 1.1. *Let X, Y be nonzero real normed spaces and let $f : S_X \rightarrow S_Y$ be a surjective isometry. Then f extends uniquely to a surjective real-linear isometry $X \rightarrow Y$.*

Any linear extension must be the radial map

$$T(0) = 0, \quad T(x) = \|x\| f\left(\frac{x}{\|x\|}\right) \quad (x \neq 0). \quad (1.1)$$

The formula gives $\|T(x)\| = \|x\|$ for every $x \in X$. It also gives

$$T\left(\|y\| f^{-1}\left(\frac{y}{\|y\|}\right)\right) = y \quad (y \in Y \setminus \{0\}).$$

Together with $T(0) = 0$, this proves surjectivity. The remaining task is to show that T preserves distances between points of different norms.

For $x, y \in S_X$ and $\lambda \in [0, 1]$, define

$$D_\lambda^f(x, y) = \|f(x) - \lambda f(y)\| - \|x - \lambda y\|, \quad \mu(f) = \sup_{\substack{x, y \in S_X \\ 0 \leq \lambda \leq 1}} |D_\lambda^f(x, y)|. \quad (1.2)$$

The superscript always identifies the sphere map whose defect is being used.

Proposition 1.2 (The zero-defect reduction). *If $\mu(f) = 0$, then the radial map (1.1) is a surjective isometry fixing zero, and hence is real-linear.*

Proof. For $x, y \in S_X$ and $\alpha \geq \beta > 0$, the one calculation needed is

$$\begin{aligned} \|T(\alpha x) - T(\beta y)\| &= \alpha \|f(x) - (\beta/\alpha)f(y)\| \\ &= \alpha \|x - (\beta/\alpha)y\| = \|\alpha x - \beta y\|. \end{aligned}$$

Interchanging the two points covers the other order of their norms. Distances to zero are preserved because T preserves norms. Surjectivity was noted above. The Mazur–Ulam theorem now gives real linearity, since $T(0) = 0$; see [5, 6]. Uniqueness follows from (1.1). $\square$

Thus the entire geometric argument has a single target:

$$\boxed{\mu(f) = 0.}$$

We will suppose that $\mu(f) > 0$, realize this value on the spheres of the ultrapowers, and derive a contradiction.

1.1 The bounds that must survive every reduction

For $x, y \in S_X$ and $\lambda, \lambda' \in [0, 1]$, we will repeatedly use

$$D_\lambda^{f^{-1}}(f(x), f(y)) = -D_\lambda^f(x, y), \quad \mu(f^{-1}) = \mu(f), \quad (1.3)$$

and

$$|D_\lambda^f(x, y) - D_{\lambda'}^f(x, y)| \leq 2|\lambda - \lambda'|. \quad (1.4)$$

The inverse identity uses $f^{-1}(f(x)) = x$ and $f^{-1}(f(y)) = y$. For the radius estimate, the reverse triangle inequality gives

$$|D_\lambda^f(x, y) - D_{\lambda'}^f(x, y)| \leq \|(\lambda - \lambda')f(y)\| + \|(\lambda - \lambda')y\| = 2|\lambda - \lambda'|.$$

Because f preserves norms and sphere distances,

$$D_0^f(x, y) = D_1^f(x, y) = 0.$$

Consequently,

$$|D_\lambda^f(x, y)| \leq 2 \min\{\lambda, 1 - \lambda\}, \quad 0 \leq \mu(f) \leq 1. \quad (1.5)$$

In particular, a positive maximum can only occur at an interior radius.

Lemma 1.3 (Completion preserves the defect). *Let $\widehat{X}, \widehat{Y}$ be the completions of X, Y . The map f extends uniquely to a surjective isometry*

$$\widehat{f}: S_{\widehat{X}} \longrightarrow S_{\widehat{Y}}, \quad \mu(\widehat{f}) = \mu(f).$$

Proof. For $x \in S_{\widehat{X}}$, choose $u_n \in X \setminus \{0\}$ with $u_n \rightarrow x$, and put $x_n = u_n/\|u_n\|$. Then

$$\|x_n - x\| \leq \|x_n - u_n\| + \|u_n - x\| = |1 - \|u_n\|| + \|u_n - x\| \longrightarrow 0.$$

Thus S_X is dense in $S_{\widehat{X}}$, and the same argument applies to Y . The uniformly continuous maps f and f^{-1} extend uniquely to these completed spheres. Their extensions are inverse maps because their compositions are the identity on dense subsets. For $x_n, y_n \in S_X$ converging to $x, y \in S_{\widehat{X}}$,

$$\|\widehat{f}(x) - \widehat{f}(y)\| = \lim_n \|f(x_n) - f(y_n)\| = \lim_n \|x_n - y_n\| = \|x - y\|.$$

Hence $\widehat{f}$ is a surjective isometry.

For $x_n, y_n \in S_X$ converging to $x, y \in S_{\widehat{X}}$ and any fixed $\lambda \in [0, 1]$,

$$D_\lambda^{\widehat{f}}(x, y) = \lim_{n \rightarrow \infty} D_\lambda^f(x_n, y_n).$$

This gives $\mu(\widehat{f}) \leq \mu(f)$; the reverse inequality follows by restricting to S_X . $\square$

It therefore suffices to prove the zero-defect assertion for Banach spaces. Once it is proved there, Lemma 1.3 and Proposition 1.2 apply to the original normed spaces.

2. Two elementary geometric tools

For a nonzero vector x in a real normed space X , write

$$\nu(x) = \{\varphi \in S_{X^*} : \varphi(x) = \|x\|\}. \quad (2.1)$$

Thus $\nu(x)$ is the set of norm-one supporting functionals at x . It is nonempty by the Hahn–Banach theorem [1]. We use the same notation in each ambient space, which is determined by x . For later use, the definition gives

$$\nu(\alpha x) = \nu(x) \quad (\alpha > 0), \quad \nu(-x) = -\nu(x). \quad (2.2)$$

Indeed, $\varphi(\alpha x) = \alpha\|x\|$ is equivalent to $\varphi(x) = \|x\|$, and $\varphi(-x) = \|x\|$ is equivalent to $(-\varphi)(x) = \|x\|$.

For vectors x, y , let $[x, y]$ denote their closed line segment and (x, y) the segment with its endpoints removed.

Lemma 2.1 (A support at an interior convex combination). *Let $x, y \in S_X$. If $(1-t)x + ty \in S_X$ for some $t \in (0, 1)$, then*

$$\nu((1-t)x + ty) \subset \nu(x) \cap \nu(y).$$

In particular, $\nu(x) \cap \nu(y) \neq \emptyset$ and $[x, y] \subset S_X$.

Proof. Let $\varphi \in \nu((1-t)x + ty)$. Then

$$0 = 1 - \varphi((1-t)x + ty) = (1-t)(1 - \varphi(x)) + t(1 - \varphi(y)).$$

Since $\varphi(x) \leq 1$, $\varphi(y) \leq 1$, and $0 < t < 1$, both summands are nonnegative and both coefficients are positive. Therefore

$$1 - \varphi(x) = 1 - \varphi(y) = 0,$$

which means that $\varphi \in \nu(x) \cap \nu(y)$. For every $s \in [0, 1]$, the same functional gives

$$1 = (1-s)\varphi(x) + s\varphi(y) \leq \|(1-s)x + sy\| \leq (1-s)\|x\| + s\|y\| = 1.$$

Hence $[x, y] \subset S_X$. □

As a consequence, if $u, v \in S_X$ and $\alpha, \beta > 0$, then

$$\|\alpha u + \beta v\| = \alpha + \beta \iff \nu(u) \cap \nu(v) \neq \emptyset. \quad (2.3)$$

For the forward implication, divide by $\alpha + \beta$ and apply Lemma 2.1 with $t = \beta/(\alpha + \beta)$. For the reverse implication, take $\varphi \in \nu(u) \cap \nu(v)$ and use

$$\alpha + \beta = \varphi(\alpha u + \beta v) \leq \|\alpha u + \beta v\| \leq \alpha + \beta.$$

Lemma 2.2 (Contact of a ray with the sphere). *Let $c \in X$ satisfy $\|c\| < 1$. For every $v \in S_X$, there is a unique positive number $\alpha_c(v)$ such that*

$$c + \alpha_c(v)v \in S_X.$$

This number satisfies

$$1 - \|c\| \leq \alpha_c(v) \leq 1 + \|c\|. \quad (2.4)$$

Moreover, for $u, v \in S_X$,

$$|\alpha_c(u) - \alpha_c(v)| \leq \frac{\alpha_c(u)\alpha_c(v)}{1 - \|c\|} \|u - v\| \leq \frac{(1 + \|c\|)^2}{1 - \|c\|} \|u - v\|. \quad (2.5)$$

Proof. Fix $v \in S_X$. The function $\alpha \mapsto \|c + \alpha v\|$ is continuous, and

$$\|c\| < 1 < 2 - \|c\| \leq \|c + 2v\|.$$

The intermediate value theorem therefore gives a positive $\alpha \in (0, 2)$ such that $\|c + \alpha v\| = 1$.

To prove uniqueness and compare different directions, suppose

$$\|c + \alpha u\| = \|c + \beta v\| = 1, \quad u, v \in S_X, \quad \alpha, \beta > 0.$$

Interchanging the two contacts if necessary, assume $\alpha \leq \beta$. The identity

$$c + \alpha u = \alpha(u - v) + \left(1 - \frac{\alpha}{\beta}\right)c + \frac{\alpha}{\beta}(c + \beta v)$$

gives

$$1 \leq \alpha \|u - v\| + \left(1 - \frac{\alpha}{\beta}\right) \|c\| + \frac{\alpha}{\beta}.$$

Subtracting α/β and multiplying by β , we obtain

$$(\beta - \alpha)(1 - \|c\|) \leq \alpha\beta \|u - v\|.$$

Thus, regardless of the order of α, β ,

$$|\alpha - \beta| \leq \frac{\alpha\beta}{1 - \|c\|} \|u - v\|. \quad (2.6)$$

Taking $u = v$ forces $\alpha = \beta$, proving uniqueness.

We may now write the unique radius as $\alpha_c(v)$. Since

$$\alpha_c(v) - \|c\| \leq \|c + \alpha_c(v)v\| = 1 \leq \|c\| + \alpha_c(v),$$

we have $1 - \|c\| \leq \alpha_c(v) \leq 1 + \|c\|$. Finally, applying (2.6) with $\alpha = \alpha_c(u)$ and $\beta = \alpha_c(v)$ gives (2.5):

$$|\alpha_c(u) - \alpha_c(v)| \leq \frac{\alpha_c(u)\alpha_c(v)}{1 - \|c\|} \|u - v\| \leq \frac{(1 + \|c\|)^2}{1 - \|c\|} \|u - v\|. \quad \square$$

3. The canonical sphere map between ultrapowers

Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For a Banach space X , write

$$X_{\mathcal{U}} = \ell_{\infty}(\mathbb{N}, X) / \mathcal{N}_{\mathcal{U}}(X), \quad \mathcal{N}_{\mathcal{U}}(X) = \{(x_n) \in \ell_{\infty}(\mathbb{N}, X) : \lim_{\mathcal{U}} \|x_n\| = 0\}.$$

We denote the class of a bounded sequence by $[x_n]_{\mathcal{U}}$, with norm

$$\|[x_n]_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_n\|.$$

This is the standard Banach-space ultrapower; it is already complete. For the construction and completeness, see [2, p. IX.1]. We use the standard existence and elementary algebraic properties of ultralimits of bounded real sequences.

The point of the next lemma is that the *given* sphere map induces a specified map between the *given* ultrapowers. It preserves the full defect bound, while turning an approximate maximum into an attained one.

Lemma 3.1 (Canonical extension to the ultrapowers). *Let X, Y be nonzero real Banach spaces and let $f : S_X \rightarrow S_Y$ be a surjective isometry. There is a canonical surjective isometry*

$$f_{\mathcal{U}} : S_{X_{\mathcal{U}}} \longrightarrow S_{Y_{\mathcal{U}}}, \quad f_{\mathcal{U}}([x_n]_{\mathcal{U}}) = [f(x_n)]_{\mathcal{U}} \quad (x_n \in S_X).$$

For all unit representatives $x_n, y_n \in S_X$ and every $\lambda \in [0, 1]$,

$$D_{\lambda}^{f_{\mathcal{U}}}([x_n]_{\mathcal{U}}, [y_n]_{\mathcal{U}}) = \lim_{\mathcal{U}} D_{\lambda}^f(x_n, y_n). \quad (3.1)$$

In particular,

$$\mu(f_{\mathcal{U}}) = \mu(f) = \max_{\substack{x, y \in S_{X_{\mathcal{U}}} \\ 0 \leq \lambda \leq 1}} |D_{\lambda}^{f_{\mathcal{U}}}(x, y)|. \quad (3.2)$$

If $\mu(f) > 0$, a maximizing radius belongs to $[\mu(f)/2, 1 - \mu(f)/2]$.

Proof. Every element of $S_{X_{\mathcal{U}}}$ has a representative consisting entirely of unit vectors. Indeed, normalize each nonzero term of an arbitrary representative, replacing zero terms by one fixed unit vector. The change in the n -th term has norm $|\|x_n\| - 1|$, whose ultralimit is zero. For unit representatives,

$$\|[f(x_n)]_{\mathcal{U}} - [f(y_n)]_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|f(x_n) - f(y_n)\| = \lim_{\mathcal{U}} \|x_n - y_n\|.$$

Thus the proposed definition is independent of representatives and is isometric. Applying f^{-1} term by term gives its inverse. The norm formula for the ultrapower gives (3.1) directly. It yields $\mu(f_{\mathcal{U}}) \leq \mu(f)$, while constant sequences give the reverse inequality.

Choose $x_n, y_n \in S_X$ and $\lambda_n \in [0, 1]$ such that

$$|D_{\lambda_n}^f(x_n, y_n)| \longrightarrow \mu(f), \quad \tau = \lim_{\mathcal{U}} \lambda_n.$$

By (1.4),

$$\lim_{\mathcal{U}} |D_{\tau}^f(x_n, y_n) - D_{\lambda_n}^f(x_n, y_n)| = 0.$$

Equation (3.1) therefore gives

$$|D_{\tau}^{f_{\mathcal{U}}}([x_n]_{\mathcal{U}}, [y_n]_{\mathcal{U}})| = \mu(f).$$

The restriction on τ follows from (1.5). □

Remark 3.2 (The sign of the attained defect). The canonical map $f_{\mathcal{U}}$ attains the maximum of the *absolute* defect. If an attained value is negative, its inverse $(f_{\mathcal{U}})^{-1} = (f^{-1})_{\mathcal{U}}$ attains the positive value $\mu(f)$, by (1.3). Thus a positive defect would produce, on these specified ultrapowers, a sphere isometry with an attained positive maximum and the same bound at *every* pair and *every* radius.

4. Darbo's fixed point theorem

We collect the fixed-point background in this section for the reader. Schauder's theorem states that every continuous self-map of a nonempty compact convex subset of a normed space has a fixed point [8]. We will apply it after constructing a compact invariant subset. The measure of noncompactness provides the required compactness criterion; the properties and fixed-point argument below are classical [4, 3].

For a bounded set H in a normed space, define

$$\chi(H) = \inf \left\{ \varepsilon > 0 : \begin{array}{l} H = H_1 \cup \cdots \cup H_m \text{ for finitely many subsets } H_i \subset H, \\ \text{diam } H_i \leq \varepsilon \text{ for } 1 \leq i \leq m \end{array} \right\}. \quad (4.1)$$

Set $\chi(\emptyset) = 0$. This is the Kuratowski measure of noncompactness, with the diameter convention of [4, 3]. Recall that a set is *totally bounded* if, for every $\varepsilon > 0$, it has a finite cover by balls of radius ε . We use the elementary metric-space criterion that a space is compact if and only if it is complete and totally bounded.

Lemma 4.1. *Let H be a bounded nonempty subset of a normed space X . Then:*

1. *If $A \subset H$, then $\chi(A) \leq \chi(H)$, and $\chi(\overline{H}) = \chi(H)$.*
2. *$\chi(H) = 0$ if and only if H is totally bounded. If X is Banach, these conditions are also equivalent to compactness of $\overline{H}$.*

$$3. \chi(\overline{\text{conv } H}) = \chi(H).$$

Proof. 1. Subsets and closure. Suppose $H = H_1 \cup \dots \cup H_m$, with $\text{diam } H_i \leq \varepsilon$. Then

$$A = \bigcup_{i=1}^m (A \cap H_i), \quad \text{diam}(A \cap H_i) \leq \varepsilon$$

after empty members are discarded. This proves monotonicity. Because the union is finite,

$$\overline{H} = \bigcup_{i=1}^m \overline{H_i}, \quad \text{diam } \overline{H_i} = \text{diam } H_i \leq \varepsilon.$$

Thus $\chi(\overline{H}) \leq \chi(H)$; the opposite inequality follows from $H \subset \overline{H}$.

2. Zero measure and compactness. Suppose $\chi(H) = 0$, and fix $\varepsilon > 0$. Take a finite cover by nonempty sets H_i of diameter at most $\varepsilon/2$, and choose $x_i \in H_i$. For every $x \in H_i$,

$$\|x - x_i\| \leq \text{diam } H_i \leq \frac{\varepsilon}{2} < \varepsilon.$$

Hence the balls of radius ε about the x_i cover H . Conversely, if H is totally bounded, cover it by finitely many balls of radius $\varepsilon/2$. Their intersections with H have diameter at most ε , so $\chi(H) \leq \varepsilon$. Letting $\varepsilon \downarrow 0$ gives $\chi(H) = 0$.

By part 1, $\chi(H) = 0$ is equivalent to $\chi(\overline{H}) = 0$, hence to total boundedness of $\overline{H}$. If X is Banach, its closed subset $\overline{H}$ is complete. The metric-space criterion therefore gives precisely the asserted equivalence with compactness.

3. Closed convex hulls. Fix $\varepsilon > \chi(H)$, and choose a finite cover $H = H_1 \cup \dots \cup H_m$ by nonempty sets of diameter at most ε . Choose $\beta > 0$ such that $\|x\| \leq \beta$ for every $x \in H$.

First, convexification does not increase the diameter of an individual H_i . Indeed, if $x = \sum_j \lambda_j x_j$ and $y = \sum_k \theta_k y_k$ are convex combinations of points in H_i , then

$$x - y = \sum_{j,k} \lambda_j \theta_k (x_j - y_k), \quad \|x - y\| \leq \sum_{j,k} \lambda_j \theta_k \varepsilon = \varepsilon.$$

Thus $\text{diam}(\text{conv } H_i) \leq \varepsilon$. Also, every point in $\text{conv } H_i$ has norm at most β .

To cover the whole convex hull, keep track of the coefficients assigned to the different H_i . Let

$$\Delta_m = \left\{ (\lambda_1, \dots, \lambda_m) \in [0, 1]^m : \sum_{i=1}^m \lambda_i = 1 \right\}.$$

Grouping the terms of a convex combination according to the set H_i containing each term gives

$$\text{conv } H = \left\{ \sum_{i=1}^m \lambda_i x_i : \lambda \in \Delta_m, \quad x_i \in \text{conv } H_i \right\}.$$

If $\lambda_i = 0$, choose any $x_i \in H_i$.

Fix $\zeta > 0$. Partition each coordinate interval $[0, 1]$ into finitely many intervals of length at most ζ/m , and intersect the resulting boxes with Δ_m . This gives a finite cover $Q_1, \dots, Q_N$ of Δ_m such that

$$\lambda, \theta \in Q_j \implies \sum_{i=1}^m |\lambda_i - \theta_i| \leq \zeta.$$

The corresponding sets

$$V_j = \left\{ \sum_{i=1}^m \lambda_i x_i : \lambda \in Q_j, \quad x_i \in \operatorname{conv} H_i \right\}$$

cover $\operatorname{conv} H$. For two points of the same V_j ,

$$\begin{aligned} \left\| \sum_{i=1}^m \lambda_i x_i - \sum_{i=1}^m \theta_i y_i \right\| &= \left\| \sum_{i=1}^m \lambda_i (x_i - y_i) + \sum_{i=1}^m (\lambda_i - \theta_i) y_i \right\| \\ &\leq \sum_{i=1}^m \lambda_i \|x_i - y_i\| + \sum_{i=1}^m |\lambda_i - \theta_i| \|y_i\| \\ &\leq \varepsilon + \beta \zeta. \end{aligned}$$

Therefore

$$\chi(\operatorname{conv} H) \leq \varepsilon + \beta \zeta.$$

Let $\zeta \downarrow 0$, then $\varepsilon \downarrow \chi(H)$. Together with monotonicity and part 1, this gives

$$\chi(H) \leq \chi(\operatorname{conv} H) = \chi(\overline{\operatorname{conv} H}) \leq \chi(H). \quad \square$$

Lemma 4.2 (Nested sets with vanishing noncompactness). *Let K_n be nonempty closed bounded subsets of a Banach space, with $K_{n+1} \subset K_n$ and $\chi(K_n) \rightarrow 0$. Then $K = \bigcap_{n \geq 0} K_n$ is nonempty and compact.*

Proof. We first prove nonemptiness by finding a convergent subsequence of points $x_n \in K_n$. Starting with the set I_0 of all indices, we choose infinite subsets $I_j \subset I_{j-1}$ such that

$$p, q \in I_j \implies \|x_p - x_q\| \leq 2^{-j}. \quad (4.2)$$

For the j -th choice, take N with $\chi(K_N) < 2^{-j}$, and cover K_N by finitely many sets of diameter at most 2^{-j} . Infinitely many indices $n \in I_{j-1}$ satisfy $n \geq N$, and $x_n \in K_n \subset K_N$ for all of them. One member of the finite cover contains infinitely many of these x_n . Let I_j be their indices. This gives (4.2).

Choose increasing indices $n_j \in I_j$. If $p, q \geq j$, then $n_p, n_q \in I_j$, so

$$\|x_{n_p} - x_{n_q}\| \leq 2^{-j}.$$

The subsequence is Cauchy and therefore converges to some x . For each fixed n , all sufficiently late terms belong to K_n ; closedness gives $x \in K_n$. Hence $x \in K$, proving $K \neq \emptyset$.

For compactness, use the inclusions $K \subset K_n$:

$$0 \leq \chi(K) \leq \chi(K_n) \longrightarrow 0.$$

Thus K is totally bounded by Lemma 4.1. It is also closed in a Banach space, hence complete. The same lemma now gives compactness. $\square$

Theorem 4.3 (Darbo). *Let C be a nonempty closed bounded convex subset of a Banach space. Suppose $g : C \rightarrow C$ is continuous and, for one constant $0 \leq \kappa < 1$,*

$$\chi(g(H)) \leq \kappa \chi(H) \quad (H \subset C). \quad (4.3)$$

Then g has a fixed point.

Proof. We construct a compact convex set on which to apply Schauder's theorem. Set

$$K_0 = C, \quad K_{n+1} = \overline{\text{conv}(g(K_n))}.$$

Since $g(C) \subset C$ and C is closed and convex, $K_1 \subset K_0$. Inductively,

$$K_n \subset K_{n-1} \implies g(K_n) \subset g(K_{n-1}) \implies K_{n+1} \subset K_n.$$

The K_n are therefore nonempty closed bounded convex subsets of C . By Lemma 4.1 and (4.3),

$$\chi(K_{n+1}) = \chi(g(K_n)) \leq \kappa \chi(K_n), \quad \chi(K_n) \leq \kappa^n \chi(C) \longrightarrow 0.$$

By Lemma 4.2, $K = \bigcap_{n \geq 0} K_n$ is nonempty and compact. It is convex because every K_n is convex. For $x \in K$ and every n ,

$$g(x) \in g(K_n) \subset K_{n+1}.$$

Also $g(x) \in C = K_0$, so $g(K) \subset K$. Schauder's theorem [8] applied to $g|_K$ gives a fixed point. $\square$

5. Aligning two extremal chords

Lemma 3.1 realizes the absolute defect as a maximum, and Remark 3.2 makes its sign positive if necessary. The next proposition applies to any sphere isometry with such a maximum. Its purpose is to find two endpoints so that the two spheres have chords through the prescribed inner points τy and $\tau f(y)$.

Proposition 5.1 (Aligned extremal chords). *Let $f : S_X \rightarrow S_Y$ be a surjective isometry between the spheres of nonzero real Banach spaces. Suppose $\mu = \mu(f) > 0$ and*

$$D_\tau^f(x_0, y) = \mu \quad (x_0, y \in S_X, 0 < \tau < 1).$$

Then there are $x, z \in S_X$ such that

$$\tau y \in (x, z), \quad \tau f(y) \in (f(x), f(z)), \quad (5.1)$$

and

$$D_\tau^f(x, y) = \mu, \quad D_\tau^f(z, y) = -\mu. \quad (5.2)$$

Proof. Throughout the proof, f, y, τ, μ are fixed. We construct the chord in Y first. Taking its opposite endpoint back by f^{-1} defines a map $g : S_X \rightarrow S_X$ of directions in X ; a fixed point of g will give the chord through τy . We then find a closed convex subset of S_X on which Darbo's theorem applies.

1. Define the endpoints and the map of directions. Let $v \in S_X$. Since $\|\tau y\| = \tau < 1$, Lemma 2.2 gives the unique positive number $\alpha_{\tau y}(v)$ such that

$$x(v) := \tau y + \alpha_{\tau y}(v)v \in S_X. \quad (5.3)$$

The direction from $\tau f(y)$ to $f(x(v))$ is

$$w(v) := \frac{f(x(v)) - \tau f(y)}{\|f(x(v)) - \tau f(y)\|} \in S_Y. \quad (5.4)$$

Apply Lemma 2.2 in Y , with centre $\tau f(y)$ and direction $-w(v)$, and define

$$z(v) := f^{-1}\left(\tau f(y) - \alpha_{\tau f(y)}(-w(v)) w(v)\right) \in S_X. \quad (5.5)$$

The two identities

$$\begin{aligned} f(x(v)) - \tau f(y) &= \|f(x(v)) - \tau f(y)\| w(v), \\ f(z(v)) - \tau f(y) &= -\alpha_{\tau f(y)}(-w(v)) w(v) \end{aligned}$$

have positive coefficients on opposite rays. Therefore

$$\tau f(y) \in (f(x(v)), f(z(v))).$$

Define the corresponding direction in X by

$$g(v) := \frac{\tau y - z(v)}{\|\tau y - z(v)\|} \in S_X. \quad (5.6)$$

Since

$$x(v) = \tau y + \alpha_{\tau y}(v)v, \quad z(v) = \tau y - \|\tau y - z(v)\| g(v),$$

we have

$$g(v) = v \iff \tau y \in (x(v), z(v)). \quad (5.7)$$

To see both implications explicitly, first suppose that $g(v) = v$. The two endpoint formulas give

$$\tau y = \frac{\|\tau y - z(v)\| x(v) + \alpha_{\tau y}(v) z(v)}{\|\tau y - z(v)\| + \alpha_{\tau y}(v)}.$$

Both coefficients in this convex combination are strictly positive, so $\tau y \in (x(v), z(v))$. Conversely, suppose $\tau y = (1 - \lambda)x(v) + \lambda z(v)$ for some $0 < \lambda < 1$. Then

$$\tau y - z(v) = \frac{1 - \lambda}{\lambda} (x(v) - \tau y) = \frac{1 - \lambda}{\lambda} \alpha_{\tau y}(v)v.$$

The coefficient of the unit vector v is positive. Normalizing this identity in (5.6) gives $g(v) = v$. Thus a fixed point gives exactly the missing segment condition.

All these maps are continuous. Indeed, the radial-contact functions are continuous by Lemma 2.2, f and f^{-1} are isometries, and the denominators satisfy

$$\begin{aligned} \|f(x(v)) - \tau f(y)\| &\geq \|f(x(v))\| - \tau \|f(y)\| = 1 - \tau > 0, \\ \|\tau y - z(v)\| &\geq \|z(v)\| - \tau \|y\| = 1 - \tau > 0. \end{aligned}$$

We seek a fixed point among the maximizing directions

$$E = \{v \in S_X : D_\tau^f(x(v), y) = \mu\}. \quad (5.8)$$

Continuity shows that E is closed in S_X , hence in X . To check nonemptiness, put

$$v_0 = \frac{x_0 - \tau y}{\|x_0 - \tau y\|} \in S_X.$$

Here $\|x_0 - \tau y\| \geq 1 - \tau > 0$, and

$$x_0 = \tau y + \|x_0 - \tau y\| v_0 \in S_X.$$

Uniqueness in Lemma 2.2 gives

$$\alpha_{\tau y}(v_0) = \|x_0 - \tau y\|, \quad x(v_0) = \tau y + \alpha_{\tau y}(v_0)v_0 = x_0.$$

Consequently,

$$D_\tau^f(x(v_0), y) = D_\tau^f(x_0, y) = \mu, \quad v_0 \in E.$$

2. Use extremality to obtain and propagate common supports. For $v \in E$, the segment condition in Y and the isometry give

$$\|x(v) - z(v)\| = \|f(x(v)) - \tau f(y)\| + \|f(z(v)) - \tau f(y)\|.$$

Subtract this identity from the triangle inequality in X :

$$\begin{aligned} 0 &\leq \|x(v) - \tau y\| + \|z(v) - \tau y\| - \|x(v) - z(v)\| \\ &= -D_\tau^f(x(v), y) - D_\tau^f(z(v), y) \\ &= -\mu - D_\tau^f(z(v), y) \leq 0. \end{aligned} \tag{5.9}$$

The last inequality uses $D_\tau^f(z(v), y) \geq -\mu$, which is part of the global bound (1.2). Every inequality is therefore an equality, so

$$D_\tau^f(z(v), y) = -\mu \quad (v \in E). \tag{5.10}$$

Moreover,

$$x(v) - z(v) = \|x(v) - \tau y\| v + \|z(v) - \tau y\| g(v)$$

has norm equal to the sum of its two positive coefficients. By (2.3),

$$\nu(v) \cap \nu(g(v)) \neq \emptyset \quad (v \in E). \tag{5.11}$$

To construct a convex subset of E , we need the following propagation property:

$$v \in E, \quad u \in S_X, \quad \nu(u) \cap \nu(g(v)) \neq \emptyset \implies u \in E. \tag{5.12}$$

Fix $v \in E$, so that the point $z(v)$ stays fixed while u varies. The assumed common support and (2.3) give

$$\begin{aligned} \|x(u) - \tau y\| + \|z(v) - \tau y\| &= \|x(u) - z(v)\| \\ &= \|f(x(u)) - f(z(v))\| \\ &\leq \|f(x(u)) - \tau f(y)\| + \|f(z(v)) - \tau f(y)\|. \end{aligned}$$

After rearrangement and (5.10), this reads

$$\mu = -D_\tau^f(z(v), y) \leq D_\tau^f(x(u), y) \leq \mu.$$

Thus $u \in E$, proving (5.12). In particular, $u = g(v)$ satisfies the common-support assumption because $\nu(g(v)) \neq \emptyset$. Hence $g(E) \subset E$.

3. Construct a closed convex set preserved by g . We first prove that, for every convex $K \subset E$ and every $u \in K$,

$$\text{conv}(K \cup \{g(u)\}) \subset E. \tag{5.13}$$

The point is to extend from the known support relation between u and $g(u)$ to every segment joining K to $g(u)$.

Fix $v \in K$. For $0 < \lambda < 1$, set

$$u_\lambda = (1 - \lambda)u + \lambda v \in K \subset E \subset S_X.$$

By (5.11), choose

$$\varphi_\lambda \in \nu(u_\lambda) \cap \nu(g(u_\lambda)).$$

Since u_λ is an interior convex combination of $u, v \in S_X$, Lemma 2.1 gives

$$\varphi_\lambda \in \nu(u) \cap \nu(v), \quad \varphi_\lambda(v) = \varphi_\lambda(g(u_\lambda)) = 1.$$

For $0 \leq \theta \leq 1$, it follows that

$$1 = \varphi_\lambda((1 - \theta)v + \theta g(u_\lambda)) \leq \|(1 - \theta)v + \theta g(u_\lambda)\| \leq (1 - \theta) + \theta = 1.$$

Thus every point of $[v, g(u_\lambda)]$ belongs to S_X and has the common support φ_λ with $g(u_\lambda)$. Apply (5.12) with base direction u_λ :

$$[v, g(u_\lambda)] \subset E.$$

For each $\theta \in [0, 1]$, continuity of g gives

$$(1 - \theta)v + \theta g(u_\lambda) \longrightarrow (1 - \theta)v + \theta g(u) \quad (\lambda \downarrow 0).$$

Since E is closed, $[v, g(u)] \subset E$. As $v \in K$ was arbitrary and K is convex,

$$\text{conv}(K \cup \{g(u)\}) = \bigcup_{v \in K} [v, g(u)] \subset E.$$

We may add finitely many points of $g(K)$ in the same way: after each addition, the enlarged set is convex, lies in E , and still contains the original K . Every convex combination uses finitely many points, so

$$\text{conv}(K \cup g(K)) \subset E. \quad (5.14)$$

Starting from $v_0 \in E$, define

$$C_0 = \{v_0\}, \quad C_{n+1} = \text{conv}(C_n \cup g(C_n)), \quad C = \overline{\bigcup_{n \geq 0} C_n}. \quad (5.15)$$

By (5.14), the C_n are increasing convex subsets of E . Their union is convex, and E is closed, so $C \subset E \subset S_X$ is nonempty, closed, bounded and convex. Finally, continuity and $g(C_n) \subset C_{n+1}$ give

$$g(C) \subset g\left(\overline{\bigcup_{n \geq 0} C_n}\right) \subset \overline{\bigcup_{n \geq 0} C_{n+1}} = C.$$

4. Estimate the change of direction. To apply Theorem 4.3, we will show that g contracts χ . The calculation below separates a uniform contraction from errors depending on two bounded real functions. A finite partition of their ranges will make these errors as small as needed on each member of a cover.

For $v \in C$, define the four endpoint distances

$$\begin{aligned} \alpha(v) &= \|x(v) - \tau y\|, & \beta(v) &= \|z(v) - \tau y\|, \\ \gamma(v) &= \|f(x(v)) - \tau f(y)\|, & \delta(v) &= \|f(z(v)) - \tau f(y)\|. \end{aligned} \quad (5.16)$$

Thus $\alpha(v) = \alpha_{\tau y}(v)$ and $\delta(v) = \alpha_{\tau f(y)}(-w(v))$. For unit vectors a, b in the same space, the triangle inequalities give

$$1 - \tau = \|a\| - \tau\|b\| \leq \|a - \tau b\| \leq \|a\| + \tau\|b\| = 1 + \tau.$$

Each of the four distances in (5.16) therefore belongs to $[1 - \tau, 1 + \tau]$. Since $v \in C \subset E$, the defining defect and (5.10) give

$$\gamma(v) - \alpha(v) = \mu, \quad \beta(v) - \delta(v) = \mu. \quad (5.17)$$

The point formulas, with $w(v)$ as defined in (5.4), are

$$\begin{aligned} x(v) &= \tau y + \alpha(v)v, & z(v) &= \tau y - \beta(v)g(v), \\ f(x(v)) &= \tau f(y) + \gamma(v)w(v), & f(z(v)) &= \tau f(y) - \delta(v)w(v). \end{aligned} \quad (5.18)$$

Let $u, v \in C$. Start with the difference in (5.6), retaining both normalizing denominators:

$$\begin{aligned} g(u) - g(v) &= \frac{\tau y - z(u)}{\beta(u)} - \frac{\tau y - z(v)}{\beta(v)} \\ &= \frac{z(v) - z(u)}{\beta(u)} + \left(\frac{1}{\beta(u)} - \frac{1}{\beta(v)} \right) (\tau y - z(v)) \\ &= \frac{z(v) - z(u)}{\beta(u)} + \frac{\beta(v) - \beta(u)}{\beta(u)} g(v). \end{aligned} \quad (5.19)$$

The remaining differences follow directly from (5.18):

$$\begin{aligned} f(z(u)) - f(z(v)) &= -\delta(u)(w(u) - w(v)) + (\delta(v) - \delta(u))w(v), \\ \gamma(u)(w(u) - w(v)) &= f(x(u)) - f(x(v)) + (\gamma(v) - \gamma(u))w(v), \\ x(u) - x(v) &= \alpha(u)(u - v) + (\alpha(u) - \alpha(v))v. \end{aligned} \quad (5.20)$$

By (5.17),

$$\beta(u) - \beta(v) = \delta(u) - \delta(v), \quad \gamma(u) - \gamma(v) = \alpha(u) - \alpha(v).$$

Taking norms in (5.19) and the first line of (5.20), and using that f is an isometry, gives

$$\begin{aligned} \beta(u)\|g(u) - g(v)\| &\leq \|z(u) - z(v)\| + |\beta(u) - \beta(v)| \\ &= \|f(z(u)) - f(z(v))\| + |\delta(u) - \delta(v)| \\ &\leq \delta(u)\|w(u) - w(v)\| + 2|\delta(u) - \delta(v)|. \end{aligned}$$

The other two lines similarly give

$$\begin{aligned} \gamma(u)\|w(u) - w(v)\| &\leq \|f(x(u)) - f(x(v))\| + |\gamma(u) - \gamma(v)| \\ &= \|x(u) - x(v)\| + |\alpha(u) - \alpha(v)| \\ &\leq \alpha(u)\|u - v\| + 2|\alpha(u) - \alpha(v)|. \end{aligned}$$

Substitution yields

$$\begin{aligned} \beta(u)\|g(u) - g(v)\| &\leq 2|\delta(u) - \delta(v)| \\ &\quad + \frac{\delta(u)}{\gamma(u)}(\alpha(u)\|u - v\| + 2|\alpha(u) - \alpha(v)|). \end{aligned} \quad (5.21)$$

The positive defect creates the contraction coefficient:

$$\frac{\alpha(u)\delta(u)}{\beta(u)\gamma(u)} = \left(1 - \frac{\mu}{\beta(u)}\right) \left(1 - \frac{\mu}{\gamma(u)}\right) \leq \kappa := \left(1 - \frac{\mu}{1+\tau}\right)^2 < 1. \quad (5.22)$$

Indeed, $0 < \mu < \beta(u), \gamma(u) \leq 1 + \tau$ by (5.17). For the error coefficients, note that $\beta(u) - \delta(u) = \mu > 0$ by (5.17), so $\delta(u) < \beta(u)$. Together with $\beta(u), \gamma(u) \geq 1 - \tau$, this gives

$$\frac{1}{\beta(u)} \leq \frac{1}{1-\tau}, \quad \frac{\delta(u)}{\beta(u)\gamma(u)} \leq \frac{1}{\gamma(u)} \leq \frac{1}{1-\tau}.$$

Dividing (5.21) by $\beta(u)$, we obtain

$$\begin{aligned} \|g(u) - g(v)\| &\leq \kappa \|u - v\| \\ &\quad + \frac{2}{1-\tau} (|\alpha(u) - \alpha(v)| + |\delta(u) - \delta(v)|) \quad (u, v \in C). \end{aligned} \quad (5.23)$$

Now let $H \subset C$ be any nonempty subset. Fix $\varepsilon > \chi(H)$ and $\zeta > 0$. Take a finite cover $H = \bigcup_i H_i$ with $\text{diam } H_i \leq \varepsilon$. Cover the bounded ranges of α and δ by finitely many intervals I_j and J_k , respectively, each of length at most ζ . The sets

$$H_{i,j,k} = H_i \cap \alpha^{-1}(I_j) \cap \delta^{-1}(J_k)$$

form a finite cover of H . If u, v belong to the same member, then

$$\|u - v\| \leq \varepsilon, \quad |\alpha(u) - \alpha(v)| \leq \zeta, \quad |\delta(u) - \delta(v)| \leq \zeta.$$

Thus (5.23) gives

$$\text{diam } g(H_{i,j,k}) \leq \kappa \varepsilon + \frac{4\zeta}{1-\tau}.$$

These images form a finite cover of $g(H)$, so

$$\chi(g(H)) \leq \kappa \varepsilon + \frac{4\zeta}{1-\tau}.$$

Let $\zeta \downarrow 0$, and then $\varepsilon \downarrow \chi(H)$. The empty case is immediate. We have proved

$$\chi(g(H)) \leq \kappa \chi(H) \quad (H \subset C). \quad (5.24)$$

5. Apply the fixed-point theorem. The set C is nonempty, closed, bounded and convex, and $g : C \rightarrow C$ is continuous. Together with (5.24) and $0 \leq \kappa < 1$, these are the hypotheses of Theorem 4.3. Choose a fixed point $v_* \in C \subset E$, and set

$$x = x(v_*), \quad z = z(v_*).$$

The identity $g(v_*) = v_*$ gives $\tau y \in (x, z)$ by (5.7). The construction of $z(v_*)$ gives $\tau f(y) \in (f(x), f(z))$. Finally, $v_* \in E$ and (5.10) give $D_\tau^f(x, y) = \mu$ and $D_\tau^f(z, y) = -\mu$. The endpoint x obtained this way need not equal the original maximizing point x_0 . $\square$

6. The contradiction from the two chord coordinates

Let $f : S_X \rightarrow S_Y$ be a surjective isometry between the spheres of nonzero real Banach spaces. Suppose $D_\tau^f(x_0, y) = \mu(f) > 0$, and write $\mu = \mu(f)$. By (1.5), $0 < \tau < 1$. Choose $x, z \in S_X$ as in Proposition 5.1. Thus

$$\tau y \in (x, z), \quad \tau f(y) \in (f(x), f(z)), \quad D_\tau^f(x, y) = \mu, \quad D_\tau^f(z, y) = -\mu.$$

All endpoints are now fixed. Define their distances from the two inner points by

$$\begin{aligned} \alpha &= \|x - \tau y\|, & \beta &= \|z - \tau y\|, \\ \gamma &= \|f(x) - \tau f(y)\|, & \delta &= \|f(z) - \tau f(y)\|. \end{aligned}$$

These are positive constants, each at least $1 - \tau$. With

$$v = \frac{x - \tau y}{\alpha} \in S_X, \quad w = \frac{f(x) - \tau f(y)}{\gamma} \in S_Y,$$

the two segment conditions give

$$\begin{aligned} x &= \tau y + \alpha v, & z &= \tau y + \beta v, \\ f(x) &= \tau f(y) + \gamma w, & f(z) &= \tau f(y) + \delta w. \end{aligned} \tag{6.1}$$

The defect equalities become

$$\gamma - \alpha = D_\tau^f(x, y) = \mu, \quad \beta - \delta = -D_\tau^f(z, y) = \mu.$$

Since the endpoints lie on opposite rays from each inner point, distance preservation gives

$$\alpha + \beta = \|x - z\| = \|f(x) - f(z)\| = \gamma + \delta \leq 2.$$

We will use these relations in the form

$$\gamma - \alpha = \beta - \delta = \mu, \quad \rho := \alpha + \beta = \gamma + \delta \leq 2. \tag{6.2}$$

Define the two chord coordinates and the remaining radial distance by

$$\xi = \frac{\alpha}{\rho}, \quad \eta = \frac{\delta}{\rho}, \quad \varepsilon = 1 - \tau. \tag{6.3}$$

Their meaning follows by substituting (6.1):

$$\begin{aligned} (1 - \xi)x + \xi z &= \tau y + (\alpha - \xi(\alpha + \beta))v = \tau y, \\ \eta f(x) + (1 - \eta)f(z) &= \tau f(y) + (\eta(\gamma + \delta) - \delta)w = \tau f(y). \end{aligned}$$

Thus

$$\tau y = (1 - \xi)x + \xi z, \quad \tau f(y) = \eta f(x) + (1 - \eta)f(z). \tag{6.4}$$

The positive defect appears as a gap between the coordinates:

$$\xi, \eta > 0, \quad \xi + \eta = \frac{\alpha + \delta}{\rho} = \frac{\alpha + \beta - \mu}{\rho} = 1 - \frac{\mu}{\rho} < 1. \tag{6.5}$$

We first derive a support estimate from the global bound on the defect. That estimate will show that $\xi, \eta \leq \varepsilon$: a larger coordinate would give a radial move that increases the defect. Once both coordinates are at most ε , the two identities in (6.4) give incompatible bounds on the preserved distance between y and z .

6.1 A support estimate from cancellation of one norm

Decrease the radius at x by σ , and increase it at z by $(\delta/\gamma)\sigma$. The resulting target vectors are

$$\begin{aligned} f(x) - (\tau - \sigma)f(y) &= \gamma w + \sigma f(y), \\ f(z) - (\tau + (\delta/\gamma)\sigma)f(y) &= -\frac{\delta}{\gamma}(\gamma w + \sigma f(y)). \end{aligned}$$

Their norms therefore cancel in a weighted difference of defects. Choose

$$\sigma = \min \left\{ \tau, \frac{\gamma\varepsilon}{\delta} \right\}, \quad (6.6)$$

so that

$$0 \leq \tau - \sigma < \tau < \tau + \frac{\delta}{\gamma}\sigma \leq 1,$$

and at least one of the two new radii equals 0 or 1. The defect at that radius is zero.

Lemma 6.1 (The support estimate). *For every $\varphi \in \nu(\alpha v + \sigma y)$,*

$$\frac{\varepsilon}{\alpha} \leq \varphi(v) \leq \begin{cases} 1 - \eta, & \eta \leq \varepsilon, \\ \eta, & \eta > \varepsilon. \end{cases} \quad (6.7)$$

Proof. The support set is nonempty: the reverse triangle inequality gives

$$\|\alpha v + \sigma y\| \geq \|\alpha v + \tau y\| - (\tau - \sigma) = 1 - \tau + \sigma = \varepsilon + \sigma > 0.$$

Let $\varphi \in \nu(\alpha v + \sigma y)$. Using $\varphi(y) \leq 1$, we obtain

$$\begin{aligned} 1 &= \|\alpha v + \tau y\| \leq \|\alpha v + \sigma y\| + (\tau - \sigma)\|y\| \\ &= \alpha\varphi(v) + \sigma\varphi(y) + \tau - \sigma \\ &\leq \alpha\varphi(v) + \tau. \end{aligned}$$

This proves $\varepsilon/\alpha \leq \varphi(v)$.

For the upper bound, write both defects with the common target norm:

$$\begin{aligned} D_{\tau-\sigma}^f(x, y) &= \|\gamma w + \sigma f(y)\| - \|\alpha v + \sigma y\|, \\ -\frac{\gamma}{\delta} D_{\tau+(\delta/\gamma)\sigma}^f(z, y) &= \left\| \frac{\gamma\beta}{\delta} v + \sigma y \right\| - \|\gamma w + \sigma f(y)\|. \end{aligned} \quad (6.8)$$

The coefficient in the difference of the two source vectors is

$$\gamma\beta - \alpha\delta = \beta(\gamma - \alpha) + \alpha(\beta - \delta) = \mu(\alpha + \beta) = \mu\rho$$

by (6.2). Adding (6.8) and evaluating the first norm with the support φ therefore gives

$$\begin{aligned} D_{\tau-\sigma}^f(x, y) - \frac{\gamma}{\delta} D_{\tau+(\delta/\gamma)\sigma}^f(z, y) &= \left\| \frac{\gamma\beta}{\delta} v + \sigma y \right\| - \|\alpha v + \sigma y\| \\ &\geq \varphi \left(\frac{\gamma\beta}{\delta} v + \sigma y \right) - \varphi(\alpha v + \sigma y) = \frac{\mu\rho}{\delta} \varphi(v). \end{aligned} \quad (6.9)$$

The branch in the definition of σ is determined by η :

$$\eta \leq \varepsilon \iff \delta \leq \varepsilon(\gamma + \delta) \iff \tau\delta \leq \varepsilon\gamma.$$

If $\eta \leq \varepsilon$, then $\sigma = \tau$. The first defect in (6.9) is $D_0^f(x, y) = 0$, so the global defect bound gives

$$\frac{\mu\rho}{\delta}\varphi(v) \leq -\frac{\gamma}{\delta}D_{\tau+(\delta/\gamma)\sigma}^f(z, y) \leq \frac{\gamma\mu}{\delta}.$$

Dividing by $\mu\rho/\delta > 0$ yields $\varphi(v) \leq \gamma/\rho = 1 - \eta$.

If $\eta > \varepsilon$, then $\sigma = \gamma\varepsilon/\delta < \tau$, and $\tau + (\delta/\gamma)\sigma = 1$. The second defect is zero, so

$$\frac{\mu\rho}{\delta}\varphi(v) \leq D_{\tau-\sigma}^f(x, y) \leq \mu.$$

This gives $\varphi(v) \leq \delta/\rho = \eta$. □

6.2 Interchanging the coordinates by using the inverse map

The proof of Lemma 6.1 uses only the two aligned chords (6.1), the extremal length relations (6.2), and the global bound

$$|D_\lambda^f(a, b)| \leq \mu \quad (a, b \in S_X, \ 0 \leq \lambda \leq 1).$$

It does not use how Proposition 5.1 produced the endpoints. We may therefore apply the estimate to any configuration satisfying these conditions.

The two already aligned chords exchange roles when we pass to f^{-1} . The radius τ is unchanged, but the signs of the defects are reversed. We consequently exchange the endpoints: take $f(z)$ as the first endpoint, $f(x)$ as the opposite endpoint, and $f(y)$ as the base point. The same two chords are now written as

$$\begin{aligned} f(z) &= \tau f(y) + \delta(-w), & f(x) &= \tau f(y) - \gamma(-w), \\ z &= \tau y + \beta(-v), & x &= \tau y - \alpha(-v). \end{aligned}$$

The two defects have the required signs:

$$\begin{aligned} D_\tau^{f^{-1}}(f(z), f(y)) &= -D_\tau^f(z, y) = \beta - \delta = \mu, \\ D_\tau^{f^{-1}}(f(x), f(y)) &= -D_\tau^f(x, y) = \alpha - \gamma = -\mu. \end{aligned}$$

By (1.3), $\mu(f^{-1}) = \mu$, so the global defect bound is unchanged as well. The directions become $-w, -v$, and

$$(\alpha, \beta, \gamma, \delta) \mapsto (\delta, \gamma, \beta, \alpha), \quad (\xi, \eta) \mapsto (\eta, \xi). \quad (6.10)$$

The constants $\rho, \tau, \varepsilon, \mu$ stay the same. Thus all the hypotheses of Lemma 6.1 hold in the inverse configuration, with the two chord coordinates interchanged. Its auxiliary displacement is chosen anew by (6.6); there is no need for the displacements in the two orientations to agree.

6.3 A chord coordinate cannot exceed ε

Lemma 6.2. *The chord coordinates satisfy $\xi, \eta \leq \varepsilon$.*

Proof. Because the inverse configuration exchanges ξ and η , it is enough to exclude $\eta > \varepsilon$. Suppose this inequality holds. Then $\sigma = \gamma\varepsilon/\delta < \tau$, and Lemma 6.1 gives

$$\frac{\varepsilon}{\alpha} \leq \varphi(v) \leq \eta \quad (\varphi \in \nu(\alpha v + \sigma y)).$$

Since the support set is nonempty,

$$\varepsilon \leq \alpha\eta = \xi\delta, \quad \sigma = \frac{\gamma\varepsilon}{\delta} \leq \frac{\gamma\alpha}{\rho} = \alpha(1 - \eta). \quad (6.11)$$

Consequently, for any such φ ,

$$\|\alpha v + \sigma y\| = \alpha\varphi(v) + \sigma\varphi(y) \leq \alpha\eta + \sigma \leq \alpha. \quad (6.12)$$

Thus the radial move does not increase the source norm.

We next prove the sign assertion

$$\eta > \varepsilon \implies \varphi(y) < 0 \quad \text{for every } \varphi \in \nu(v). \quad (6.13)$$

If $\varphi \in \nu(v)$ and $\varphi(y) \geq 0$, then

$$\alpha \leq \alpha + \sigma\varphi(y) = \varphi(\alpha v + \sigma y) \leq \|\alpha v + \sigma y\| \leq \alpha.$$

Every inequality is an equality. Hence $\varphi \in \nu(\alpha v + \sigma y)$, and Lemma 6.1 would give

$$1 = \varphi(v) \leq \eta < 1,$$

a contradiction. This proves (6.13).

Choose $\varphi \in \nu(v)$ and apply it to $-z = \beta v - \tau y$. The sign just proved implies

$$\beta < \beta - \tau\varphi(y) = \varphi(-z) \leq \|z\| = 1.$$

Since $\beta - \delta = \mu > 0$, this gives $\delta < \beta < 1$. Hence (6.11) implies

$$\varepsilon \leq \xi\delta < \xi.$$

We now transfer only the sign implication (6.13) to the inverse configuration. To make the substitution explicit, (6.10) gives

$$\begin{aligned} \text{base point: } y &\mapsto f(y), \\ \text{source direction: } v &\mapsto -w, \\ \text{second chord coordinate: } \eta &\mapsto \xi. \end{aligned}$$

The required inequality for that second coordinate is precisely $\xi > \varepsilon$, just proved above. Thus (6.13) gives

$$\varphi(f(y)) < 0 \quad (\varphi \in \nu(-w)).$$

For any $\psi \in \nu(w)$, we have

$$\|-\psi\| = 1, \quad (-\psi)(-w) = \psi(w) = 1,$$

so $-\psi \in \nu(-w)$. Applying the preceding inequality to $-\psi$ yields

$$-\psi(f(y)) = (-\psi)(f(y)) < 0, \quad \text{hence} \quad \psi(f(y)) > 0.$$

This is a sign statement about supports at w ; it does not involve the displacement chosen for the inverse configuration. Choose $\psi \in \nu(w)$ and return to the original $\sigma = \gamma\varepsilon/\delta > 0$. The target norm now strictly increases:

$$\|\gamma w + \sigma f(y)\| \geq \psi(\gamma w + \sigma f(y)) = \gamma + \sigma\psi(f(y)) > \gamma.$$

Together with (6.12), this yields

$$D_{\tau-\sigma}^f(x, y) = \|\gamma w + \sigma f(y)\| - \|\alpha v + \sigma y\| > \gamma - \alpha = \mu.$$

The radius $\tau - \sigma$ belongs to $(0, 1)$, so this contradicts the definition of μ . $\square$

6.4 The remaining coordinates contradict a preserved distance

Proposition 6.3. *Let $f : S_X \rightarrow S_Y$ be a surjective isometry between the spheres of nonzero real Banach spaces. There are no $x, y \in S_X$ and $\tau \in [0, 1]$ such that*

$$D_\tau^f(x, y) = \mu(f) > 0.$$

Proof. Suppose such a maximum exists. By (1.5), its radius lies in $(0, 1)$; Proposition 5.1 then gives the configuration used above. Recall the chord length and coordinates from (6.2)–(6.3):

$$\begin{aligned} \rho &= \|x - z\| = \alpha + \beta = \gamma + \delta \leq 2, \\ \xi &= \frac{\alpha}{\rho}, \quad \eta = \frac{\delta}{\rho}, \quad \varepsilon = 1 - \tau. \end{aligned}$$

In particular, (6.5) gives

$$\xi, \eta > 0, \quad \xi + \eta = 1 - \frac{\mu}{\rho} < 1.$$

By Lemma 6.2, $\xi, \eta \leq \varepsilon$. Applying Lemma 6.1 in the original and inverse configurations gives, respectively,

$$\frac{\varepsilon}{\alpha} \leq 1 - \eta, \quad \frac{\varepsilon}{\delta} \leq 1 - \xi.$$

Substituting $\alpha = \rho\xi$ and $\delta = \rho\eta$, we obtain

$$\varepsilon \leq \rho\xi(1 - \eta), \quad \varepsilon \leq \rho\eta(1 - \xi). \tag{6.14}$$

We first check that $\varepsilon < 1/2$, which will make the coefficients in the final triangle inequalities nonnegative. If $\xi \leq \eta$, then $2\xi \leq \xi + \eta < 1$, so $\xi < 1/2$, and

$$\varepsilon \leq \rho\xi(1 - \eta) \leq 2\xi(1 - \xi) < \frac{1}{2}.$$

If $\eta \leq \xi$, use the other inequality in (6.14):

$$\varepsilon \leq \rho\eta(1 - \xi) \leq 2\eta(1 - \eta) < \frac{1}{2}.$$

Here $\rho \leq 2$ comes from (6.2). We have proved

$$\varepsilon < \frac{1}{2}, \quad \xi \leq \varepsilon < 1 - \varepsilon = \tau, \quad \eta \leq \varepsilon. \tag{6.15}$$

The first chord identity $\tau y = (1 - \xi)x + \xi z$ in (6.4) gives a lower bound for $\|y - z\|$:

$$(1 - \xi)x = (\tau - \xi)y + \xi(y - z). \quad (6.16)$$

Because $\tau - \xi \geq 0$, taking norms gives

$$1 - \xi = \|(\tau - \xi)y + \xi(y - z)\| \leq \tau - \xi + \xi\|y - z\|.$$

Since $\xi > 0$ and $\varepsilon = 1 - \tau$, this becomes

$$\frac{\varepsilon}{\xi} \leq \|y - z\|.$$

The second identity $\tau f(y) = \eta f(x) + (1 - \eta)f(z)$ gives an upper bound in Y :

$$(1 - \eta)(f(y) - f(z)) = (\varepsilon - \eta)f(y) + \eta f(x). \quad (6.17)$$

Here $\varepsilon - \eta \geq 0$, so

$$(1 - \eta)\|f(y) - f(z)\| = \|(\varepsilon - \eta)f(y) + \eta f(x)\| \leq (\varepsilon - \eta) + \eta = \varepsilon.$$

Using $1 - \eta > 0$ and preservation of the sphere distance, we obtain

$$\frac{\varepsilon}{\xi} \leq \|y - z\| = \|f(y) - f(z)\| \leq \frac{\varepsilon}{1 - \eta}. \quad (6.18)$$

But $\xi + \eta < 1$ means $\xi < 1 - \eta$. Since $\varepsilon > 0$, this gives $\varepsilon/\xi > \varepsilon/(1 - \eta)$, contrary to (6.18). $\square$

Proof of Theorem 1.1. Let $f : S_X \rightarrow S_Y$ be the original sphere isometry, now with X, Y merely normed. By Lemma 1.3, its extension $\hat{f} : S_{\hat{X}} \rightarrow S_{\hat{Y}}$ satisfies $\mu(\hat{f}) = \mu(f)$.

Suppose $\mu(f) > 0$. Choose a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. By Lemma 3.1, the canonical map

$$(\hat{f})_{\mathcal{U}} : S_{(\hat{X})_{\mathcal{U}}} \longrightarrow S_{(\hat{Y})_{\mathcal{U}}}$$

satisfies

$$\mu((\hat{f})_{\mathcal{U}}) = \mu(\hat{f}) = \mu(f) > 0,$$

and its absolute defect attains this value. By Remark 3.2, either this map or its inverse attains a positive defect equal to the same global maximum. Both are surjective sphere isometries between Banach spaces, contradicting Proposition 6.3.

Thus $\mu(f) = 0$. By Proposition 1.2, the radial extension is a surjective isometry fixing zero, hence is real-linear and is the unique linear extension of f . $\square$

Further reading

For other presentations of the argument, see [Open Mathematics Courses: Tingley's problem](#) and [Latent Seminar: Tingley's sphere-isometry extension](#). These are supplementary reading suggestions; the attribution of the underlying argument remains to the OpenAI preprint cited above.

Use of ChatGPT

The entire manuscript was drafted and revised using ChatGPT.

References

- [1] S. Banach, *Sur les fonctionnelles linéaires*, *Studia Mathematica* **1** (1929), 211–216. doi:10.4064/sm-1-1-211-216.
- [2] D. Dacunha-Castelle, *Ultraproduits d’espaces de Banach*, Séminaire Goulaouic–Schwartz (1971–1972), exposé no. 9, 1–11. NUMDAM, exposé IX.
- [3] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto*, *Rendiconti del Seminario Matematico della Università di Padova* **24** (1955), 84–92. NUMDAM, volume 24.
- [4] C. Kuratowski, *Sur les espaces complets*, *Fundamenta Mathematicae* **15** (1930), 301–309. doi:10.4064/fm-15-1-301-309.
- [5] S. Mazur and S. Ulam, *Sur les transformations isométriques d’espaces vectoriels normés*, *Comptes Rendus de l’Académie des Sciences, Paris* **194** (1932), 946–948.
- [6] B. Nica, *The Mazur–Ulam theorem*, *Expositiones Mathematicae* **30** (2012), 397–398. Author version: [arXiv:1306.2380](https://arxiv.org/abs/1306.2380).
- [7] OpenAI, *A positive solution to Tingley’s problem*, preprint dated 23 September 2026. Source PDF in the OpenAI mathematics repository (fixed revision).
- [8] J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, *Studia Mathematica* **2** (1930), 171–180. doi:10.4064/sm-2-1-171-180.
- [9] D. Tingley, *Isometries of the unit sphere*, *Geometriae Dedicata* **22** (1987), 371–378. doi:10.1007/BF00147942.